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Geometrical measure of entanglement for three-particle *W* states

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***Abstract*. The geometrical measure of entanglement of the *W* states is introduced and exact analytic expressions are obtained for it. Based on numerical calculations, the degree of entanglement is considered for some states of this class which are used as a quantum channel in problems of quantum theory of information.**

Keywords: *entangled states, measure of entanglement, quantum channel.*

1. Introduction

Entangled states have specific correlation which is of interest for problems of quantum theory of information. Such states are used as a quantum channel in processes of teleportation, dense coding, key distribution, etc. The quality of a quantum channel and, hence, its potential is determined first of all by the degree of correlations or entanglement. Thus, in the dense coding problem the capacity of a quantum channel formed by the EPR (Einstein – Podolsky – Rosen) pair is directly determined by the degree of its entanglement, which is equal to the entropy of one of the particles in the case of a pure two-particle state. However, if a quantum system is in a mixed state or consists of more than two particles, the universal measure of entanglement is unknown because correlations in a multiparticle system can be various from the physical point of view.

Despite the absence of the universal measure, there is need for entanglement criteria. First of all this concerns experiments where the entangled states are generated which should be identified. To determine most completely the properties of the found state, all possible criteria should be used. In this connection the development of criteria and measures characterising entangled states is an important problem.

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The geometrical measure of entanglement was introduced in [1] and discussed in [2]. However, the authors of these papers have not obtained convenient analytic results characterising specified states. In this paper, by following [3], we introduced the geometrical measure of entanglement for a special class of states – the so-called *W* states and obtained analytic expressions for the measure of entanglement of these states. This set of states is of interest for quantum information processes, some of them are realised and studied in experiments, and their properties and applications and the scheme of their generation are considered in [4].

2. The *W* states

Consider the *W* states for which we introduce below the measure of entanglement. These states describe multiparticle two-level systems, including system with one excited particle, and are reduced to the Dicke states in particular cases [5].

The three-particle *W* state has the form

$$\Psi = p_{000}|000\rangle + p_{100}|100\rangle + p_{010}|010\rangle + p_{001}|001\rangle, \qquad (1)$$

where $|p_{000}|^2 + |p_{100}|^2 + |p_{010}|^2 + |p_{001}|^2 = 1$. A particular case $W = (1/\sqrt{3})(|100\rangle + |010\rangle + |001\rangle)$ is known in the quantum theory of information as the *W* state [6]. For $p_{000} = 0$ and the condition $p_{100} + p_{010} + p_{001} = 0$, the Dicke states appear with $j = 3/2$ and $m = 1/2$, where j and m are the eigenvalues corresponding to the eigenvectors of two collective operators $J^2 = J_1^2 + J_2^2 + J_3^2$, and operators J_k ($k = 1, 2, 3$) satisfy commutation relations for the angular momentum operators [4]. In this case, the representation in terms of antisymmetric wave functions is valid:

$$\eta = \sqrt{2}\left[p_{010}|\Psi^-\rangle_{12}|0\rangle_3 + p_{001}|\Psi^-\rangle_{13}|0\rangle_2\right], \qquad (2)$$

where $\Psi^- = (1/\sqrt{2})(|01\rangle - |10\rangle)$.

Because the measure of entanglement does not change upon local unitary transformations, along with state (1) a number of other states can be considered. Thus, by making the replacement $0 \to 1$, we find $\Psi_1 = p_{011}|011\rangle + p_{101}|101\rangle + p_{110}|110\rangle$. This wave function describes the state which is obtained, for example, after the distribution of two excitations between three two-level particles and can be realised upon parametric interaction of light in a transparent nonlinear medium. To do this requires three simultaneous down-conversion processes, in each of them three classical pump waves being transformed to photon pairs $a-b$, $a-c$, and $b-c$. This interaction can be described by the effective Hamiltonian $H_{\rm eff}={\rm i}\hbar(k_1a^\dagger b^\dagger+k_2a^\dagger c^\dagger+k_3b^\dagger c^\dagger-k_1ab-k_2ac-k_3bc)$, where a, b, and c are the mode annihilation operators and k_x ($x=1, 2, 3$) are coupling constants. This process was considered in [7], and its experimental realisation is known when the states $|0\rangle$ and $|1\rangle$ are the Fock states of light.

The W states differ from well-known GHZ (Grinberger–Horne–Zeilinger) states, in particular, from the typical GHZ state ${\rm GHZ}=(1/\sqrt{2})(|000\rangle+|111\rangle)$. The main difference is that the W states cannot be related by local unitary transformations [6], and, hence, the type of entanglement in them is different. As an example, we present in the explicit form the relation between the states GHZ and $W^*=(1/\sqrt{2})|011\rangle+(1/2)(|010\rangle+|100\rangle)$, which is performed by the nonlocal unitary two-particle operator

$$(1\otimes V)|{\rm GHZ}\rangle_{ABC}=|W^*\rangle, \quad (3)$$

where $V=|\Psi^+\rangle\langle 11|+|00\rangle\langle 10|+|\Psi^-\rangle\langle 01|+|11\rangle\langle 00|$ and $\Psi^\pm=(|10\rangle\pm|01\rangle)/\sqrt{2}$. It is known that the GHZ state can be used as a quantum channel for the teleportation of one particle [8] or the unknown entangled state [9]. The quantum channel formed by W states in the same cases was studied in papers [10] and [11], respectively.

