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Eigenfrequencies of a nonreciprocal nonuniformly filled ring optical resonator

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***Abstract*. A nonreciprocal ring optical resonator with the simplest type of nonuniformity in the form of two interacting scatterers is considered. A boundary problem of a periodic type allowing one to write the characteristic equation for determining eigenfrequencies is formulated. Analytic expressions for eigenfrequencies and the distribution coefficient of travelling waves in the mode are derived under the assumption of relative smallness of the reflection coefficient and the nonreciprocity parameter. The peculiarities of splitting these frequencies and the structure of the eigenfrequencies (resonator modes) corresponding to them are discussed.**

Keywords: *ring resonator, eigenfrequencies, modes, frequency nonreciprocity.*

1. Usually the theory of ring lasers is formulated based on the equations for the expansion coefficients (oscillation amplitudes) of the generated field in travelling waves. The latter ones are the modes of a uniform reciprocal (ideal) ring resonator with a doubly degenerate frequency spectrum. To substantiate the appearance of the internal synchronisation (locking) of travelling waves and to explain the peculiarities of the regime of their mutual beatings, a linear coupling is introduced phenomenologically in the dynamic equations for the expansion coefficients. At the same time, it is possible and even natural to use the expansion of the field over the modes of a real ring resonator, nonreciprocal (in some cases) and usually nonuniform (due to the finite aperture of reflectors, the presence of a diaphragm, the nonuniformity of the refractive index, etc.). In the case of this expansion, the dynamic equations for the oscillation amplitudes will be coupled only due to the mutual influence through the active medium, which allows the approach to the analysis of the locking and beat phenomena to be somewhat changed.

To use the expansion in the modes of a real resonator, it is necessary to solve quite accurately the spectral problem for this resonator. This was the aim of Refs [1, 2], where the real resonator was considered reciprocal and a diaphragm in the form of a thin cylinder served as the nonuniformity. Modes and the spectrum of a reciprocal resonator with another type of nonuniformity was studied in [3], i.e. two partially reflecting infinite facets (sites of the reflection coefficient jump in the filling medium) taking multiple reflections between them into account. The nonuniformity was localised in [1, 2], which specified the description of the modes in the paraxial approximation. The approximation of plane waves is adequate to the formulation of the problem in [3].

In this paper, we summarise the results of paper [3] for the case of not only a nonuniform but also a nonreciprocal resonator in order to construct later the theory of a laser gyroscope based on the expansion of its radiation over the modes of such a real resonator. Losses in the filling medium are neglected. A problem is stated to obtain within the framework of the resonator theory an analytic dependence of the frequency splitting on the nonreciprocity and nonuniformity parameters. To avoid the undesirable cumbersome expressions and to take into account, at the same time, the main peculiarity of the nonuniformity (the possibility of mutual multiple reflections in the resonator), its simplest model as in paper [3] was taken: filling the resonator with a piecewise-uniform dielectric with the step reflective index only in two maximally remote sections. Only forward and backward radiation scattering is possible in this representation and, hence, losses due to the radiation exit from the resonator are absent. The formulation of the problem used here, despite the ultimate simplicity of the mode (as well as due to it) transfers some important issues from the level of the laser theory to the level of a simpler linear resonator theory.

2. Let ω_p be the degenerate eigenfrequency of a reciprocal uniform ring resonator (p is a natural number). The degeneracy is removed in a nonreciprocal ring resonator. One travelling wave corresponds to each of different eigenfrequencies $\omega_p^{(\pm)}$. Below, we briefly describe the scheme used to remove the degeneracy because it is this scheme that we will use to take the nonuniformity into account.

The field E (for example, the transverse component of a polarised electric field) in a one-dimensional ring resonator is described by the wave equation

$$\frac{\partial^2 E}{\partial z^2} + 2\frac{a}{c_0}\frac{\partial^2 E}{\partial z \partial t} - \frac{n^2}{c_0^2}\frac{\partial^2 E}{\partial t^2} = 0, \qquad (1)$$

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where c_0 is the speed of light; n is the refractive index of the filling medium; a is the dimensionless nonreciprocity parameter {if the axial contour of the resonator has the form of a circle with a perimeter L, and the nonreciprocity is produced by its rotation with a constant angular velocity Ω with respect to the normal to the resonator plane, then $a = L\Omega/(2\pi c_0) \ll 1$ [4]}. The field in the ring resonator without losses should satisfy the boundary conditions of a periodic type:

$$E(0,t) = E(L,t), \quad \frac{\partial E(0,t)}{\partial z} = \frac{\partial E(L,t)}{\partial z}. \tag{2}$$

