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Fluctuations of the orbital angular momentum of a laser beam, carrying an optical vortex, in the turbulent atmosphere*

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Abstract. **The evolution of the orbital angular momentum (OAM) of a Laguerre – Gaussian beam interacting with turbulent inhomogeneities of the atmosphere is studied theoretically. The integral representations are obtained for the OAM in terms of the distributions of the random intensity and random field of the permittivity of the medium, and also for OAM statistical characteristics in terms of corresponding correlation functions. It is found that the average OAM value is preserved during the propagation of the laser beam in a random medium. The dependence of the dispersion of OAM fluctuations on the atmospheric turbulence and beam parameters is calculated. It is shown that the dependence of the OAM dispersion on the initial angular momentum of the laser beam disappears in the case of very strong turbulence.**

Keywords: *optical vortex, orbital angular momentum, atmospheric turbulence, Laguerre – Gaussian beams, wave-front dislocations.*

1. Introduction

It has been found theoretically and experimentally that light beams not only transfer energy but also have the linear and angular momenta. The total angular momentum can contain both the spin component related to polarisation and the orbital component related to the spatial distributions of the intensity and phase [1, 2]. This angular momentum can be also imparted to material particles, producing their rotation. This property has important applications in quite different fields such as biology [3] and micromechanics [4]. The angular momentum of light can be also used for information coding and processing [5], in particular, in optical communication systems [6]. By normalising the angular momentum of light to the energy carried by a light beam and measuring the energy in photon energy units $\hbar\omega$, each photon can be characterised by the spin orbital momentum $\sigma_z\hbar$, where $\sigma_z = \pm 1$ for circularly polarised light and $\sigma_z = 0$ for linearly polarised light. The orbital angular momentum (OAM) is related to the energy circulation in the light beam and is independent of the light polarisation. The properties of the OAM are manifested most distinctly in a beam carrying an optical vortex (OV) due to its helical wave front [7]. Laguerre – Gaussian light beams are the typical example of the beams carrying OVs. The longitudinal component of the OAM in such beams is an integer (topological charge l or the optical vortex strength) multiplied by $\hbar$. The beam OAM characterises the properties of OVs existing in the beam. This stimulates increasing interest in the peculiarities of OAM transformations in various optical systems [8 – 11].

The aim of this paper is to study the OAM transformation in a medium with random inhomogeneities of the permittivity, in particular, in the turbulent atmosphere. The object of our analysis is a linearly polarised coherent light beam representing the Laguerre – Gaussian mode $\mathrm{LG}_0^{(l)}$ with $l = 1,\ 2,\ 0$.

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2. Basic equations

Let $u(\boldsymbol{r}; z)$ be the complex amplitude of the light field of a coherent paraxial beam propagating along the z axis and $\boldsymbol{r}(x, y)$ be the vector in the xy plane. It is known that the z component of the OAM can be written in the form [2, 5]

$$L_z = \frac{\mathrm{i}}{2\omega}\,\frac{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\boldsymbol{r}\times(u\nabla_\perp u^* - u^*\nabla_\perp u)]\boldsymbol{n}\,\mathrm{d}x\mathrm{d}y}{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u(x, y; z)|^2\mathrm{d}x\mathrm{d}y}, \qquad (1)$$

where

$$\nabla_\perp = \boldsymbol{l}\frac{\partial}{\partial x} + \boldsymbol{m}\frac{\partial}{\partial y};$$

$\boldsymbol{l}$, $\boldsymbol{m}$, and $\boldsymbol{n}$ are the unit vectors directed along the x, y, and z axes, respectively; and ω is the angular frequency. By introducing the function $U_{1;1}(\boldsymbol{r}_1, \boldsymbol{r}_2; z) = u(\boldsymbol{r}_1; z)u^*(\boldsymbol{r}_2; z)$ [$U_{1;1}(\boldsymbol{r}_1, \boldsymbol{r}_1; z) = I(\boldsymbol{r}_1; z)$ is the beam intensity], making the change of variables $\boldsymbol{r} = (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$, $\boldsymbol{\rho} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, and denoting $U_{1;1}(\boldsymbol{r} + \boldsymbol{\rho}/2, \boldsymbol{r} - \boldsymbol{\rho}/2; z)$ by $U_2(\boldsymbol{r}; \boldsymbol{\rho}; z)$, we obtain

$$u\nabla_\perp u^* - u^*\nabla_\perp u = 2\nabla_{\perp\boldsymbol{\rho}}U_2(\boldsymbol{r}, \boldsymbol{\rho}; z)|_{\rho=0},$$

where

$$\nabla_{\perp\boldsymbol{\rho}} = \boldsymbol{l}\frac{\partial}{\partial\rho_x} + \boldsymbol{m}\frac{\partial}{\partial\rho_y}.$$

By using this equality, we obtain instead of (1) the expression

$$L_z = \frac{1}{\mathrm{i}\omega}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\boldsymbol{r}\times[\nabla_{\perp\boldsymbol{\rho}}U_2(\boldsymbol{r},\boldsymbol{\rho};z)|_{\rho=0}]\}\boldsymbol{n}\,\mathrm{d}x\mathrm{d}y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}I(x,y;z)\mathrm{d}x\mathrm{d}y}. \quad (2)$$

We will use the field distribution in the initial plane ($z=0$) as in [8]

$$u_{0l}(\boldsymbol{r}) = \frac{1}{a}\sqrt{\frac{8\Phi}{c}}\sqrt{\frac{1}{|l|!}}\left(\frac{x+\mathrm{i}y}{a}\right)^{|l|}\exp\left(-\frac{r^2}{2a^2}\right), \quad (3)$$

where c is the speed of light, Φ is the total energy flux, and a is the effective radius of the beam. By using (3) and (2), it is easy calculate that the orbital angular momentum L_z for $l = 1, 2, 0$ is $1/\omega$, $2/\omega$, and 0, respectively. Consider now the change in L_z during the propagation of laser beam in an inhomogeneous medium.

