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Dynamic synchronisation regions of a ring laser with the use of a periodic support

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***Abstract*. The method is proposed for calculating dynamic synchronisation regions of a ring laser with a periodic frequency support of a special type. The proposed algorithm in essence taking into account the special type of the support allows the search for minimal widths of regions when the support parameters change. The widths of the regions are calculated as an example for the case of the harmonic carrier modulation as a harmonic envelope (three-frequency support) and an envelope of the 'soft' meander type (multifrequency support).**

Keywords: *ring laser, dynamic synchronisation regions, alternating support.*

1. Introduction

A single-mode (two-wave) ring laser can operate in the two-frequency generation regime (each wave is generated at its own frequency) and in the single-frequency regime (both waves are generated at one frequency, i.e. mutually phase-locked). Realisation of these regimes depends on the ratio of the difference in the resonator eigenfrequencies $\Delta\Omega$ (frequency nonreciprocity) to the value Ω_{st} [static synchronisation threshold (SST)]. The single-frequency regime occurs for $\Delta\Omega/\Omega_{st} \leqslant 1$, while the two-frequency regime – in the opposite case. If the SST is inadmissibly large, a periodic frequency support (frequency modulation) at the frequency ν_m with the amplitude Ω_m is introduced, which substantially exceeds the SST. The support is the part of the difference in frequencies generated by counterpropagating waves, which is formed by the control signal (i.e. completely known). As the measured (i.e. unknown in advance) part of this frequency difference, the support is generated by the nonreciprocity of the ring resonator. In the presence of a periodic support, the frequency difference $\bar{\Omega}$ being measured is a result of some operation performed above the value set of the phase difference in the counterpropagating waves $\varphi(t)$ in certain discrete instants of time. In the simplest case, the phase difference is quantized with the step π, the specified instants of time correspond to the quantized values of the phase and the expression

$$\bar{\Omega} = \frac{\varphi(nT_m) - \varphi(0)}{nT_m}$$

is used to estimate the frequency nonreciprocity, where $T_m = 2\pi/\nu_m$ is the support period; n is an interger (in theory, usually $n = 1$). It is possible to estimate rather accurately in this way the frequency nonreciprocity, which is smaller than the SST, but even in this case it should exceed the value Ω_{dyn} called the dynamic synchronisation threshold (DST).

It has been also found that in the vicinity of the values $\Delta\Omega$ multiple of the support frequency, the estimate of the nonreciprocity drastically deteriorates due to the appearance of dynamic synchronisation regions (DSRs) of different orders. The appearance of such regions is the result of synchronisation of a coupled system of two generators in the regime of beatings on the support frequency overtone. The main peculiarities of using the periodic support show up in the dependence $\bar{\Omega}$ ($\Delta\Omega$), which can be treated as an output characteristic of the ring laser as a measuring converter. This characteristic was studied in many papers. Early papers [1 – 5] mainly described the DST; however, in [6] the DSRs of the lowest regions were studied in a numerical experiment. The theory was further developed in the direction of more exact and detailed description of the entire complex of the DSRs and the characteristic behaviour between them.

The phase difference of counterpropagating waves in the presence of a periodic support $\Omega_m H(\nu_m t)$ satisfies the known expression

$$\frac{d\varphi}{dt} = \Delta\Omega + \Omega_m H(\nu_m t) - \Omega_{st}\cos\varphi. \quad (1)$$

Usually, either a harmonic support or a support of the meander type is used (this is determined by specific technical decisions). In the first case one frequency is present in its spectrum, while in the second case, the spectrum is a multifrequency one. For this reason, it is the multifrequency periodic support that is studied in many papers. In particular, the general theory of a ring laser with a periodic support was constructed in [7, 8] but for $\nu_m \gg \Omega_{st}$. Not only DSRs of the integral orders (in the linear coupling approximation) [7] but also DSRs of the

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half-integer orders (in the quadratic coupling approximation) [8] were studied.

