

NONLINEAR OPTICAL PHENOMENA

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Interaction of a probe pulse with a 'dressed' Bose – Einstein condensate of rarefied atomic gases

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***Abstract*. Semiclassical equations describing the interaction of a probe pulse with a 'dressed' Bose – Einstein condensate of a rarefied atomic gas are proposed. The analytic solution of these equations is obtained for low-intensity pulses. The conditions of the appearance of a diffraction grating from recoil atoms are found. The existence of induced superradiance at the probe-beam frequency is predicted. The pulse propagation velocity in the condensate is determined as a function of its energy. The limits of the applicability of the two-level model of a 'dressed' atom are estimated.**

Keywords: *'dressed' Bose – Einstein condensate, diffraction grating of recoil atoms, probe pulse delay, induced superradiance, multiple excitation of atoms.*

1. Introduction

Optical phenomena observed upon interaction of laser radiation with the Bose – Einstein condensate (BEC) of various substances attract recent attention of many researchers [1 – 20]. This interest is caused by the fact that a BEC has two very important properties: a long (hundreds of microseconds) transverse relaxation time of the induced dipole moment in atoms and extremely small (a few cm s^{-1}) thermal velocities of atoms. These properties are manifested differently depending on the medium geometry, radiation intensity, etc. [3 – 6].

In this paper, the experiment on the interaction of the 'dressing' and probe laser beams with the condensate of sodium vapour confined in a magnetic trap of diameter 20 μm and length 200 μm and containing a few millions of atoms is considered [5]. The dressing beam frequency is shifted to the red with respect to the resonance frequency of the $3S_{1/2}$, $F = 1 \rightarrow 3P_{3/2}$, $F = 0, 1, 2$ transition by 1.7 GHz, while the probe beam frequency is in turn is lower by 91 kHz than the dressing beam frequency. Both beams lie in the plane perpendicular to the elongation axis of the condensate and intersect at the angle 135°. The main result of the experiment was the observation of the probe beam amplification when the dressing beam was switched on. The amplification was most noticeable for the relatively weak probe beam. As the probe beam power was increased, the amplification decreased, and when the intensities of the beams became comparable, the amplification changed to decaying oscillations. For a comparatively high dressing beam intensity, the residual radiation at the probe beam frequency was detected after the irradiation of the condensate. In addition, the delay of the probe pulse at the output was observed, which was interpreted as a decrease in the group velocity of light.

The interaction of two laser beams with a condensate is described, as a rule, semiclassically [5, 19, 20]. At present the basic properties of the probe beam amplification are determined [5, 19, 20]. In addition, the delay mechanism of probe pulses was qualitatively explained [5, 19] and the existence of the residual radiation was interpreted [5].

The aim of this paper is to analyse quantitatively the two latter effects and estimate the limits of applicability of the two-level model of a 'dressed' atom used for their description. In addition, the conditions of multiple excitation of atoms and, first of all, the conditions of population of states providing the reemission of photons from the probe wave to the dressing wave are studied

2. Equations of the model

Because the interaction of two laser beams with an atom is quasi-resonant, we consider the atom as a system with two electronic states with the wave functions ϕ_a (ground state) and ϕ_b (excited state). During reemission, a recoil momentum is imparted to the atom. By neglecting the finite size of a trap, we describe the translational motion of the atom with the momentum $\hbar\boldsymbol{k}$ by the de Broglie wave. The corresponding wave function of the atom can be written in the form

$$|s;\boldsymbol{k}\rangle = \frac{1}{V^{1/2}}\exp(\mathrm{i}\boldsymbol{kr})\phi_s \ \ (s = a, b), \tag{1}$$

where V is the condensate volume.

The wave function Ψ of the atom in an arbitrary state can be defined by the expansion

$$\Psi = \sum_{s=a,b;\boldsymbol{k}} c_{s,\boldsymbol{k}}|s;\boldsymbol{k}\rangle, \tag{2}$$

where $c_{s,\boldsymbol{k}}$ are time-dependent expansion coefficients. Note that in basis (1) chosen above, the electronic wave functions ϕ_s also depend on time [through the time phase factor

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$\exp(-\mathrm{i}\omega_s t)$, where $\hbar\omega_s$ is the electronic energy of the atom in the state s].

The total laser field can be written as the superposition of the fields of the dressing (E_d) and probe (E_p) waves propagating perpendicular to the elongation direction of the condensate (the z axis) and intersecting at the angle 135° in the xy plane:

$$E = \mathrm{i}\sum_{f=\mathrm{d,p}} E_f \exp(-\mathrm{i}\omega_f t + \mathrm{i}\boldsymbol{k}_f\boldsymbol{r}) + \mathrm{c.c.} \quad (3)$$

Here, $\boldsymbol{k}_\mathrm{d} = k_\mathrm{d}(\boldsymbol{i}-\boldsymbol{j})/\sqrt{2}$; k_d and k_p are the wave numbers; $\boldsymbol{i}$ and $\boldsymbol{j}$ are the unit vectors along the x and y axes, respectively. The polarisation of all waves is assumed to be directed along the z axis.

By using the rotating wave method, we can obtain in the dipole approximation the following equations for the coefficients $c_{s,\boldsymbol{k}}$ and amplitudes E_f of the average fields in the condensate [20]:

$$\dot{c}_{a,\boldsymbol{k}} = \frac{d}{\hbar}\left[\bar{E}_\mathrm{d} c_{b,\boldsymbol{k}+\boldsymbol{k}_\mathrm{d}} + \bar{E}_\mathrm{p}\exp(-\mathrm{i}\delta\omega t) c_{b,\boldsymbol{k}+\boldsymbol{k}_\mathrm{p}}\right] \times \exp(-\mathrm{i}\Delta\omega t) - \mathrm{i}w_k c_{a,\boldsymbol{k}},$$

$$\dot{c}_{b,\boldsymbol{k}} = -\frac{d}{\hbar}\left[E_\mathrm{d} c_{a,\boldsymbol{k}-\boldsymbol{k}_\mathrm{d}} + E_\mathrm{p}\exp(-\mathrm{i}\delta\omega t) c_{a,\boldsymbol{k}-\boldsymbol{k}_\mathrm{p}}\right] \times \exp(\mathrm{i}\Delta\omega t) - \mathrm{i}\left(\frac{\Gamma_\mathrm{R}}{2} + w_k\right) c_{b,\boldsymbol{k}}, \quad (4)$$

$$\frac{D}{2c}\dot{E}_\mathrm{p} = E_\mathrm{p}^\mathrm{e} - E_\mathrm{p} + \frac{\hbar}{d\tau_\mathrm{R}}\exp[-\mathrm{i}(\delta\omega + \Delta\omega t)] \sum_{\boldsymbol{k}'-\boldsymbol{k}=\boldsymbol{k}_\mathrm{p}} \bar{c}_{a,\boldsymbol{k}} c_{b,\boldsymbol{k}'}, \quad (5)$$