Waves $E(z,t) = u(z)\exp(\mathrm{i}\omega t)$ satisfy boundary problems (1), (2), where $u(z)$ are the solutions of the stationary boundary problem

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + \mathrm{i}2a\frac{\omega}{c_0}\frac{\mathrm{d}u}{\mathrm{d}z} + n^2\frac{\omega^2}{c_0^2}u = 0, \tag{3}$$

$$u(0) = u(L), \quad \frac{\mathrm{d}u(0)}{\mathrm{d}z} = \frac{\mathrm{d}u(L)}{\mathrm{d}z}. \tag{4}$$

A spatial travelling wave $u(z) = E'\exp(-\mathrm{i}kz)$ is the solution of wave equation (3), if the frequency ω and the wave number k satisfy the dispersion equation

$$k^2 - k2a\chi - \chi^2 n^2 = 0, \quad \chi = \frac{\omega}{c_0}. \tag{5}$$

If the wave number $k = k_p^{(\pm)} = \pm k_{p0} = \pm 2\pi p/L$, the travelling wave, satisfying boundary conditions (4) is the solution of boundary problem (3), (4), i.e. this wave is a mode of a ring resonator. Eigenfrequencies are found from expression (5). Let us solve it with respect to wave numbers:

$$k_p^{(\pm)} = -a\chi_p^{(\pm)} \pm \left[\left(a\chi_p^{(\pm)}\right)^2 + n^2\chi_p^{(\pm)2}\right]^{1/2} \approx \pm n\chi_p^{(\pm)} - a\chi_p^{(\pm)}.$$

We obtain here the doublets of eigenfrequencies $\omega_p^{(\pm)} = c_0\chi_p^{(\pm)} = c_0 k_{p0}/(n \mp a)$. They depend both on the degenerate frequency $\omega_{p0} = c_0 k_{p0}$ in the absence of nonreciprocity and on the nonreciprocity parameter, which causes the splitting of the degenerate frequency and the doublet formation. Thus, all eigenfrequencies $\omega_p^{(\pm)}$ are different and a natural travelling wave (mode) $E_p^{(\pm)}(z,t) = E_p^{(\pm)}\exp(\mp \mathrm{i}k_{p0}z) \times \exp(\mathrm{i}\omega_p^{(\pm)}t)$ corresponds to each of them.

3. Consider a nonreciprocal nonuniform ring resonator according to the scheme in clause 2. Let the refractive index of the filling medium be:

$$n(z) = \begin{cases} n_1 & \text{for} \quad 0 \leqslant z < \dfrac{L}{2}, \\ n_2 & \text{for} \quad \dfrac{L}{2} \leqslant z < L. \end{cases}$$

It was shown in [3] that this nonuniformity, similarly to nonreciprocity, removes the degeneracy of the spectrum of eigenfrequencies of a uniform reciprocal ring resonator. One should expect that the combined effect of the nonreciprocity and nonuniformity will also lead to a nondenegerate (simple) spectrum, the influence of both factors increasing the frequency splitting of doublets compared to the influence of any of them. The splitting of eigenfrequencies will be found to prove this assumption. The parameter of the reduced nonreciprocity $2\pi pa/n_{\mathrm{av}}$, where $n_{\mathrm{av}} = (n_1 + n_2)/2$, and the reflection coefficient $R_1 = (n_1 - n_2)/(n_1 + n_2)$ are considered the small quantities of the same order.

4. The field in a nonuniform nonreciprocal ring resonator without losses satisfies the wave equation

$$\frac{\partial^2 E}{\partial z^2} + \frac{a}{c_0}\frac{\partial^2 E}{\partial z \partial t} - \frac{n^2(z)}{c_0^2}\frac{\partial^2 E}{\partial t^2} = 0 \tag{6}$$

and boundary conditions of periodic type (2). We will seek natural oscillations in the form $E(z,t) = u(z)\exp(\mathrm{i}\omega t)$. The mode $u(z)$ should be the solution of the boundary problem

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + \mathrm{i}2a\frac{\omega}{c_0}\frac{\mathrm{d}u}{\mathrm{d}z} + n^2(z)\frac{\omega^2}{c_0^2}u = 0 \tag{7}$$

with boundary conditions (4) supplemented with conditions

$$u\left(\frac{L_+}{2}\right) = u\left(\frac{L_-}{2}\right), \quad \frac{\mathrm{d}u}{\mathrm{d}z}\left(\frac{L_+}{2}\right) = \frac{\mathrm{d}u}{\mathrm{d}z}\left(\frac{L_-}{2}\right). \tag{8}$$

Subscripts '+' and '−' at L in expression (8) mean the right and left vicinities of the point $z = L/2$, respectively.