The specified restriction is not always fulfilled in practice. To make up for this deficiency, the Floquet theory was used in [9] instead of the averaging method. To apply this theory, nonlinear equation (1) was reduced to the second-order differential equation for some artificial 'oscillation' with a periodic coefficient and phase close to $\varphi(t)$. However, the authors of [9] succeeded in using the general theory for the support of the meander type. Note that in [7] the general theory was in fact used to calculate only the harmonic support and the meander support, although the possibilities of the averaging method are far from being restricted by this. Further significant generalisations are presented in papers [10, 11], where the characteristic $\bar{\Omega}(\Delta\Omega)$ is described in detail for the case of the meander support, based on [7], i.e. on the averaging method. In particular, not only the DSR dimensions but also the positions of their centres are specified (which refines the scale factor of the characteristic). Some special types of the unmodulated support (including a harmonic and a meander support) were studied in [11]. Other papers quoted above are the generalisation of the obtained results for the case of the support involving noise components. A special place is held by paper [12], where a physically illustrative method (vector model) convenient for applications was developed, which is efficient, in particular, for studying a periodic support with a noise component.

All papers known to us are devoted to a periodic support satisfying an important condition, which was formulated in [11]: any horizontal straight line is intersected by a curve describing the support modulation no more than two times at the period of this modulation. In this case, the shape of this support at the period can be arbitrary – from meander to harmonic (including triangle, saw-like, exponential, etc.). However, apart from the support of this type, a basically different type is possible, i.e. a modulated support including fast (carrier) and slow components so that several carrier periods fitted the period of the slow component. In this case, it is reasonable in the construction of the theory to focus on the high-frequency component of the support and instead of the averaging method to use an alternative technique, which takes into account the special character of the modulated support. This paper is devoted to the description and application of this method. For definiteness, we consider the supports of two types: with the modulation of a harmonic signal by a harmonic signal or an exponentially smoothed meander. Such supports, as far as we know, have not been studied in detail so far. The method realised below has a limited field of application but within this field it is efficient for numerical calculations, which is important for practical applications.

2. Basic equation and its transformation

If a modulated support is used to form the characteristic $\bar{\Omega}(\Delta\Omega)$, phase equation (1) can be written in the form:

$$\frac{d\varphi}{dt} = \Delta\Omega - \Omega_m \cos(\nu_c t)[1 + \varepsilon H(\nu_m t)] - \Omega_{st}\cos\varphi. \quad (2)$$

In this particular case, the support represents a harmonic carrier at the frequency ν_c with a periodic modulation by the function $H(\nu_m t)$ ($|H(\nu_m t)| \leqslant 1$) and the modulation depth $\varepsilon < 1$. Eqaution (2) can be reduced to the from

$$\frac{d\varphi}{dt} = \Delta\Omega - \Omega_m \cos(\nu_c t) + \Omega_m \varepsilon f(t) - \Omega_{st}\cos\varphi, \quad (3)$$

where

$$f(t) = \cos(\nu_c t)H(\nu_m t). \quad (4)$$

By using the dimensionless time $\tau = \Omega_{st}t$, Eqn (3) can be written in the from

$$\frac{d\varphi}{d\tau} = \gamma - \sigma\cos(\alpha\tau) + \varepsilon\sigma\bar{f}(\alpha\tau) - \cos\varphi(\tau), \quad (5)$$

where $\bar{f}(\alpha\tau) = 2\sin(\alpha\tau)H(M^{-1}\alpha\tau)$ and dimensionless parameters

$$\alpha = \frac{\nu_c}{\Omega_{st}}; \quad \sigma = \frac{\Omega_m}{\Omega_{st}} \geqslant 1; \quad \gamma = \frac{\Delta\Omega}{\Omega_{st}}; \quad M = \frac{\nu_c}{\nu_m} \geqslant 2 \quad (6)$$

are introduced.

It is easy to assume that the support period is $T_m = 2\pi/\nu_m$, although as we will see below it is not obligatory (the support can be almost periodic). Let us represent solution (5) in the from:

$$\varphi(\tau) = m\alpha\tau - \frac{\sigma}{\alpha}\sin(\alpha\tau) + \varepsilon\frac{\sigma}{\alpha}F(\alpha\tau) + \varphi_m(\tau), \quad (7)$$

where the integer $m \geqslant 0$ is the DSR order;

$$F(\alpha\tau) = \int \bar{f}(\zeta)d\zeta$$

is the periodic function; $\zeta = \alpha\tau$. By substituting (7) into expression (5), it is easy to derive the equation for the additive phase $\varphi_m(\tau)$:

$$\frac{d\varphi_m(\tau)}{d\tau} = \gamma - m\alpha - \cos[m\alpha\tau - \eta\sin(\alpha\tau) + \varepsilon\eta F(\alpha\tau) + \varphi_m(\tau)],$$

where $\eta = \Omega_m/\nu_c \geqslant 1$.