$$\frac{D}{2c}\dot{E}_\mathrm{d} = E_\mathrm{d}^\mathrm{e} - E_\mathrm{d} + \frac{\hbar}{d\tau_\mathrm{R}}\exp(-\mathrm{i}\Delta\omega t) \sum_{\boldsymbol{k}'-\boldsymbol{k}=\boldsymbol{k}_\mathrm{p}} \bar{c}_{a,\boldsymbol{k}} c_{b,\boldsymbol{k}'}, \quad (6)$$

where E_d^e and E_p^e are the field amplitudes of the dressing and probe beams at the input to the condensate; $\Delta\omega = \omega_{ba} - \omega_\mathrm{d}$ is the electronic resonance mismatch; $\delta\omega = \omega_\mathrm{d} - \omega_\mathrm{p}$ is the frequency difference of the dressing and probe waves; Γ_R is the spontaneous decay probability of the excited atomic state; $\hbar w_k$ is the kinetic recoil energy of an atom with the momentum $\hbar\boldsymbol{k}$; d is the matrix element of the dipole transition moment; $\tau_\mathrm{R} = c\hbar/(\pi\omega_\mathrm{d} d^2 N_0 D)$ is the superradiance lifetime; N_0 is the concentration of atoms; D is the transverse size of the condensate; $\omega_{ba} = \omega_b - \omega_a$; and the horizontal bar denotes complex conjugation.

Note that the field amplitudes E_d and E_p change in the condensate linearly and, therefore, their difference at the output and input is twice as large as their average values.

The Rabi frequency in experiments on the interaction of laser radiation with the BEC of alkali atoms [3–6] is much smaller than the resonance mismatch. Therefore, the amplitude of the excited electronic state can be written in the form $c_{b,\boldsymbol{k}} = C_{b,\boldsymbol{k}}\exp(\mathrm{i}\Delta\omega t)$, where $C_{b,\boldsymbol{k}}$ is a slowly varying function of time [14]. In this case, its time derivative can be neglected, the amplitude can be expressed from the second equation in (4) and substituted to the remaining equations. As a result, we obtain for $C_{a,\boldsymbol{k}} = c_{a,\boldsymbol{k}}$ and the field amplitudes E_d and E_p the equations

$$\dot{C}_{a,\boldsymbol{k}} = \frac{d^2(\mathrm{i}-v)}{\hbar^2\Delta\omega}\left[\left(|E_\mathrm{p}|^2 + |E_\mathrm{d}|^2\right)C_{a,\boldsymbol{k}} + E_\mathrm{d}\bar{E}_\mathrm{p}\exp(-\mathrm{i}\delta\omega t) \times C_{a,\boldsymbol{k}+\boldsymbol{k}_\mathrm{p}-\boldsymbol{k}_\mathrm{d}} + E_\mathrm{p}\bar{E}_\mathrm{d}\exp(\mathrm{i}\delta\omega t) C_{a,\boldsymbol{k}+\boldsymbol{k}_\mathrm{d}-\boldsymbol{k}_\mathrm{p}}\right] - \mathrm{i}w_k C_{a,\boldsymbol{k}}, \quad (7)$$

$$\frac{D}{2c}\dot{E}_\mathrm{p} = E_\mathrm{p}^\mathrm{e} - E_\mathrm{p} + \frac{\mathrm{i}-v}{\varDelta}\sum_{\boldsymbol{k}}\left[|C_{a,\boldsymbol{k}}|^2 E_\mathrm{p} + \exp(-\mathrm{i}\delta\omega t)\bar{C}_{a,\boldsymbol{k}} C_{a,\boldsymbol{k}+\boldsymbol{k}_\mathrm{p}-\boldsymbol{k}_\mathrm{d}} E_\mathrm{d}\right], \quad (8)$$

$$\frac{D}{2c}\dot{E}_\mathrm{d} = E_\mathrm{d}^\mathrm{e} - E_\mathrm{d} + \frac{\mathrm{i}-v}{\varDelta}\sum_{\boldsymbol{k}}\left[|C_{a,\boldsymbol{k}}|^2 E_\mathrm{d} + \exp(\mathrm{i}\delta\omega t)\bar{C}_{a,\boldsymbol{k}} C_{a,\boldsymbol{k}+\boldsymbol{k}_\mathrm{d}-\boldsymbol{k}_\mathrm{p}} E_\mathrm{p}\right], \quad (9)$$

where $\varDelta = \Delta\omega\tau_\mathrm{R}$ and $v = \Gamma_\mathrm{R}/2\Delta\omega$.

Let us introduce the field transformation $E_f \to> E_f \times \exp(2\mathrm{i}ct/D\varDelta)$ ($f = \mathrm{d, p}$). Then, Eqn (7) remains invariable, and (8) and (9) take the form

$$\frac{D}{2c}\dot{E}_\mathrm{p} = E_\mathrm{p}^\mathrm{e} - E_\mathrm{p} - \frac{\mathrm{i}}{\varDelta}\Big(1 - \sum_{\boldsymbol{k}}|C_{a,\boldsymbol{k}}|^2\Big)E_\mathrm{p} + \frac{\mathrm{i}-v}{\varDelta}\exp(-\mathrm{i}\delta\omega t)\sum_{\boldsymbol{k}}\bar{C}_{a,\boldsymbol{k}} C_{a,\boldsymbol{k}+\boldsymbol{k}_\mathrm{p}-\boldsymbol{k}_\mathrm{d}} E_\mathrm{d}, \quad (10)$$

$$\frac{D}{2c}\dot{E}_\mathrm{d} = E_\mathrm{d}^\mathrm{e} - E_\mathrm{d} - \frac{\mathrm{i}}{\varDelta}\Big(1 - \sum_{\boldsymbol{k}}|C_{a,\boldsymbol{k}}|^2\Big)E_\mathrm{d} + \frac{\mathrm{i}-v}{\varDelta}\exp(\mathrm{i}\delta\omega t)\sum_{\boldsymbol{k}}\bar{C}_{a,\boldsymbol{k}} C_{a,\boldsymbol{k}+\boldsymbol{k}_\mathrm{d}-\boldsymbol{k}_\mathrm{p}} E_\mathrm{p}. \quad (11)$$

Here, we take into account that $v\sum_{\boldsymbol{k}}|C_{a,\boldsymbol{k}}|^2 \ll \varDelta$ under the experimental conditions.