For a travelling wave $u(z) = E'\exp(-\mathrm{i}kz)$ on each interval, where the refractive index is constant, the dispersion equation

$$k^2 - 2a\frac{\omega}{c_0}k - \frac{\omega^2}{c_0^2}n_1^2 = 0 \quad \text{for} \quad 0 \leqslant z < \frac{L}{2},$$

$$k^2 - 2a\frac{\omega}{c_0}k - \frac{\omega^2}{c_0^2}n_2^2 = 0 \quad \text{for} \quad \frac{L}{2} \leqslant z < L$$

is valid. Two wave numbers are found here for each of the mentioned intervals, the wave numbers corresponding to the specified frequency:

$$k_1^{(\pm)} = \chi\left[\pm(a^2 + n_1^2)^{1/2} + a\right] \approx \pm\chi(n_1 \pm a) \quad \text{for} \quad 0 \leqslant z < \frac{L}{2}, \tag{9}$$

$$k_2^{(\pm)} = \chi\left[\pm(a^2 + n_2^2)^{1/2} + a\right] \approx \pm\chi(n_2 \pm a) \quad \text{for} \quad \frac{L}{2} \leqslant z < L. \tag{10}$$

The distribution $u(z)$ satisfying (7) should be searched for in the form of a wave with different ratios of amplitudes of counterpropagating travelling waves within each interval with a constant refractive index:

$$u_1(z) = E_1^{(+)}\exp\left(-\mathrm{i}k_1^{(+)}z\right) + E_1^{(-)}\exp\left(-\mathrm{i}k_1^{(-)}z\right) \quad \text{for} \quad 0 \leqslant z < \frac{L}{2}, \tag{11}$$

$$u_2(z) = E_2^{(+)}\exp\left(-\mathrm{i}k_2^{(+)}z\right) + E_2^{(-)}\exp\left(-\mathrm{i}k_2^{(-)}z\right) \quad \text{for} \quad \frac{L}{2} \leqslant z < L, \tag{12}$$

where $E_{1,2}^{(\pm)}$ are unknown amplitudes of travelling waves. Their choice is determined by boundary conditions (4) and (8).

Boundary condition of periodicity (4) allows one to obtain the first relation in the form

$$E_1^{(+)} = R_1 E_1^{(-)} + T_2 E_2^{(+)} \exp\left(-\mathrm{i}k_2^{(+)}L\right),$$
$$E_2^{(-)} \exp\left(-\mathrm{i}k_2^{(-)}L\right) = -R_1 E_2^{(+)} \exp\left(-\mathrm{i}k_2^{(+)}L\right) + T_1 E_1^{(-)}. \tag{13}$$

Boundary condition of continuity (8) allows one to obtain the second relation in the form

$$E_1^{(-)} \exp\left(-\mathrm{i}k_1^{(-)}\frac{L}{2}\right) = R_1 E_1^{(+)} \exp\left(-\mathrm{i}k_1^{(+)}\frac{L}{2}\right) + T_2 E_2^{(-)} \exp\left(-\mathrm{i}k_2^{(-)}\frac{L}{2}\right),$$
$$E_2^{(+)} \exp\left(-\mathrm{i}k_2^{(+)}\frac{L}{2}\right) = -R_1 E_2^{(-)} \exp\left(-\mathrm{i}k_2^{(-)}\frac{L}{2}\right) + T_1 E_1^{(+)} \exp\left(-\mathrm{i}k_1^{(+)}\frac{L}{2}\right). \tag{14}$$

Here, $T_2 = n_2/n_{\mathrm{av}}$ and $T_1 = n_1/n_{\mathrm{av}}$ are the transmission coefficients. The ratio between them and the reflection coefficient R_1 is obvious: $T_1T_2 = 1 - R_1^2$.