If we integrate this equation in the dimensionless time at the interval $[-M\pi/\alpha, M\pi/\alpha]$, which corresponds to the modulation period, we obtain the expression for the phase increment at the specified interval:

$$\Delta\varphi_m = (\gamma - m\alpha)2M\frac{\pi}{\alpha} - \int_{-M\pi/\alpha}^{M\pi/\alpha} \cos[m\alpha\tau - \eta\sin(\alpha\tau) + \varepsilon\eta F(\alpha\tau) + \varphi_m(\tau)]d\tau. \quad (8)$$

It is easy to find from (7) and (8) that

$$\bar{\Omega} = \frac{\Delta\varphi(\tau)}{2M\pi/\alpha} = m\alpha + \frac{\Delta\varphi_m(\tau)}{2M\pi/\alpha}.$$

By definition the quantity $\bar{\Omega}$ is taken as the estimate of the frequency nonreciprocity. One can see that for all $\Delta\Omega$ so that $\Delta\varphi_m(\tau)/(2M\pi/\alpha) = 0$, the mth order synchronisation takes place: $\bar{\Omega} = m\alpha$. The synchronisation width $2S_m$ can be determined from the value of the integral in expression (8).

3. Estimate of the DSR width in the general form

Let us introduced in the study the function $\Re_m(\eta, \varepsilon)$ of a large parameter η:

$$\Re_m(\eta,\varepsilon) = \frac{1}{2M\pi}\int_{-M\pi}^{M\pi} \exp\{\mathrm{i}[m\zeta - \eta\sin\zeta + \varepsilon\eta F(\zeta)]\}\mathrm{d}\zeta. \quad (9)$$

It follows from the estimates for $\eta \gg 1$ and $\varepsilon < 1$ that the second term in the exponent substantially exceeds two others at points $\zeta_k = (2k+1)\pi/2$. For $-M \leqslant k \leqslant M-1$, these points are stationary for integral (9). Therefore, an asymptotic approximation [13]

$$\Re_m(\eta,\varepsilon) \approx \frac{1}{2M\pi}\sqrt{\frac{2\pi}{\eta}}\sum_{k=-M}^{k=M-1}\exp\left\{\mathrm{i}\left[m\zeta_k - \eta(-1)^k + \frac{\pi}{4}(-1)^k + \varepsilon\eta F(\zeta_k)\right]\right\} \quad (10)$$

can be used for it. Extreme stationary points are neglected because terms corresponding to them in (10) are proportional to $2\pi/\eta$, i.e. noticeably lower than the terms taken into account. It follows from (8) that synchronisation occurs under the condition

$$|\gamma - m\alpha| \leqslant S_m(\eta,\varepsilon) = \frac{1}{2M\pi}\int_{-M\pi}^{M\pi}\cos[m\zeta - \eta\sin\zeta + \varepsilon\eta F(\zeta)]\mathrm{d}\zeta. \quad (11)$$

We obtain from (11) and (10) an approximate expression for the DSR half-width of the mth order in the general form:

$$S_m(\eta,\varepsilon) = |\mathrm{Re}\,\Re_m(\eta,\varepsilon)| = \frac{1}{2M\pi}\sqrt{\frac{2\pi}{\eta}} \times\left|\sum_{k=-M}^{k=M-1}\cos\left[m\zeta_k - \eta(-1)^k + \frac{\pi}{4}(-1)^k + \varepsilon\eta F(\zeta_k)\right]\right|. \quad (12)$$

Note that expression (11) is equivalent to the condition $|\Omega - m\nu_\mathrm{c}| \leqslant S_m(\eta,\varepsilon)\Omega_\mathrm{st}$, i.e. $S_m(\eta,\varepsilon)$ is the DSR half-width normalised to the SST and characterising the support influence.