3. Measure of entanglement

The geometrical measure of entanglement is defined as the distance between the W state defined by (1) and the set of all three-particle factorised states. Then, the measure of entanglement or this distance is calculated by solving the variational problem because the minimal distance to the set of factorised or nonnentangled states should be found.

The set of nonentangled states has the form

$$\phi=\varphi_1\otimes\varphi_2\otimes\varphi_3, \quad (4)$$

where $\varphi_k=u_k|0\rangle+z_k|1\rangle$, and the normalisation condition is determined by the relation $\bigotimes_k(|u_k|^2+|z_k|^2)=1$, $k=1$, 2, 3. We will consider the distance

$$E(123)(\Psi)=\min\nolimits_{\Phi\in\Omega(123)}{\rm dist}(\Phi,\Psi) \quad (5)$$

as the measure of entanglement of the specified state, where ${\rm dist}(\Phi,\Psi)=(||\Phi-\Psi||^2)^{1/2}$, $\Omega(123)=\phi$. This is the problem of the search for the minimum of the function ${\rm dist}^2(\Phi,\Psi)=||\Phi-\Psi||^2=(\Psi-\ \Phi;\Psi-\Phi)$ on the set ϕ. Condition (5) has a simple meaning, being equivalent to the maximum value of $(\Psi;\Phi)$, which is known in the quantum theory of information as the fidelity showing with which probability the state Φ contains the state Ψ, or vice versa.

By using the indefinite Lagrange factors, problem (5) is reduced to the determination of the stationary point of the function

$$Q=2-(\Psi;\varphi)-(\varphi;\Psi)+\lambda\bigotimes_k\left(|u_k|^2+|z_k|^2\right)-1=0. \quad (6)$$

The corresponding variational equations lead to the system of algebraic equations

$$\lambda u_1^*(|u_2|^2+|z_2|^2)(|u_3|^2+|z_3|^2)=p_{000}^*+p_{010}^*z_2u_3+p_{001}^*u_2z_3,$$

$$\lambda z_1^*(|u_2|^2+|z_2|^2)(|u_3|^2+|z_3|^2)=p_{100}^*u_2u_3,$$

$$\lambda u_2^*(|u_1|^2+|z_1|^2)(|u_3|^2+|z_3|^2)=p_{000}^*+p_{010}^*z_1u_3+p_{001}^*u_1z_3,$$

$$\lambda z_2^*(|u_1|^2+|z_1|^2)(|u_3|^2+|z_3|^2)=p_{010}^*u_1u_3,$$

$$\lambda u_3^*(|u_1|^2+|z_1|^2)(|u_2|^2+|z_2|^2)=p_{000}^*+p_{010}^*z_2u_1+p_{100}^*u_2z_1,$$

$$\lambda z_3^*(|u_1|^2+|z_1|^2)(|u_3|^2+|z_3|^2)=p_{001}^*u_2u_1.$$

By using the change of variables $c_k=z_k/u_k$ ($k=1, 2, 3$) we write this system in the form

$$c_1=\frac{p_{100}^*}{p_{000}+p_{010}c_2^*+p_{001}c_3^*},$$

$$c_2=\frac{p_{010}^*}{p_{000}+p_{010}c_1^*+p_{001}c_3^*},$$

$$c_3=\frac{p_{001}^*}{p_{000}+p_{010}c_2^*+p_{001}c_1^*}.$$

By solving these equations, we find the required quantity $E(1,2,3)(W)=[2\,(1-F_{\max})]^{1/2}$, where $F_{\max}$ is the fidelity maximum

$$|(\Psi,\varphi)|=\frac{p_{000}+|p_{100}|\rho_1+|p_{010}|\rho_2+|p_{001}|\rho_3}{\left[(1+\rho_1^2)(1+\rho_2^2)(1+\rho_3^2)\right]^{1/2}}. \quad (7)$$

Here, the quantities $\rho_{1,2,3}$ are related by the equation

$$|p_{100}|Z(\rho_1)=|p_{010}|Z(\rho_2)=|p_{001}|Z(\rho_3), \quad (8)$$

where $Z(x)=x+1/x$. For one of the unknowns, we can write the closed equation

$$p_{000}=\pm T(\rho_1), \quad (9)$$

where

$$T(\rho)=\frac{|p_{100}|}{\rho}-|p_{010}|Z^{-1}\left[\frac{|p_{100}|}{|p_{010}|}Z(\rho)\right]$$

$$-|p_{001}|Z^{-1}\left[\frac{|p_{100}|}{|p_{001}|}Z(\rho)\right].$$

Equation (9) is obtained by assuming that p_{000} is real and $p_{000}\geqslant 0$, $|p_{100}|\geqslant|p_{010}|$, $|p_{001}|$.