5. We will use below the reduced amplitudes of travelling waves

$$\tilde{E}_1^{(\pm)} = E_1^{(\pm)} \exp\left(-\mathrm{i}k_1^{(\pm)}\frac{L}{2}\right),\ \tilde{E}_2^{(\pm)} = E_2^{(\pm)} \exp\left(-\mathrm{i}k_2^{(\pm)}\frac{L}{2}\right).$$

In these notations, boundary conditions (13) and (14) will take the from:

$$\begin{pmatrix} \tilde{E}_2^{(-)} \\ \tilde{E}_2^{(+)} \end{pmatrix} = \frac{1}{T_2} \times \begin{pmatrix} \exp\left[\mathrm{i}\left(k_1^{(-)}+k_2^{(-)}\right)\frac{L}{2}\right] & -R_1\exp\left[\mathrm{i}\left(k_1^{(+)}+k_2^{(-)}\right)\frac{L}{2}\right] \\ -R_1\exp\left[\mathrm{i}\left(k_1^{(-)}+k_2^{(+)}\right)\frac{L}{2}\right] & \exp\left[\mathrm{i}\left(k_1^{(+)}+k_2^{(+)}\right)\frac{L}{2}\right] \end{pmatrix}, \tag{15}$$

$$\begin{pmatrix} \tilde{E}_1^{(-)} \\ \tilde{E}_1^{(+)} \end{pmatrix} = \frac{1}{T_1}\begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix}\begin{pmatrix} \tilde{E}_2^{(-)} \\ \tilde{E}_2^{(+)} \end{pmatrix}. \tag{16}$$

Let us combine these two equations into one by using in this case the ratio between the transmission and reflection coefficients presented above. As a result, we obtain the condition of self-reproduction of the vector of reduced amplitudes after a round trip in the resonator:

$$\begin{pmatrix} \tilde{E}_1^{(-)} \\ \tilde{E}_1^{(+)} \end{pmatrix} = \hat{W}\begin{pmatrix} \tilde{E}_1^{(-)} \\ \tilde{E}_1^{(+)} \end{pmatrix}. \tag{17}$$

This condition is typical of ring structures of different nature. By substituting (15) into (16), we obtain the transformation matrix (per transit)

$$\hat{W} = \frac{1}{T_2}\frac{1}{T_1}\begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix} \times \begin{pmatrix} \exp\left[\mathrm{i}\left(k_1^{(-)}+k_2^{(-)}\right)\frac{L}{2}\right] & -R_1\exp\left[\mathrm{i}\left(k_1^{(+)}+k_2^{(-)}\right)\frac{L}{2}\right] \\ -R_1\exp\left[\mathrm{i}\left(k_1^{(-)}+k_2^{(+)}\right)\frac{L}{2}\right] & \exp\left[\mathrm{i}\left(k_1^{(+)}+k_2^{(+)}\right)\frac{L}{2}\right] \end{pmatrix}, \tag{18}$$

which depends on the resonator parameters (including the nonreciprocity parameter) and on the unknown yet frequency.

The zero vector of reduced amplitudes in Eqn (17) exists only for frequencies, which are the roots of the characteristic equation

$$\mathrm{Det}\,(\hat{W} - \hat{I}) = 0, \tag{19}$$

where $\hat{I}$ is the unit matrix. The frequencies determined in this way are the eigenfrequencies of the resonator. Mixed waves (11), (12) corresponding to them are eigenwaves (modes) for boundary problem (7), (4) and (8). Let us derive the expression for eigenfrequencies.

6. It is easy to show that $\mathrm{Det}\,(\hat{W} - \hat{I}) = 1 - \mathrm{Sp}\,\hat{W} + \mathrm{Det}\,\hat{W}$. Let us find the trace and determinant of transformation matrix (17):

$$\mathrm{Det}\,W = \frac{1-R_1^2}{(T_1T_2)^2}\left\{\exp\left[\mathrm{i}\left(k_1^{(-)}+k_2^{(-)}+k_1^{(+)}+k_2^{(+)}\right)\frac{L}{2}\right] - R_1^2\exp\left[\mathrm{i}\left(k_1^{(+)}+k_2^{(-)}+k_1^{(-)}+k_2^{(+)}\right)\frac{L}{2}\right],\right.$$

$$\mathrm{Sp}\,\hat{W} = \frac{1}{T_1T_2} \times \left\{\exp\left[\mathrm{i}\left(k_1^{(-)}+k_2^{(-)}\right)\frac{L}{2}\right] - R_1^2\exp\left[\mathrm{i}\left(k_1^{(-)}+k_2^{(+)}\right)\frac{L}{2}\right] - R_1^2\exp\left[\mathrm{i}\left(k_1^{(+)}+k_2^{(-)}\right)\frac{L}{2}\right] + \exp\left[\mathrm{i}\left(k_1^{(+)}+k_2^{(+)}\right)\frac{L}{2}\right]\right\}.$$