By using a phenomenological relaxation model, we can write the system of equations for elements of the density matrix $R_{a,k;a,k'} = C_{a,k}\bar{C}_{a,k'} = R_{\boldsymbol{k},\boldsymbol{k}'}$ and the amplitudes E_d and E_p of the average fields:

$$\dot{R}_{\boldsymbol{k},\boldsymbol{k}'} = \frac{d^2(\mathrm{i}-v)}{\hbar^2\Delta\omega}\left[E_\mathrm{d}\bar{E}_\mathrm{p}\exp(-\mathrm{i}\delta\omega t)R_{\boldsymbol{k}-\boldsymbol{k}_\mathrm{d}+\boldsymbol{k}_\mathrm{p},\boldsymbol{k}'} + E_\mathrm{p}\bar{E}_\mathrm{d} \times \exp(\mathrm{i}\delta\omega t)R_{\boldsymbol{k}+\boldsymbol{k}_\mathrm{d}-\boldsymbol{k}_\mathrm{p},\boldsymbol{k}'}\right] - \frac{d^2(\mathrm{i}+v)}{\hbar^2\Delta\omega} \times\left[E_\mathrm{d}\bar{E}_\mathrm{p}\exp(-\mathrm{i}\delta\omega t)R_{\boldsymbol{k}'+\boldsymbol{k}_\mathrm{d}-\boldsymbol{k}_\mathrm{p},\boldsymbol{k}} + E_\mathrm{p}\bar{E}_\mathrm{d}\exp(\mathrm{i}\delta\omega t)\bar{R}_{\boldsymbol{k}'-\boldsymbol{k}_\mathrm{d}+\boldsymbol{k}_\mathrm{p},\boldsymbol{k}}\right] - \left[\mathrm{i}w_{k,k'} + \Gamma_{\boldsymbol{k},\boldsymbol{k}'} + \frac{vd^2}{\hbar^2\Delta\omega}\left(|E_\mathrm{p}|^2 + |E_\mathrm{d}|^2\right)\right]R_{\boldsymbol{k},\boldsymbol{k}'}, \quad (12)$$

$$\frac{D}{2c}\dot{E}_\mathrm{p} = E_\mathrm{p}^\mathrm{e} - E_\mathrm{p} - \frac{\mathrm{i}}{\varDelta}\Big(1 - \sum_{\boldsymbol{k}} R_{\boldsymbol{k},\boldsymbol{k}}\Big)E_\mathrm{p} + \frac{\mathrm{i}-v}{\varDelta}\exp(-\mathrm{i}\delta\omega t)\sum_{\boldsymbol{k}} R_{\boldsymbol{k}+\boldsymbol{k}_\mathrm{p}-\boldsymbol{k}_\mathrm{d},\boldsymbol{k}} E_\mathrm{d}, \quad (13)$$

$$\frac{D}{2c}\dot{E}_\mathrm{d} = E_\mathrm{d}^\mathrm{e} - E_\mathrm{d} - \frac{\mathrm{i}}{\varDelta}\Big(1 - \sum_{\boldsymbol{k}} R_{\boldsymbol{k},\boldsymbol{k}}\Big)E_\mathrm{d} +$$

$$+\frac{\mathrm{i}-\nu}{\Delta}\exp(\mathrm{i}\delta\omega t)\sum_{\boldsymbol{k}} R_{\boldsymbol{k}+\boldsymbol{k}_{\mathrm{d}}-\boldsymbol{k}_{\mathrm{p}},\boldsymbol{k}} E_{\mathrm{p}}, \tag{14}$$

where $\Gamma_{\boldsymbol{k},\boldsymbol{k}'}$ are the transverse relaxation rates and $w_{k,k'} = w_k - w_{k'}$ are the multiphoton transition frequencies.

By neglecting spontaneous scattering ($\nu = 0$), one can easily see that the $\sum_{\boldsymbol{k}} |C_{a,\boldsymbol{k}}|^2 = 1$ and the third term in Eqns (10), (11), (13), and (14) vanishes.

Because the law of conservation of momentum takes place during the interaction of an atom with an electromagnetic wave, the momentum of the atom in the state $|a,\boldsymbol{k}\rangle$ can be written in the form

$$\boldsymbol{p} = \hbar\boldsymbol{k} = \hbar n(\boldsymbol{k}_{\mathrm{d}} - \boldsymbol{k}_{\mathrm{p}}), \quad n = 0, \pm1, \pm2, \ldots, .$$

Upon absorption of a photon with the wave vector $\boldsymbol{k}_{\mathrm{d}}$ and emission of a photon with the wave vector $\boldsymbol{k}_{\mathrm{p}}$, the number n increases, while in the case of the inverse process, this number decreases by unity.

Let us make the substitution $|a,\boldsymbol{k}\rangle \to \Psi_n$, $R_{\boldsymbol{k},\boldsymbol{k}'} \to R_{m,n}$, $w_{k,k'} \to w_{m,n}$, $\Gamma_{\boldsymbol{k},\boldsymbol{k}'} \to \Gamma_{m,n}$ and introduce the transformation of elements of the density matrix $R_{m,n} \to R_{m,n}\times \exp[\mathrm{i}(n-m)\delta\omega t]$. Then, the factors containing $\exp(\mathrm{i}\delta\omega t)$ in (12)–(14) will disappear. Let us also introduce the dimensionless time and field amplitudes [20]: $\tau = t/\tau_{\mathrm{amp}}$ and $\varepsilon_f = E_f/E_{\max}$ ($f = \mathrm{d}$, p), where $\tau_{\mathrm{amp}} = \Delta\omega\hbar^2/(d^2E^2_{\max})$ is the characteristic time scale and $E_{\max}$ is a maximum amplitude of laser fields used in the experiment. In this case, Eqns (12)–(14) can be written explicitly in the fiorm

$$\dot{R}_{m,n} = (\mathrm{i}-\nu)(\varepsilon_{\mathrm{d}}\bar{\varepsilon}_{\mathrm{p}}R_{m-1,n} + \varepsilon_{\mathrm{p}}\bar{\varepsilon}_{\mathrm{d}}R_{m+1,n})$$
$$-(\mathrm{i}+\nu)(\varepsilon_{\mathrm{d}}\bar{\varepsilon}_{\mathrm{p}}R_{m,n+1} + \varepsilon_{\mathrm{p}}\bar{\varepsilon}_{\mathrm{d}}R_{m,n-1})$$
$$-\left[\mathrm{i}\beta_{m,n} + \gamma_{m,n} + \nu\left(|\varepsilon_{\mathrm{p}}|^2 + |\varepsilon_{\mathrm{d}}|^2\right)\right]R_{m,n}, \tag{15}$$

$$\kappa\dot{\varepsilon}_{\mathrm{p}} = \varepsilon^{\mathrm{e}}_{\mathrm{p}} - \varepsilon_{\mathrm{p}} - \frac{\mathrm{i}}{\Delta}\Big(1 - \sum_n R_{n,n}\Big)\varepsilon_{\mathrm{p}} + \frac{\mathrm{i}-\nu}{\Delta}\sum_n R_{n-1,n}\varepsilon_{\mathrm{d}}, \tag{16}$$

$$\kappa\dot{\varepsilon}_{\mathrm{d}} = \varepsilon^{\mathrm{e}}_{\mathrm{d}} - \varepsilon_{\mathrm{d}} - \frac{\mathrm{i}}{\Delta}\Big(1 - \sum_n R_{n,n}\Big)\varepsilon_{\mathrm{d}} + \frac{\mathrm{i}-\nu}{\Delta}\sum_n R_{n+1,n}\varepsilon_{\mathrm{d}}. \tag{17}$$

Here, $\beta_{m,n} = f_{m,n} - (m-n)\delta$ is the multiphoton mismatch of the resonance; $f_{m,n} = w_{m,n}/\tau^{-1}_{\mathrm{amp}}$; $\delta = \delta\omega/\tau^{-1}_{\mathrm{amp}}$; $\gamma_{m,n} = \Gamma_{m,n}\times \tau_{\mathrm{amp}}$; $\kappa = D/(2c\tau_{\mathrm{amp}})$; $w_{m,n} = \boldsymbol{p}^2/(2M\hbar) = (1+\sqrt{2})\hbar k^2_{\mathrm{d}}\times (m^2 - n^2)(2M)^{-1}$; and M is the atom mass.