In the absence of modulation ($\varepsilon = 0$), we obtain the expression

$$S_m(\eta,\varepsilon) = |\mathrm{Re}\,\Re_m(\eta,\varepsilon)| = \frac{1}{2M\pi}\sqrt{\frac{2\pi}{\eta}}\left|\sum_{k=-M}^{k=M-1}\cos\left[m\zeta_k - \eta(-1)^k + \frac{\pi}{4}(-1)^k\right]\right|.$$

It is easy to show that it does not depend on M and therefore, we can set $M = 1$. As a result. We obtain

$$S_m(\eta,0) = \sqrt{\frac{2}{\pi\eta}}\left|\cos\left(m\frac{\pi}{2} - \eta + \frac{\pi}{4}\right)\right|.$$

This is a known asymptotic expression for the DSR half-width of the mth order in the case of the harmonic support, which indirectly confirms the validity of the main result described by expression (12). The modulation effect is reasonably described by the influence coefficient

$$\Sigma_m(\eta,\varepsilon) = \frac{S_m(\eta,\varepsilon)}{S_m(\eta,0)}. \quad (13)$$

The quantity $\Sigma_0(\eta,\varepsilon)$ related to the region of mutual phase matching of waves (the main DSR) is of special interest.

4. Estimate of the DSR widths for the specified types of the support modulation

Let us use general expressions to calculate the coefficient $\Sigma_m(\eta,\varepsilon)$ for the specific cases.

4.1 A three-frequency support

Let the function $H(\nu_\mathrm{m}t)$ be equal to $\cos(\nu_\mathrm{m}t)$ in Eqn (2). This support contains three harmonics in its spectrum. In this case, $\bar{f}(\alpha\tau) = \cos(\alpha\tau)H(M^{-1}\alpha\tau)$ and its prototype is

$$F(\alpha\tau) = \int\bar{f}(\zeta)\mathrm{d}\zeta = \frac{1}{2}\left\{\frac{\sin[(1+M^{-1})\alpha\tau]}{1+M^{-1}} + \frac{\sin[(1-M^{-1})\alpha\tau]}{1-M^{-1}}\right\}.$$

By substituting this expression into (13), we will calculate $\Sigma_m(\eta,\varepsilon)$, $m = 0$, 1 for different values of M, η, and ε. Typical values of these parameters are $4 \leqslant M \leqslant 8$, $100 \leqslant \eta \leqslant 200$, $0 \leqslant \varepsilon \leqslant 0.1$. The results of these calculations are presented in Figs 1–3.

It follows from Fig. 1 that if four periods of the carrier fit one period of the envelope, the influence of the modulation depth on the DSR width is the same in the regions of different orders in a broad range of the excess of the modulation frequency Ω_m by the average frequency ν_m. There exist modulation depths ε and parameters η for which the DSR widths drastically decrease. For example, for $\varepsilon = 0.068$, $\eta = 100$, the influence coefficient is 0.01, while for $\varepsilon = 0.082$, $\eta = 200$ it is 0.009, i.e. the DSR widths decrease by approximately 100 times due to the modulation.

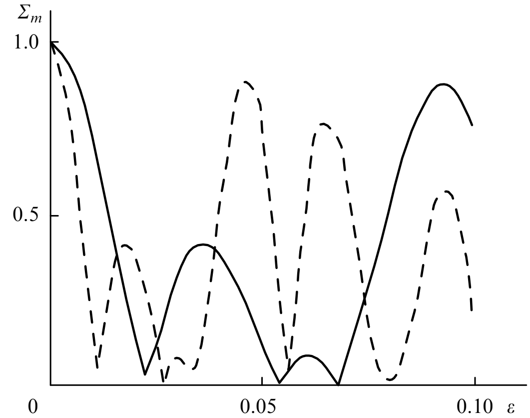

Figure 1. Dependences of the influence coefficient $\Sigma_m(\eta,\varepsilon)$ on the modulation depth ε for lowest DSRs ($m = 0$, 1) at $\eta = 100$ (solid curve) and 200 (dashed curve), $M = 4$.