By using (9), (10) and the ratio between the reflection and transmission coefficients as well as some cumbersome transformations, we reduce the latter expressions to the form

$$\mathrm{Det}\,\hat{W} = \exp\left(-\mathrm{i}\chi 4a\frac{L}{2}\right),$$

$$\mathrm{Sp}\,\hat{W} = 2\exp\left(-\mathrm{i}\chi 2a\frac{L}{2}\right)(1-R_1^2)^{-1}\left\{\cos\left[n_1+n_2)\frac{L}{2}\right] - R_1^2\cos\left[\chi(n_1-n_2)\frac{L}{2}\right]\right\}.$$

Taking these expressions into account, characteristic equation (19) can be written in the form:

$$2\exp\left(-\mathrm{i}\chi 2a\frac{L}{2}\right)\frac{1}{1-R_1^2}\left\{(1-R_1^2)\cos\left(\chi 2a\frac{L}{2}\right)\right.$$
$$\left.-\cos\left[\chi(n_1+n_2)\frac{L}{2}\right]-R_1^2\cos\left[\chi(n_1-n_2)\frac{L}{2}\right]\right\}=0.$$

It is convenient to represent this equation in the equivalent form:

$$\sin\left(\chi\frac{n_1+n_2-2a}{2}\frac{L}{2}\right)\sin\left(\chi\frac{n_1+n_2+2a}{2}\frac{L}{2}\right)$$
$$-R_1^2\sin\left(\chi\frac{n_1-n_2-2a}{2}\frac{L}{2}\right)\sin\left(\chi\frac{n_1-n_2+2a}{2}\frac{L}{2}\right)=0. \quad (20)$$

If $a=0$ and $R_1=0$, the roots of Eqn (20) are doubly degenerate and form a sequence $k_p=2\pi p/(n_{\mathrm{av}}L)$ (formally $n_1=n_2$) for natural p. We again arrived at the above-mentioned frequency spectrum of a reciprocal uniformly filled ring resonator.

It is natural to search for the reduced eigenfrequencies χ with the help of the theory of degenerate spectrum perturbations at small parameters of nonreciprocity and nonuniformity (in the above sense). Let us write for the first term in (20) by using a linear approximation for its factors:

$$\sin\left(\chi\frac{n_1+n_2-2a}{2}\frac{L}{2}\right)\sin\left(\chi\frac{n_1+n_2+2a}{2}\frac{L}{2}\right)$$
$$\approx\left(\frac{n_1+n_2}{2}\frac{L}{2}\right)^2(\chi-k_p^{(+)})(\chi-k_p^{(-)}), \quad (21)$$

where $k_p^{(\pm)}=2\pi p/[(n_{\mathrm{av}}\mp a)L]$ are the roots of the first and second cofactors, respectively.

The second term in (20) is small. To preserve the order of smallness, the parameter χ in it can be replaced by k_p; and a number of approximate transformations shown below can be performed. First,

$$R_1^2\sin\left(\chi\frac{n_1-n_2-2a}{2}\frac{L}{2}\right)\sin\left(\chi\frac{n_1-n_2+2a}{2}\frac{L}{2}\right)$$
$$\approx R_1^2\sin^2\left(k_p\frac{n_1-n_2}{2}\frac{L}{2}\right). \quad (22)$$

In addition, for χ close to k_p, Eqn (20) under expressions (21) and (22) can be replaced by the quadratic equation in the form

$$(\chi-k_p^{(+)})(\chi-k_p^{(-)})\approx R_1^2\sin^2\left(R_1k_{p0}\frac{L}{2}\right)\left(n_{\mathrm{av}}\frac{L}{2}\right)^{-2},$$

where $k_{p0}=2\pi p/L=2\pi/\lambda$.

By solving this equation we obtain two values $\chi_p^{(\pm)}$:

$$\chi_p^{(\pm)}=\frac{k_p^{(+)}+k_p^{(-)}}{2}$$
$$\pm\left[\left(\frac{k_p^{(+)}-k_p^{(-)}}{2}\right)^2+R_1^2\sin^2\left(R_1k_{p0}\frac{L}{2}\right)\left(n_{\mathrm{av}}\frac{L}{2}\right)^{-2}\right]^{1/2}. \quad (23)$$

In the zero approximation as follows from above the expression for splitting frequencies caused by the degenerate frequency perturbation has the form:

$$\Delta\omega_p=\omega_p^{(+)}-\omega_p^{(-)}=(\chi_p^{(+)}-\chi_p^{(-)})c_0$$
$$=2\frac{c_0}{Ln_{\mathrm{av}}}\left[(ak_{p0}L)^2+4R_1^2\sin^2\left(R_1k_{p0}\frac{L}{2}\right)\right]^{1/2}. \quad (24)$$

This expression within the framework of the assumptions performed takes into account the effect of both the nonreciprocity and nonuniformity of filling a ring resonator.