We assume that first

$$R_{0,0}(0) = 1, \quad R_{m,n}(0) = 0 \tag{18}$$

$$(m, n = 0, \pm1, \pm2, \ldots, \pm l;\, m = n \neq 0),$$

where l is the maximum excitation multiplicity of condensate atoms. One can easily see that the number of equations in system (15)–(17) is $(2l+1)(l+1)+2$.

Let us estimate the values of parameters entering Eqns (15)–(17), for example, for experimental conditions in [5]. Let $E_{\max} = 1.45\times10^{-2}$ CGS units (the field amplitude for the laser radiation intensity equal to 100 mW cm^{-2}). Because $d = 0.52\times10^{-17}$ CGS units (effective dipole moment) and $\Delta\omega = 1.07\times10^{10}$ s^{-1}, the characteristic time is $\tau_{\mathrm{amp}} = 2.10\times10^{-6}$ s and the energy unit is $\hbar\tau^{-1}_{\mathrm{amp}} = 0.50\times10^{-21}$ erg. Let the total number of atoms in the condensate be $N = 4\times10^6$. Then, for $L = 200$ μm and $D = 20$ μm, the concentration of atoms in the condensate is $N_0 = 1.60\times10^{13}$ cm^{-3}. Because $\omega_{\mathrm{d}} = 3.20\times\ 10^{15}$ s^{-1}, we have $\tau_{\mathrm{R}} = 0.37\times10^{-8}$ s and $\Gamma_{\mathrm{R}} = 0.63\times\ 10^8$ s^{-1}. Due to absorption of a photon at a wavelength of 589 nm, the kinetic energy of the atom is $W_0 = 1.60\times10^{-22}$ erg. Thus, the typical width of the laser pulse (100 μs) and the peak field intensity in it (the power density 5 mW cm^{-2}) under these experimental conditions are approximately $47.6\tau_{\mathrm{amp}}$ and $0.22E_{\max}$, constants are $\nu = 0.29\times10^{-2}$, $\Delta = 9.95$, $\delta = 1.20$, and the energy is $W_0 = 0.32\hbar\tau^{-1}_{\mathrm{amp}}$.

3. Extended pulses in a 'dressed' condensate

Consider a BEC irradiated by pulses of duration

$$t_f \gg \frac{1}{\delta\omega},\ \frac{1}{w_{m,n}} \quad (f = \mathrm{p}, \mathrm{d};\, m \neq n = 0, \pm1, \pm2, \ldots, \pm l). \tag{19}$$

Then, the probability of the $0 \to -1$ transition is extremely small and can be neglected. In addition, if the dressing wave is not very intense [20], the repeated excitation of atoms by this wave can be also neglected. In this situation only the two states, 0 and 1, will be actual. It is these conditions that are realised in experiments [5]. Therefore, we consider here Eqns (15)–(18) only for m, $n = 0, 1$. In this case, by neglecting the influence of spontaneous Rayleigh scattering (assuming that $\nu = 0$) on the condensate dynamics and the delay of radiation in the condensate ($\kappa\dot{\varepsilon}_f \ll \varepsilon_f$), we obtain

$$\dot{R}_{0,0} = \mathrm{i}\varepsilon_{\mathrm{p}}\bar{\varepsilon}_{\mathrm{d}}R_{1,0} - \mathrm{i}\bar{\varepsilon}_{\mathrm{p}}\varepsilon_{\mathrm{d}}\bar{R}_{1,0}, \tag{20}$$

$$R_{1,1} = \mathrm{i}\bar{\varepsilon}_{\mathrm{p}}\varepsilon_{\mathrm{d}}\bar{R}_{1,0} - \mathrm{i}\varepsilon_{\mathrm{p}}\bar{\varepsilon}_{\mathrm{d}}R_{1,0}, \tag{21}$$

$$\dot{R}_{1,0} = \mathrm{i}\bar{\varepsilon}_{\mathrm{p}}\varepsilon_{\mathrm{d}}R_{0,0} - \mathrm{i}\bar{\varepsilon}_{\mathrm{p}}\varepsilon_{\mathrm{d}}\bar{R}_{1,1} - (\mathrm{i}\beta + \gamma)R_{1,0}, \tag{22}$$

$$\varepsilon_{\mathrm{p}} = \varepsilon^{\mathrm{e}}_{\mathrm{p}} + \frac{\mathrm{i}}{\Delta}\,\bar{R}_{1,0}\varepsilon_{\mathrm{d}}, \tag{23}$$

$$\varepsilon_{\mathrm{d}} = \varepsilon^{\mathrm{e}}_{\mathrm{d}} + \frac{\mathrm{i}}{\Delta}\,R_{1,0}\varepsilon_{\mathrm{p}}, \tag{24}$$

where $\beta = \beta_{1,0}$ and $\gamma = \gamma_{1,0}$. The field amplitudes $\varepsilon^{\mathrm{out}}_{\mathrm{p}}$ and $\varepsilon^{\mathrm{out}}_{\mathrm{d}}$ at the output can be obtained from the corresponding expressions (23) and (24) by doubling the second term in each of them.

In the second part of the experiment in paper [5], the condensate was irradiated by a continuous wave at frequency ω_{d}, so that $E^{\mathrm{e}}_{\mathrm{d}}(t) = E^0_{\mathrm{d}} = \mathrm{const}$ during the entire irradiation time. We will call the condensate under these conditions a 'dressed' condensate.

Consider first the interaction of the 'dressed' condensate with a probe rectangular pulse:

$$E^{\mathrm{e}}_{\mathrm{p}}(t) = \begin{cases} E^0_{\mathrm{p}}, & 0 \leqslant t \leqslant t_{\mathrm{p}}, \\ 0, & t < 0, t > t_{\mathrm{p}} \end{cases} \tag{25}$$

where E^0_{p} and t_{p} are the amplitude and duration of the probe pulse.