As an example, we consider the following case. Let us assume that all coefficients are real and introduce the parametrisation $p_{100}=q\cos\theta$, $p_{010}=q\sin\theta\cos\varphi$, $p_{001}=$

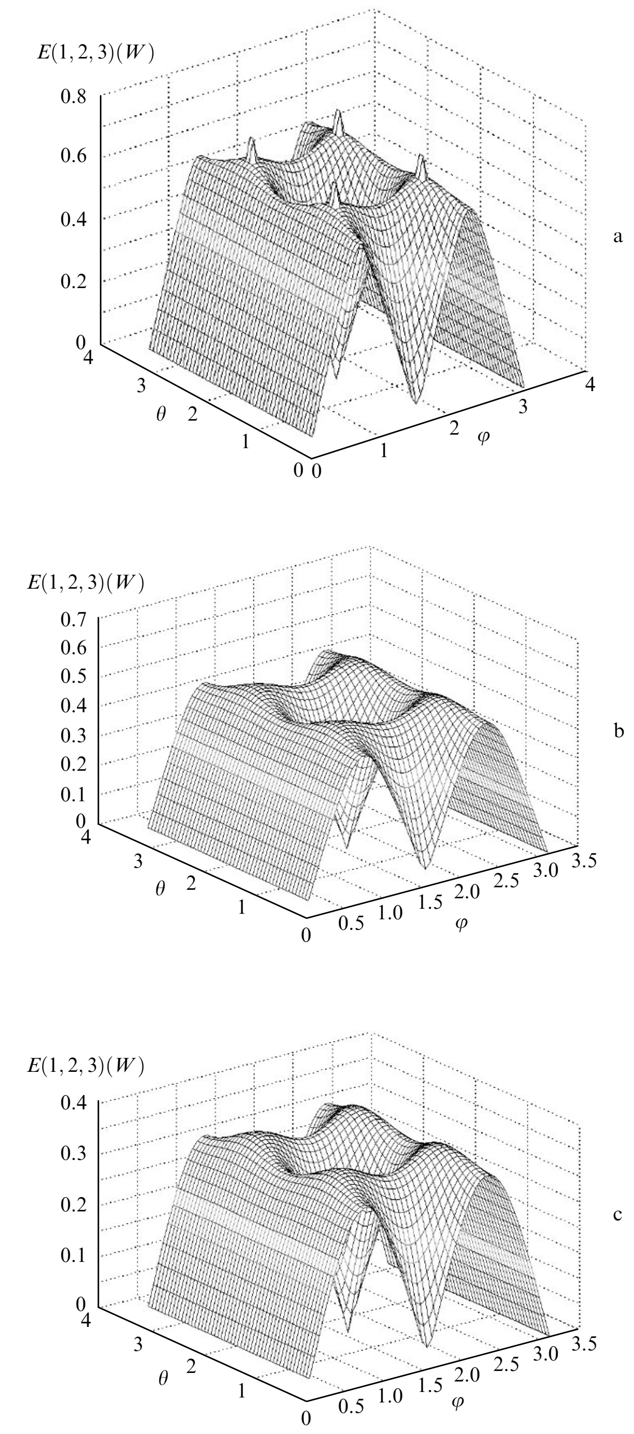

Figure 1. Geometrical measure of entanglement of the W state for $p_{000} = 0.2$ (a), 0.4 (b), and 0.6 (c).

$q \sin\theta \sin\varphi$, and $q = (1 - p_{000}^2)^{1/2}$, where q, θ, and φ are spherical coordinates. Figure 1 shows the measure of entanglement of the W state for different values of p_{000}. It has the characteristic four-peak shape with the dip down to zero at the surface centre. One can see that, the measure of entanglement $E(1, 2, 3)(W)$ decreases with increasing p_{000}. This is explained by the fact that the weight of the state $|000\rangle$ increases and the state as a whole proves to be close to this factorised state. The peaks of $E(1, 2, 3)(W)$ are formed due to the presence of the EPR pair $\Psi^{\pm} = (1/\sqrt{2})(|01\rangle \pm |10\rangle)$, which is maximally entangled in itself. Thus, by rewriting (1), taking into account the accepted parametrisation, we find, for example, that $\Psi = (1 - q^2)^{1/2}|000\rangle + q \sin\theta |0\rangle_1 (\cos\varphi |01\rangle + q \sin\varphi |10\rangle) + q\cos\theta|100\rangle$. It follows from this that for $\cos\varphi = \sin\varphi$, the EPR pairs appear. The dip at the surface centre appears because for $\theta = \varphi = \pi$ our state will be not entangled: $[(1 - q^2)^{1/2}|0\rangle + q|1\rangle]|00\rangle$.

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