7. Let us formulate the main results of this paper. The degenerate frequencies (of a reciprocal uniform resonator) form an equidistant spectrum with a frequency spacing $\omega_{p0}-\omega_{p-1\,0}=2\pi c_0/(n_1L)$. The rotation and nonuniformity of the resonator filling remove the degeneracy. Expression (23) yields an expression for a doublet of eigenfrequencies $\omega_p^{(\pm)}=c_0\chi_p^{(\pm)}$ lying near the degenerate eigenfrequency $\omega_{p0}=c_0k_{p0}$. The frequency difference for each doublet (frequency splitting) is determined by expression (24). It depends both on the number of the doublet (through k_{p0}) and on the resonator parameter, i.e. the nonreciprocity parameter and the reflection coefficient. Smaller (as well as larger) doublet frequencies form a non-equidistant sequence. A similar dependence is also typical of the difference of eigenfrequencies of coupled conservative contours with different partial resonance frequencies. Thus, a certain degenerate eigenfrequency of an 'unperturbed' ring resonator corresponds to coinciding eigenfrequencies of two identical uncoupled contours. The coupling (of the inductive and capacitive type) between the contours is a counterpart of coupling between the waves of a ring resonator due to its nonuniform filling. In the presence of coupling and 'nonreciprocity' (differences in the partial frequencies of contours), eigenfrequencies of the system of contours, as eigenfrequencies of the resonator, 'push' apart [5], each of the two factors only increasing the splitting. The internal synchronisation in the ring resonator [6] can be considered to be the compensation for total splitting of eigenfrequencies of a real resonator due to nonlinear coupling between modes (which are not travelling waves). Naturally, the theory of generation in a two-frequency ring laser is more complicated and the mentioned analogy has only a qualitative character.

Expression (24) allows one to notice an important feature of splitting doublet frequencies, which, for obvious regions, does not have an analogue in the system of coupled conservative contours. For an approximate numerical estimate by expression (24) let us specify the values of quantities entering it. If the axial contour of the ring resonator is a circle with the perimeter $L=0.4$ m and the nonreciprocity is formed by the resonator rotation with the angular velocity equal to the velocity of the Earth rotation, $a=1.55\times10^{-14}$.

Consider a resonator filled with media with $n_1=1.1+5\times10^{-4}$ and $n_2=1.1-5\times10^{-4}$ as an example explaining the effect of two sharp nonuniformities of the resonator (regarded as interacting scattering centres). In this case, $n_{\mathrm{av}}=1.1$, $\Delta n=n_1-n_2=10^{-3}$, and $R_1=10^{-5}$. By using a resonator with the central wavelength $\lambda_0=10^{-6}$ m, the degenerate wave number is $k_{p0}=6.28\times10^6$ m^{-1}. The frequency splitting depends not only on R_1 but also on the periodically changing quantities $|\sin(R_1k_{p0}L/2)|$. One can see that at comparatively low difference in the refractive indices, the contribution introduced into slpitting (24) due to reflections on the inhomogeneity, can vary from zero to the maximum value $\Delta\omega_p^{\max} \approx 4R_1 c_0/(n_{\mathrm{av}}L)$. For $n_{\mathrm{av}} = 1.1$, $L = 0.4$ m and $R_1 = 10^{-3}$, this contribution is 2.72×10^5 rad s^{-1}, which is the order of the capture region in a ring laser with a considered real resonator.

The frequency splitting varies from the minimum value depending only on the nonreciprocity parameter to the maximum value determined also by the reflection coefficient (numerical estimates are presented in Fig. 1). For clearness, the nonreciprocity parameter in Fig. 1 is $a = 3 \times 10^{-11}$. One can see that at some values of the refractive index difference (reflection coefficient) the medium filling the resonator 'is bleached': the frequency splitting is determined by the nonreciprocity only. This is the effect of interaction of scatterers, which also takes place for more realistic inhomogeneities.