Let the fraction of atoms acquiring the recoil momentum be small. Then, $R_{0,0} \approx 1$, and the second term in (22) can be neglected:

$$\dot{R}_{1,0} = \mathrm{i}\bar{\varepsilon}_{\mathrm{p}}\varepsilon_{\mathrm{d}} - (\mathrm{i}\beta + \gamma)R_{1,0}. \tag{26}$$

In addition, we assume that the depletion of the dressing beam is insignificant, i.e. $E_d \approx E_d^e$. In this case, the system of equations (23) and (26) has the analytic solution

$$R_{1,0}(\tau) = \frac{i\varepsilon_d^0 \bar{\varepsilon}_p^0}{\alpha - i\beta} \times \begin{cases} [\exp(\alpha - i\beta)\tau - 1], & \tau \leqslant \tau_p, \\ [\exp(\alpha - i\beta)\tau_p - 1]\exp[(\alpha - i\beta)(\tau - \tau_p)], & \tau > \tau_p, \end{cases} \tag{27}$$

$$\varepsilon_p(\tau) = \varepsilon_p^0 \times \begin{cases} 1 + \dfrac{g(\alpha - i\beta)}{\alpha^2 + \beta^2}[\exp(\alpha - i\beta)\tau - 1], & \tau \leqslant \tau_p, \\ \dfrac{g(\alpha - i\beta)}{\alpha^2 + \beta^2}[\exp(\alpha + i\beta)\tau_p - 1]\exp[(\alpha + i\beta)(\tau - \tau_p)], & \tau > \tau_p. \end{cases} \tag{28}$$

By substituting now expressions (27) and (28) into Eqn (21) and integrating it, we find the approximate solution for the diagonal element of the density matrix:

$$R_{1,1}(\tau) = \frac{2|\varepsilon_p^0|^2|\varepsilon_d^0|^2}{\alpha^2 + \beta^2}\left\{\frac{\exp\alpha\tau}{\alpha^2 + \beta^2}\left[2\gamma\beta\sin\beta\tau - \left(\alpha^2 + 2\gamma\alpha + \beta^2\right)\right.\right.$$

$$\left.\left.\times\cos\beta\tau\right] + \frac{g}{2\alpha}\exp 2\alpha\tau + \gamma\tau + 1 - \frac{g}{2\alpha} + \frac{2\alpha\gamma}{\alpha^2 + \beta^2}\right\}, \ \tau \leqslant \tau_p,$$

$$R_{1,1}(\tau) = R_{1,1}(\tau_p) + \frac{2\mathrm{Re}\left(\varepsilon_p^0\right)|\varepsilon_d^0|^2 g}{\alpha\left(\alpha^2 + \beta^2\right)} \tag{29}$$

$$\times(\exp 2\alpha\tau - 2\exp\alpha\tau\cos\beta\tau + 1)\{\exp[2\alpha(\tau - \tau_p)] - 1\}, \ \tau > \tau_p.$$

Similarly, we can obtain the expression

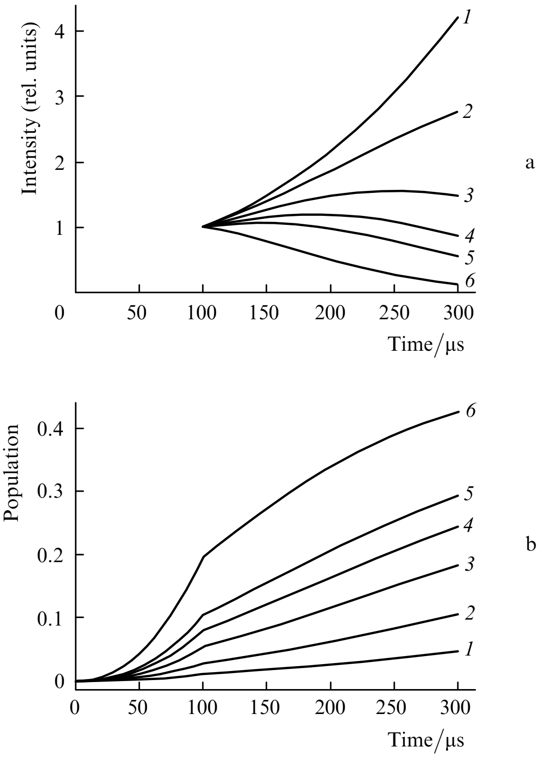

Figure 1. Interference of light from a recoil atom grating induced upon irradiation of a condensate by a 100-μs rectangular Bragg pulse of intensity $I_p^0 = 0.002$ (*1*), 0.005 (*2*), 0.01 (*3*), 0.015 (*4*), and 0.04 mW cm^{-2} (*6*); (a) the intensity of diffracted radiation normalised to its value for $t = t_p$, (b) the excited-level population. The transverse relaxation rate is 0.5×10^{-4} s^{-1}, the dressing beam intensity is 20 mW cm^{-2}.

$$\varepsilon_s(\tau) = \varepsilon_d(\tau) - \varepsilon_d^0 = \frac{-\varepsilon_d^0|\varepsilon_p^0|^2}{\Delta\left(\alpha^2 + \beta^2\right)}\begin{cases} (\alpha + i\beta)[\exp(\alpha + i\beta)\tau - 1]\left\{1 + \dfrac{g(\alpha - i\beta)}{\alpha^2 + \beta^2}[\exp(\alpha + i\beta)\tau - 1\right\}, & \tau \leqslant \tau_p, \\ g(\exp 2\alpha\tau_p - 2\exp\alpha\tau_p\cos\beta\tau_p + 1)\exp[2\alpha(\tau - \tau_p)], & \tau > \tau_p. \end{cases} \tag{30}$$

for the field amplitude of the scattered dressing radiation. Here, $\tau_p = t_p/\tau_{amp}$; $\varepsilon_f^0 = E_f^0/E_{max}$ are the dimensionless peak values of the field amplitudes of the double beam ($f =$ d, p) at the condensate input; $\alpha = g - \gamma$; $g = |\varepsilon_d^0|^2/\Delta$; $\beta\tau = (w - \delta\omega)t$; $(g - \gamma)\tau = (G - \Gamma)t$; $w = w_{1,0}$ is the 1–0 transition frequency; $\Gamma = \Gamma_{1,0}$ is the transverse relaxation rate of the matrix element $R_{1,0}$; $G = g/\tau_{amp} = 3NR_d/Sk_d^2$ is the probe radiation gain per unit time; R_d is the Rayleigh scattering rate of the dressing beam; S is the condensate cross section perpendicular to the dressing beam propagation direction.

It follows from solutions of (27) and (29) that in the case of the exact two-phonon resonance, a dynamic grating of atoms moving in the same direction with the same velocity is formed by the instant of the probe beam switching off. Because the dressing radiation is continuous, it will diffract from this grating at times $t > t_p$. For $G < \Gamma$, as follows from (30), the diffracted light intensity I_s ($I_s \sim |\varepsilon_s|^2$) will decrease, the decrease being faster with increasing the difference $\Gamma - G$. This means that in the case of a low-intensity dressing beam, the dynamic grating decays with time. The situation becomes critical from this point of view when $G = \Gamma$.

In the case of the high-power dressing radiation ($G > \Gamma$), the radiation scattering probability increases with time. However, this is not always the case. The numerical solutions of the total system of equations (20)–(24) (Fig. 1) show that, as the ratio I_p^0/I_d^0 increases, which is accompanied by the increase in the number of recoil atoms, the increase in the scattering probability slows down and ceases at all after the transition of approximately 10 % of atoms to level 1. For even larger values of I_p^0/I_d^0, the value of I_s decreases with time. Thus, the additional factor affecting the possibility of the existence of a dynamic grating of recoil atoms is the degree of nonlinearity of the interaction of the probe pulse with a 'dressed' BEC, specified by the ratio I_p^0/I_d^0.