An increase of M up to 8, as follows from Fig. 2, confirms that the modulation depth similarly affects the DSR widths of different orders. For $\varepsilon = 0.085$ and $\eta = 100$, the DSR widths decrease by 200 times, while for 0.07 and $\varepsilon = 0.07$, $\eta = 200$ – approximately by 100 times.

Calculations show (Fig. 3) that the effect of the modulation depth on the DSR widths of different orders is absolutely the same. As a parameter, we selected the ratio of the envelope amplitude to the quadruple carrier frequency $\varepsilon\Omega_\mathrm{m}/(4\nu_\mathrm{c}) = \varepsilon\eta/4$ (for $\eta = 200$, $M = 4$), which is convenient for comparison with the experimental results. At some

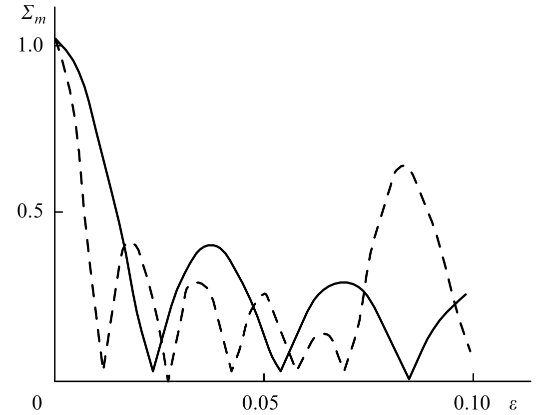

Figure 2. Dependences of the influence coefficient $\Sigma_m(\eta, \varepsilon)$ on the modulation depth ε for lowest DSRs ($m = 0$, 1) at $\eta = 100$ (solid curve) and 200 (dashed curve), $M = 8$.

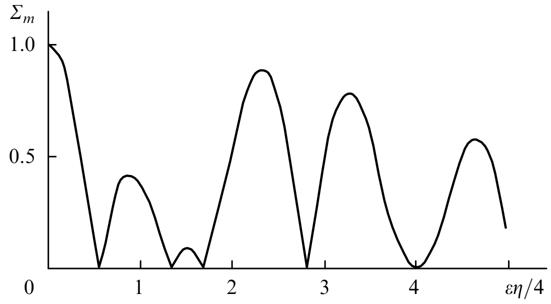

Figure 3. Dependences of the influence coefficient $\Sigma_m(\eta, \varepsilon)$ on $\varepsilon\eta/4$ for $m = 0$ and 1, $M = 4$, $\eta = 200$.

modulation depths, a significant decrease in the DSR widths is possible. For example, for $\varepsilon\eta/4 = 1.69$ the modulation leads to a decrease in these widths by more than 1500 times.

The results of calculations give an idea of the influence of different parameters of a harmonically modulated support on the DSR width. One can see that the choice of the modulation width ε allows one to substantially decrease these widths (equally for the regions of different orders). By varying the parameters M, ε, and η we can determine the maximum decrease in the DSR widths.

4.2 A multifrequency support

Consider the modulation of a harmonic carrier by the function shown in Fig. 4. Analytically this function can be written in the form:

$$H(t) = \left\{ h_0 - (1 + h_0) \exp\left[-\beta\left(t + \frac{T_\mathrm{m}}{2} \right) \right] \right\} \times \left[\Phi\left(t + \frac{T_\mathrm{m}}{2} \right) - \Phi(t) \right] + [-h_0 + (1 + h_0)\exp(-\beta t)] \times \left[\Phi(t) - \Phi\left(t - \frac{T_\mathrm{m}}{2} \right) \right], \tag{14}$$

where

$$h_0 = \left[1 + \exp\left(-\frac{\beta}{v_\mathrm{c}} \pi M \right) \right] \left[1 - \exp\left(-\frac{\beta}{v_\mathrm{c}} \pi M \right) \right]^{-1};$$