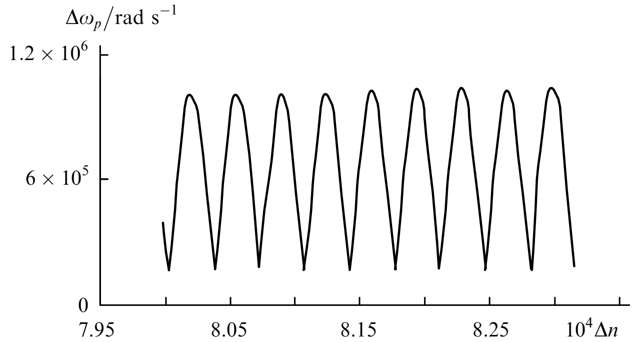

Figure 1. Dependence of the eigenfrequency difference $\Delta\omega_p$ (24) on the refractive index difference Δn for $a = 3 \times 10^{-11}$ and $k_{p0}L = 3.99\times 10^6$.

8. The description of modes of the resonator under study is also of certain interest. The modes are determined with the accuracy up to a constant value and only the distribution coefficient, i.e. the ratio of amplitudes of counterpropagating travelling waves forming this mode, is their quantitative characteristic. The ring resonator problem under study is similar to the quantum-mechanical spectral problem with a periodic potential and periodic boundary conditions. The solutions of this problem can be (depending on the potential properties) periodic eigenfunctions of the type of the standing or travelling wave.

In our case we can expect similar results. In this case, we should take into account that even insignificant but jumpwise changes in the refractive index Δn are efficient equivalent point reflectors. In the presence of two such reflectors, a system of two coupled linear resonators, which are combined into one ring resonator, is formed. Depending on Δn and geometrical parameters of the system, the coupling of such linear resonators can be weak (up to complete independence) and strong (up to the formation of a uniform ring resonator). In the first case, the modes will be similar to standing waves (the modulus of the distribution coefficient is comparable to unity) and, in the second case, the mode is similar to a travelling wave (the modulus of the distribution coefficient is either much larger than unity or close to zero). The degree of coupling in the system of interacting linear resonators periodically depends on Δn (other parameters being fixed). Indeed, the equivalent dimensions of the resonators change with changing Δn, which, in turn, obviously leads to a periodic reproducibility of all the properties of the system.

As follows from (17), the coefficient of the mode distribution in the region of the medium with $n = n_1$ for the reduced eigenfrequency $\chi = \chi_p^{(\pm)}$ can be derived from expression

$$\frac{E_1^{(-)}}{E_1^{(+)}} = \frac{\tilde{\bar{E}}_1^{(-)}}{\tilde{E}_1^{(+)}} = \frac{W_{12}}{1 - W_{11}}. \qquad (25)$$

The mentioned elements of the matrix $\hat{W}$ depend only on χ. By using (18), we can show that

$$W_{12} = \frac{R_1}{T_1 T_2} \times$$

$$\times \left\{ \exp\left[\mathrm{i}\left(k_1^{(+)} + k_2^{(+)}\right)\frac{L}{2} \right] - \exp\left[\mathrm{i}\left(k_1^{(+)} + k_2^{(-)}\right)\frac{L}{2} \right] \right\}$$

$$= \frac{R_1}{T_1 T_2} \left[\exp\left(\mathrm{i}\chi L \frac{n_1 + n_2}{2} \right) \exp(-\mathrm{i}\chi L a) \right.$$

$$\left. - \exp\left(\mathrm{i}\chi L \frac{n_1 - n_2}{2} \right) \right],$$

$$W_{11} = \frac{1}{T_1 T_2}$$

$$\times \left\{ \exp\left[\mathrm{i}\left(k_1^{(-)} + k_2^{(-)}\right)\frac{L}{2} \right] - R_1^2 \exp\left[\mathrm{i}\left(k_1^{(-)} + k_2\,(+)\right)\frac{L}{2} \right] \right\}$$

$$= \frac{1}{T_1 T_2} \left[\exp\left(-\,\mathrm{i}\chi L \frac{n_1 + n_2}{2} \right) \exp(\mathrm{i}\chi L a) \right.$$

$$\left. - R_1^2 \exp\left(-\,\mathrm{i}\chi L \frac{n_1 - n_2}{2} \right) \right].$$

By omitting the terms proportional to R_1^2 and using two last expressions, Eqn (25) can be written in the form:

$$\frac{E_1^{(-)}}{E_1^{(+)}}$$

$$\approx R_1 \frac{\exp(\mathrm{i}\chi L n_{\mathrm{av}})\exp(-\mathrm{i}\chi L a) - \exp[\mathrm{i}\chi L(n_1 - n_2)/2]}{1 - \exp(-\mathrm{i}\chi L n_{\mathrm{av}})\exp(\mathrm{i}\chi L a)}. \qquad (26)$$

It is easy to obtain from (23) after some cumbersome transformations that

$$\exp(\mathrm{i}\chi_p L n_{\mathrm{av}})$$

$$= \exp\left\{ \pm\,\mathrm{i}\left[(a k_{p0} L)^2 + 4R_1^2 \sin^2\left(R_1 k_{p0} \frac{L}{2} \right) \right]^{1/2} \right. \qquad (27)$$

[one can easily see that expression (24) follows from (27)].