For $G > \Gamma$, the intensity of scattered dressing radiation increases with decreasing the parameter I_s (Fig. 1). Because $\hbar w_p \ll \hbar\omega_d$, the scattered energy is mainly concentrated in the p mode. The probe-wave intensity can be made due to grating generation many times higher than the input intensity. Switching off the laser at frequency ω_p under such conditions will not affect in fact the probe-pulse dynamics, resulting in the appearance of superradiance at this frequency. The role of the pulse E_p^e (25) will be reduced only to the induction of weak polarisation in the BEC whose value follows from expression (27):

$$R_{1,0}(\tau_p) = \frac{i\varepsilon_d^0 \bar{\varepsilon}_p^0}{\alpha - i\beta}[\exp(\alpha - i\beta)\tau_p - 1]. \qquad (31)$$

Let the dressing field be very strong, so that $G \gg \Gamma$. Then, by omitting the first term in the right-hand side of (24) for times $t > t_p$ and substituting this expression into (20)–(22), we obtain the equations

$$\dot{R} = -2gZR - i\beta R, \qquad (32)$$

$$\dot{Z} = 2g|R|^2, \qquad (33)$$

where $Z = R_{1,1} - R_{0,0}$; $R = 2R_{1,0}$.

Equations (32) and (33) have the integral of motion

$$|R(\tau)|^2 + Z(\tau)^2 = |R(\tau_p)|^2 + Z(\tau_p)^2 = 1. \qquad (34)$$

By using it, we can readily obtain the solution of these equations:

$$Z(\tau) = \tanh\frac{\tau - \tau_p - \tau_0}{\tau_s},$$

$$R(\tau) = \operatorname{sech}\frac{\tau - \tau_p - \tau_0}{\tau_s}\exp[-i\beta(\tau - \tau_p)],$$

where

$$\tau_0 = -\frac{\tau_s}{2}\ln\left[\frac{1 + Z(\tau_p)}{1 - Z(\tau_p)}\right]; \qquad (35)$$

$\tau_s = \frac{1}{2}g$.

If $|R(\tau_p)| \ll 1$, then $Z(\tau_p) = -[1 - |R(\tau_p)|^2]^{1/2} \approx -1 + 0.5|R(\tau_p)|^2$ and $\tau_0 \approx -\tau_s \ln|R_{1,0}(\tau_p)|$.

Thus, when $G \gg \Gamma$ and the phase relaxation can be neglected, Eqns (20)–(24) have the solution

$$R_{1,0}(\tau) = \frac{i\varepsilon_d^0 \bar{\varepsilon}_p^0}{g - i\beta} \times \begin{cases} [\exp(g - i\beta)\tau - 1], & \tau \leq \tau_p, \\ \dfrac{1}{2}\operatorname{sech}\dfrac{\tau - \tau_p - \tau_0}{\tau_s}\exp[-i\beta(\tau - \tau_p)], & \tau > \tau_p, \end{cases} \qquad (36)$$

$$Z(\tau) = \begin{cases} \dfrac{4|\varepsilon_p^0|^2|\varepsilon_d^0|^2}{g^2 + \beta^2}(\exp 2g\tau - 2\exp 2g\tau\cos\beta\tau + 1) - 1, & \tau \leq \tau_p, \\ \tanh\dfrac{\tau - \tau_p - \tau_0}{\tau_s}, & \tau > \tau_p, \end{cases} \qquad (37)$$

$$\varepsilon_p(\tau) = \begin{cases} \varepsilon_p^0\left\{1 + \dfrac{g(g - i\beta)}{g^2 + \beta^2}[\exp(g + i\beta)\tau - 1]\right\}, & \tau \leq \tau_p, \\ \dfrac{i\varepsilon_d^0}{2\Delta}\operatorname{sech}\dfrac{\tau - \tau_p - \tau_0}{\tau_s}\exp[i\beta(\tau - \tau_p)], & \tau > \tau_p. \end{cases} \qquad (38)$$

One can see from these expressions that the nondiagonal element $R_{1,0}(\tau)$ of the density matrix and the population difference $Z(\tau)$ at the instant $\tau = \tau_p$ are continuous, but the field $\varepsilon_p(\tau)$ experiences a break. As pointed out above, the smaller I_p^0 and the larger I_d^0, the smaller is the field jump.

Thus, for $I_p^0 \ll I_d^0$ and weak phase relaxation, radiation in the p mode appears in the form of a bell-shaped pulse of width τ_s delayed by the time τ_0 with respect to the instant τ_p.

If the gain G is moderate, exceeding Γ only by several times, the main properties of the probe radiation remain the same, as follows from the numerical solution of the total system of equations (20)–(24) (Fig. 2). In particular, probe-pulse width and peak intensity are independent of its input energy, while the delay time decreases with increasing input energy. The pulse delay time and duration are inversely proportional to the dressing radiation power, while its maximum intensity is proportional to this power.

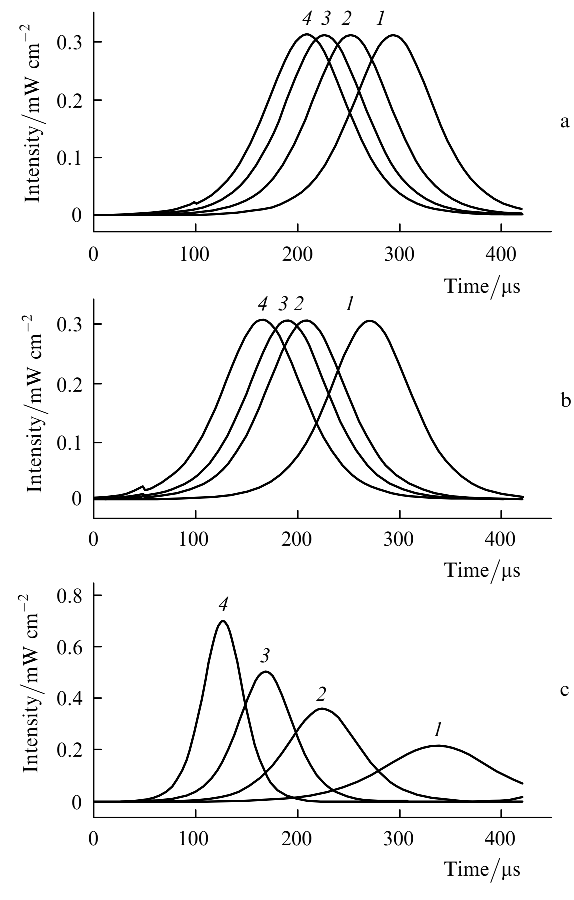

Figure 2. Shape of an induced superradiance pulse for probe pulse durations $t_p = 10$ (*1*), 25 (*2*), 50 (*3*), and 100 μs (*4*) (a), its intensities $I_p^0 = 0.00005$ (*1*), 0.0002 (*2*), 0.0004 (*3*), and 0.001 mW cm^{-2} (*4*) (b), and the dressing laser radiation intensities $I_d^0 = 40$ (*1*), 55 (*2*), 70 (*3*), and 90 mW cm^{-2} (*4*) (c). The beam intensities are $I_d^0 = 50$ mW cm^{-2} (a, b), $I_p^0 = 0.0001$ mW cm^{-2} (a), pulse durations are $t_p = 50$ (b) and 25 μs (c).