$\Phi(t)$ is the Heaviside function [14]. The whole family of curves – from an almost exact meander to a triangle shape – correspond to the function $H(t)$. The shape parameter is an addition support parameter $\beta = v_\mathrm{c}/Q$, which is convenient to replace by the Q factor. The spectrum of the support modulated in this way is rather broad (it does not depend on Q). The prototype $F(\alpha\tau) = \int \bar{F}(\zeta) \mathrm{d}\zeta$ used in (12) can be found from expression (14):

$$F(\alpha\tau) = \left\{ h_0 \sin(\alpha\tau) - (1 + h_0) \exp\left(-\frac{1}{Q} \pi M \right) \times \frac{\exp(-\alpha\tau/Q)[-Q^{-1}\cos(\alpha\tau) + \sin(\alpha\tau)]}{1 + (\beta/v_\mathrm{c})^2} \right\} \times [\Phi(\alpha\tau + \pi M) - \Phi(\alpha\tau)]$$
$$+ \left\{ -h_0 \sin(\alpha\tau) + (1 + h_0) \exp\left(-\frac{1}{Q} \alpha\tau \right) \times \frac{\exp(-\alpha\tau/Q)[-Q^{-1}\cos(\alpha\tau) + \sin(\alpha\tau)]}{1 + (\beta/v_\mathrm{c})^2} \right\} \times [\Phi(\alpha\tau) - \Phi(\alpha\tau - \pi M)].$$

By substituting this expression into (13), we will calculate $\Sigma_m(Q, \eta, \varepsilon)$ for $m = 0$, 1 at different values of M, η, ε, and Q. Typical values of these parameters are as follows: $4 \leqslant M \leqslant 8$, $100 \leqslant \eta \leqslant 200$, $0 \leqslant \varepsilon \leqslant 0.1$, and $2 \leqslant Q \leqslant 6$. The results of calculations are presented in Figs 5–7.

It follows from Fig. 5 that when the modulation depth changes, its influence on the DSR width is the same for the regions of different orders. There exist values of the modulation depth ε and parameter η at which the DSR

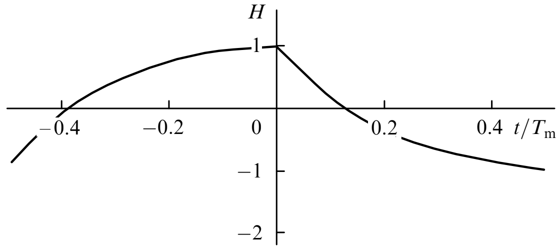

Figure 4. Modulation by the 'soft' meander $H(t)$ for the factor $Q = 5$ ($\beta = 377$).

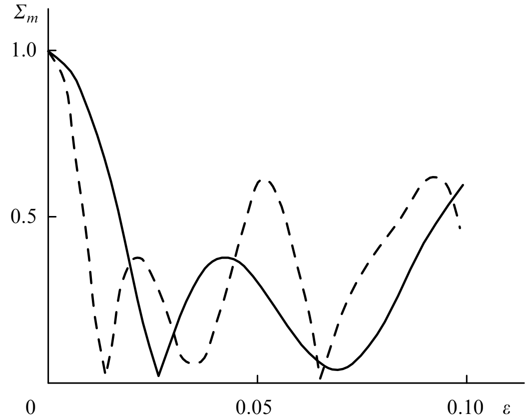

Figure 5. Dependences of the influence coefficient $\Sigma_m(Q, \eta, \varepsilon)$ on the modulation depth ε for lowest DSRs ($m = 0$, 1) at $\eta = 100$ (solid curve) and 200 (dashed curve), $M = 4$, $Q = 5$.

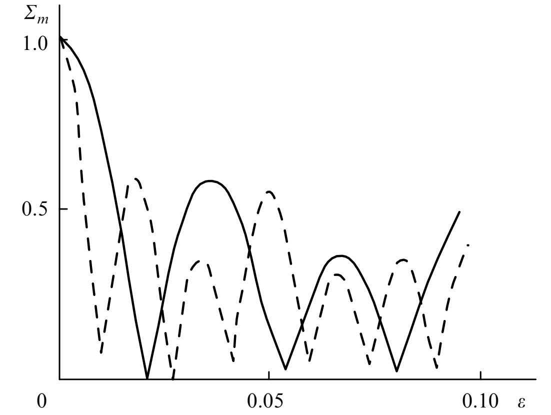

Figure 6. Dependences of the influence coefficient $\Sigma_m(Q, \eta, \varepsilon)$ on the modulation depth ε for lowest DSRs ($m = 0$, 1) at $\eta = 100$ (solid curve) and 200 (dashed curve), $M = 8$, $Q = 5$.