Expression (27) was obtained under the condition that $ak_{p0}L$ and $2R_1$ have the same order of smallness; this condition is not fully correct in the limiting case $R_1 = 0$ and $\alpha \neq 0$, when one should take advantage of significantly simpler relation, which was stated above. The use of (27) in (26) allows one to give numerical estimates of the modulus of distribution coefficient (26) in the specified approximation. Figure 2 shows the dependence of this quantity on Δn for the same values of n_{av}, a, L, k_{p0}, and λ as in clause 7.

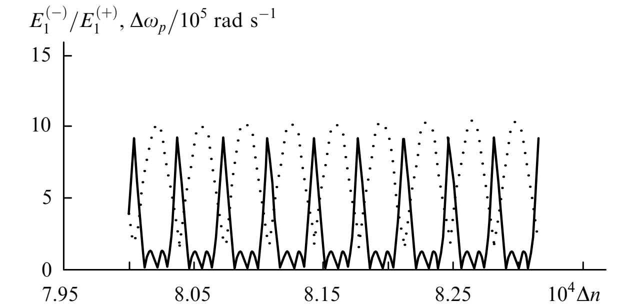

Figure 2. Dependences of the modulus of the distribution coefficient $E_1^{(-)}/E_1^{(+)}$ (26) of one of the modes (solid curve) and eigenfrequency difference $\Delta\omega_p$ (24) (dashed curve) on the refractive index difference Δn.

For comparison, Fig. 2 also presents the dependence $\Delta\omega_p(\Delta n)$. The calculation was performed only for one reduced eigenfrequency $\chi_p^{(+)} = \omega_p^{(+)}/c_0$. A similar dependence for another eigenfrequency (for the corresponding mode) can be calculated by using expression (24).

All the above mentioned peculiarities of a nonuniform ring resonator are seen in Fig. 2. These modes are on average close to standing waves. However, for those Δn when the eigenfrequency difference is determined by the nonreciprocity only (resonator 'bleaching'), the mode is close to a travelling wave (the modulus of the distribution coefficient achieves 10). For some values of Δn in the region of a strong influence of the nonuniformity (when the frequency difference is mainly caused by the resonator nonuniformity), the modulus of the distribution coefficient achieves the lower limit of 0.4, which corresponds to a slight difference of the mode from a standing wave. It is possibly caused by the approximations used but the physical explanation cannot be excluded: the prevalence of the opposite direction of wave propagation in the mode for some Δn can be caused by a complex character of interaction of waves in the system of two coupled linear resonators under the equivalence of both propagation directions (this is obvious if the nonreciprocity is absent).

Figure 3 shows for completeness the dependence of the modulus of the distribution coefficient on the nonreciprocity parameter for fixed Δn providing the mode character similar to that of a standing wave. This dependence in a wide range of variations in the nonreciprocity parameter is weak and monotonous. For other Δn, the regularity is the same, i.e. the mode structure is mainly determined by the resonator nonuniformity.

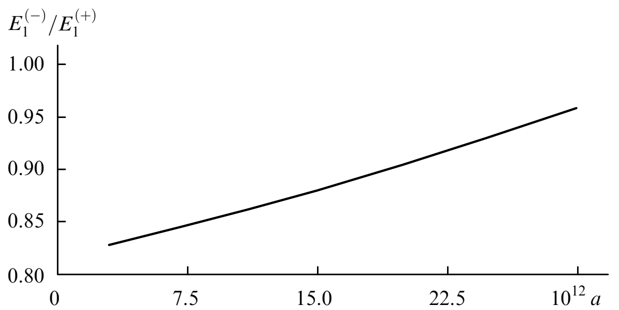

Figure 3. Dependences of the modulus of the distribution coefficient $E_1^{(-)}/E_1^{(+)}$ (26) on the nonreciprocity parameter a for $\Delta n = 8.158 \times 10^{-4}$ and $k_{p0}L = 3.99 \times 10^6$.

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