The following analysis depends on the probe radiation spectrum. We will assume here that the probe pulse has a Gaussian shape

$$E_p^e = E_p^0 \exp\left[-2\ln 2\left(\frac{t - t_0}{t_p}\right)^2\right], \qquad (39)$$

where the time t_0 corresponds to the pulse maximum I_p^0 and t_p is the pulse FWHM.

In a linear case, when the fraction of atoms receiving the recoil pulse is small and the depletion of the dressing wave is insignificant, Eqns (20)–(24) for the probe pulse of type (39), as for pulse (25), admit the analytic solution.

In the case of the exact resonance, we have the non-diagonal element of the density matrix in the form

$$R_{1,0}(\tau) = \frac{\mathrm{i}\sqrt{\pi}\varepsilon_d^0\,\bar{\varepsilon}_p^0}{2}\exp\left[a(\tau-\tau_0)-\frac{b^2}{4a^2}\right]$$

$$\times\left\{\Phi\left[\frac{a}{2b}+b(\tau-\tau_0)\right]-\Phi\left(\frac{a}{2b}-b\tau_0\right)\right\}, \qquad (40)$$

and the average field amplitude for the p mode is

$$\varepsilon_p(\tau) = \varepsilon_p^0\exp\left[-b^2(\tau-\tau_0)^2+\frac{\sqrt{\pi}g\varepsilon_p^0}{2}\right.$$

$$\times\exp\left[a(\tau-\tau_0)-\frac{b^2}{4a^2}\right]$$

$$\times\left\{\Phi\left[\frac{a}{2b}+b(\tau-\tau_0)\right]-\Phi\left(\frac{a}{2b}-b\tau_0\right)\right\}, \qquad (41)$$

where $a=\alpha$; $b=(2\ln 2)^{1/2}/\tau_p$; $\tau_p=t_p/\tau_{amp}$; $\tau_0=t_0/\tau_{amp}$; and $\Phi(t)$ in the integral Poisson function.

The output probe field amplitude ε_p^{out} is obtained from expression (41) by doubling the second term, which is always positive. That is why the intensity maximum $I_p\sim|\varepsilon_p^{out}|^2$ coincides with the amplitude maximum ε_p^{out} and is achieved at $t_0^{out}>t_0$. This means that the probe pulse propagates through the BEC for a finite time $t_0^{out}-t_0$. In addition, $I_p(t_0^{out})>I_p(t_0)$, i.e. the pulse is amplified. The propagation velocity, width, and amplification of the probe pulse in the condensate depend on its energy and the dressing wave intensity.

Let us assume that the probe pulse duration t_p and the inverse linear amplification time $1/G$ are smaller than or comparable to the transverse relaxation time $T_2=1/\Gamma$ for the 0–1 transition. If the intensity I_p^0 is smaller than I_d^0 by one–two orders of magnitude, the propagation of the probe pulse through the BEC is accompanied by its weak amplification (by several times), the amplification increasing with increasing the dressing beam intensity (Fig. 3a). In this case, the pulse almost does not broaden. The pulse propagation velocity first decreases with increasing the dressing wave power [curves (*1*) and (*2*)] and then increases [curves (*4*) and (*5*)]. The minimal propagation velocity [curve (*3*)] is achieved for values of I_d^0 at which approximately half the atoms undergo transitions from the state 0 to the state 1 (Fig. 3b). For parameters of the problem used, this velocity was ~ 1.4 m s^{-1}.

If the probe pulse energy is reduced so that the ratio I_p^0/I_d^0 becomes $\sim 10^{-3}$ and smaller, the main factor affecting the pulse shape will be grating generation. This leads to a considerable broadening of the pulse and significantly increases its propagation time in the BEC. The smaller is probe pulse energy, the more distinct are these effects. Under these conditions, an extremely large time delay of the probe pulse in the 'dressed' condensate can be observed. For example, for $I_d^0=50$ mW cm^{-2}, $I_p^0=0.001$ mW cm^{-2}

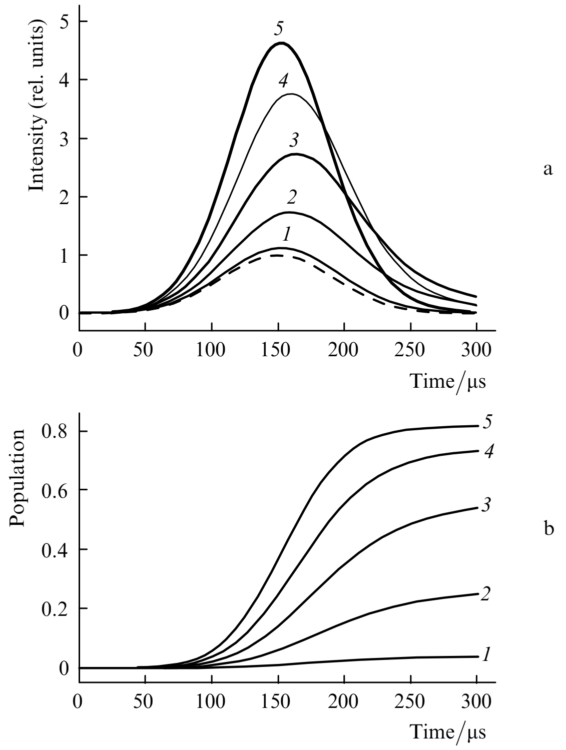

Figure 3. Evolution of a 100-μs Gaussian pulse with a peak intensity of 0.1 mW cm^{-2} (dashed curve) (a) and excitation dynamics of atoms in a condensate for $I_d^0=1$ (*1*), 5 (*2*), 10 (*3*), 15 (*4*), and 20 mW cm^{-2} (*5*) (b). The pulse intensities are normalised to the input peak value.

and $t_p=50$ μs, the time delay is ~ 113 μs (the pulse propagation velocity is ~ 0.17 m s^{-1}).

4. Short pulses in a 'dressed' condensate

So far we described the interaction of a double laser beam with a BEC by using the model of an atom with two states 0 and 1. To excite the states with $n\geqslant 2$, the dressing radiation should have a high power. The corresponding threshold [20]

$$I_{d,th}\approx\frac{1}{8}\left(\frac{c\hbar}{\pi d}\right)^2\frac{\Delta\omega\delta\omega}{I_p} \qquad (42)$$

is independent of the probe pulse duration and decreases with increasing the pulse intensity.