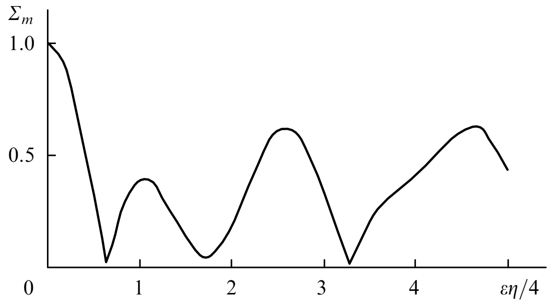

Figure 7. Dependences of the influence coefficient $\Sigma_m(Q, \eta, \varepsilon)$ on $\varepsilon\eta/4$ for $M = 4$, $\eta = 200$, $m = 0$ and 1.

widths drastically decrease, although not so fast as in the case of the harmonic modulation. For example, for $\varepsilon = 0.026$, $\eta = 100$, the influence coefficient is 0.02, while for $\varepsilon = 0.066$, $\eta = 200$, the influence coefficient is 0.011 (cf. the commentary to Fig. 3).

For $M = 8$ the dependences of the influence coefficient on the modulation depth presented in Fig. 6 change noticeably. It is important that the DSR widths decrease significantly at lower η: for $\varepsilon = 0.021$ and $\eta = 100$ the influence coefficient is 0.0009, while for $\varepsilon = 0.027$ and $\eta = 200$ it is 0.005. It follows from the comparison of Figs 7 and 3 that the passage to the modulation by a 'soft' meander' deteriorates on the whole the possibility of affecting the DSR: the number of possible values of the parameter $\varepsilon\eta/4$ decreases for which the influence coefficient is minimal and its values become substantially larger than in the case of the harmonic modulation (by more than 15 times for the best results).

Of course, the dependences presented in this paper do not cover all possible variants but numerous experiments performed by us indicate the preservation of qualitative laws.

5. Conclusions

The proposed method allows one to reveal the main properties of the output characteristic of the ring laser with a periodic support. In particular, general expressions are presented for the DSR widths (at frequencies multiple of the fundamental frequency of the support) at any types of modulation of the periodic support and within the framework of accepted assumptions (the support spectrum should be concentrated in the vicinity of the carrier frequency). However, no significant limitations are imposed on the position of this frequency with respect to $\Omega_{\rm st}$. The DSR widths in the accepted approximation depend not on the position of the first region with respect to the SST but on the quantity $\eta = \Omega_{\rm m}/\nu_{\rm c} = \sigma/\alpha \gg 1$, which makes the developed theory suitable in the range of parameters where the averaging method is not applicable. The DSR widths are determined by the quantity $M = \nu_{\rm c}/\nu_{\rm m}$. In the limiting case, when the modulation is absent ($\varepsilon = 0$), averaging can be formally performed over the period $T_{\rm m} = M2\pi/\nu_{\rm c} = 2\pi/\nu_{\rm m}$ ($\nu_{\rm m}$ means some rather low frequency corresponding to the selected period of averaging).

The general theory has been used to calculate two types of the harmonic carrier modulation. We have studied only those regions, which for $\Delta\Omega$ are located close to frequencies multiple of the carrier frequency $\nu_{\rm c}$. A more detailed analysis allows also the estimate of those regions, which lie in the vicinity of frequencies multiple of the modulation frequency $\nu_{\rm m}$; however, their widths are incommensurably smaller than the widths of the first type. Therefore, we have restricted ourselves to the study of the latter ones.

It has been found (from the ratio of the DSR widths without and with modulation) that the modulation equally affects the widths of all the DSRs independent of their order. The influence of the support is determined by the parameter $\varepsilon\eta = \varepsilon\Omega_{\rm m}/\nu_{\rm c}$, i.e. by the excess of the carrier frequency by the envelope amplitude.

The types of the support chosen as an example demonstrate the possibilities of the method and allow one to test it in experiments. However, the method is applicable for the study of more complex types of the modulation of the support of the above type.

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