Transitions from the state 0 to the state -1, from the state -1 to the state -2, etc. occur due to absorption of a photon from the p mode and its emission to the d mode. Because for central frequencies $\hbar\omega_p<\hbar\omega_d$, not only the power of the probe pulse but also its spectral width is important for excitation of the states Ψ_n with $n<0$. Therefore, we will assume that the probe pulse duration satisfies the condition

$$t_f\sim\frac{1}{\delta\omega},\ \frac{1}{w_{m,n}}\quad(f=\mathrm{p,d};\ \ m\neq n=0,\pm 1,\pm 2,\ldots,\pm l), \qquad (43)$$

and the pulse itself has Gaussian shape (39).

The numerical solution of the system of equations (15)–(17) admitting the possibility of multiple excitation of atoms shows that, indeed, beginning from some values of τ_p, the probe pulse in the presence of the dressing pulse can induce transitions from the state 0 to the state −1. Because a decrease in the probe pulse duration is accompanied by a decrease in its energy, the pulse power should be increased to preserve its energy. However, even for $t_p \sim 1$ μs and intensities I_d^0, $I_p^0 = 50 - 100$ mW cm^{-2}, only a few percent of atoms undergo transitions to the state −1. Therefore, the use of too short pulses to increase the excitation efficiency of the states −1, −2, etc. is not quite justified. We can assume that this can be achieved, as upon excitation of the states 2, 3, etc., by decreasing the one-photon detuning $\Delta\omega$ of the electronic resonance. Indeed, calculations confirmed this (Fig. 4). It was found that the smaller $\Delta\omega$ and larger the ratio I_p^0/I_d^0, the larger fraction of atoms passes through the cascade of states with negative n.

Note that the states with negative n can be also excited by reducing the two-photon atomic resonance detuning $\delta\omega$ achieved, for example, by decreasing the angle between beams in a double beam or by using a BEC of heavier atoms.

5. Conclusions

Semiclassical equations have been used to analyse the interaction of a probe pulse with a 'dressed' BEC of sodium vapour. In the case of the low-intense and broad probe pulse, only state 1, corresponding to absorption of a photon from the dressing mode and emission of this photon to the probe mode, is populated. Under these conditions, the

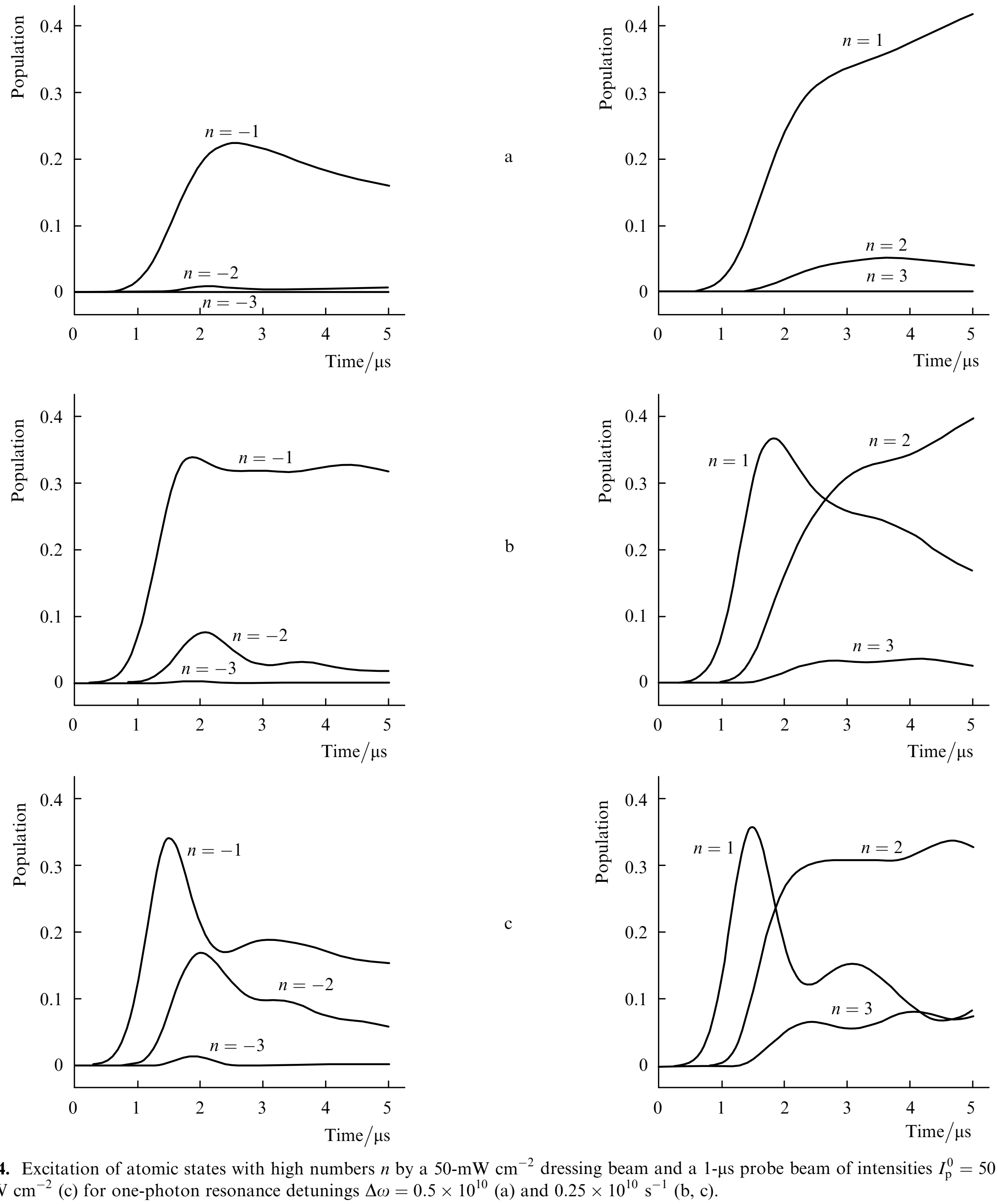

Figure 4. Excitation of atomic states with high numbers n by a 50-mW cm^{-2} dressing beam and a 1-μs probe beam of intensities $I_p^0 = 50$ (a, b) and 100 mW cm^{-2} (c) for one-photon resonance detunings $\Delta\omega = 0.5 \times 10^{10}$ (a) and 0.25×10^{10} s^{-1} (b, c).

recoil atoms form a diffraction grating when the gain exceeds the transverse relaxation rate and populations slightly deviate from equilibrium values. A pulse propagating in such a medium slows down and broadens, this effect increasing with decreasing the pulse energy. When the input pulse energy is very low and the pulse can be treated as a seed pulse, the grating generation becomes dominant and induced superradiance appears at the probe beam frequency.

Transitions from the ground state 0 to states with $n \neq 1$ are strongly nonresonant. These states can be populated only if the irradiation intensity is very high. The probe pulse width is not important for obtaining states with $n > 0$. At the same time, states with $n < 0$ are excited by short probe pulses of duration comparable with or smaller than the inverse recoil frequency of atoms. The power threshold of a cascade process in both channels can be reduced, in particular, by decreasing the one-photon electronic resonance detuning.

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