We assume that the beam propagates in the half-space $z \geqslant 0$ filled with a refracting medium with the permittivity $\varepsilon(x,y;z) = 1 + \tilde{\varepsilon}(x,y;z)$, where $\langle|\tilde{\varepsilon}|\rangle \ll 1$, and the complex amplitude of the beam satisfies the equation

$$2\mathrm{i}k\frac{\partial u}{\partial z} + \Delta_\perp u + k^2\tilde{\varepsilon}(\boldsymbol{r};z)u(\boldsymbol{r};z) = 0, \quad (4)$$

$$u(\boldsymbol{r};0) = u_{0l}(\boldsymbol{r}),$$

where $k = \omega/c$. It is known that the quantity

$$P_0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}I(x,y;z)\mathrm{d}x\mathrm{d}y,$$

which coincides accurately to a constant factor with the total energy flux of the beam, is preserved during the beam propagation in the small-angle scattering approximation. We will seek the parameters of L_z (2) in the inhomogeneous medium by the method used in [12] for calculating the statistical characteristics of the beam displacement. Let us write the equation for the random quantity U_2 in sum and difference variables:

$$\frac{\partial}{\partial z}U_2(\boldsymbol{r},\boldsymbol{\rho};z) = \frac{\mathrm{i}}{k}\nabla_{\perp\boldsymbol{\rho}}\nabla_{\perp\boldsymbol{r}}U_2 + \frac{\mathrm{i}k}{2}\left[\tilde{\varepsilon}(\boldsymbol{r}+\boldsymbol{\rho}/2;z) - \tilde{\varepsilon}(\boldsymbol{r}-\boldsymbol{\rho}/2;z)\right]U_2, \quad (5)$$

$$U_2(\boldsymbol{r},\boldsymbol{\rho};0) = u_{0l}(\boldsymbol{r}+\boldsymbol{\rho}/2)u^*_{0l}(\boldsymbol{r}-\boldsymbol{\rho}/2).$$

To derive the equation for L_z, we will subject Eqn (5) to the action of the operator $\nabla_{\perp\boldsymbol{\rho}}/(\mathrm{i}\omega P_0)$ and then set $\boldsymbol{\rho}=0$. As a result, we obtain

$$\frac{\partial}{\partial z}\frac{\nabla_{\perp\boldsymbol{\rho}}}{\mathrm{i}\omega P_0}U_2(\boldsymbol{r},\boldsymbol{\rho};z)|_{\rho=0} = \frac{1}{\omega P_0 k}\Delta_{\perp\boldsymbol{\rho}}\nabla_{\perp\boldsymbol{r}}U_2|_{\rho=0} + \frac{1}{2cP_0}\nabla_{\perp\boldsymbol{r}}\tilde{\varepsilon}(\boldsymbol{r};z)I(\boldsymbol{r};z). \quad (6)$$

Taking into account (2), we perform the vector multiplication of (6) by $\boldsymbol{r}$ and perform integration over this variable. As a result, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}L_z(z) = \frac{1}{2P_0c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}y[\boldsymbol{r}\times\nabla_{\perp\boldsymbol{r}}\tilde{\varepsilon}(\boldsymbol{r};z)]\boldsymbol{n}I(\boldsymbol{r};z). \quad (7)$$

Equation (7) should be supplemented with the boundary condition

$$L_z(0) = L_{z0}. \quad (8)$$

The equation similar to (7) was obtained earlier by Berry [13] by the quantum-mechanical method. The integration of Eqn (7) gives the z component of the beam OAM in the form

$$L_z(z) = L_{z0} + \frac{1}{2P_0c}\int_0^z\mathrm{d}\xi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}y[\boldsymbol{r}\times\nabla_{\perp\boldsymbol{r}}\tilde{\varepsilon}(\boldsymbol{r};\xi)]\boldsymbol{n}I(\boldsymbol{r};\xi). \quad (9)$$

By expanding the vector $\nabla_{\perp\boldsymbol{r}}\tilde{\varepsilon}(\boldsymbol{r};\xi)$ into the radial and azimuthal components, we obtain that the OAM of the beam propagating in the refractive medium changes under the action of the azimuthal component. It is obvious that the OAM value remains invariable in a medium with the axial symmetry. Expression (9) will be used below to study the OAM evolution in a random medium.

3. Statistical characteristics of the OAM in a random medium

We are interested in the mean $\langle L_z\rangle$ and dispersion $B_l = \langle(L_z - \langle L_z\rangle)^2\rangle$ of OAM fluctuations. To obtain $\langle L_z\rangle$, we perform statistical averaging in (9):

$$\langle L_z(z)\rangle = L_{z0} + \frac{1}{2P_0c}\int_0^z\mathrm{d}\xi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}y\left\langle I(\boldsymbol{r};\xi)\boldsymbol{n}[\boldsymbol{r}\times\nabla_{\perp\boldsymbol{r}}\tilde{\varepsilon}(\boldsymbol{r};\xi)]\right\rangle. \quad (10)$$

To calculate correlations in (10), we will use first the Markov random process approximation, assuming that the field $\tilde{\varepsilon}$ is a homogeneous random Gaussian field. In the Markov approximation, we have

$$\langle\tilde{\varepsilon}(\boldsymbol{r};z)\tilde{\varepsilon}(\boldsymbol{r}';z')\rangle = \delta(z-z')A(\boldsymbol{r}-\boldsymbol{r}'),$$

$$A(\boldsymbol{x}) = \int_{-\infty}^{\infty}\mathrm{d}\xi\langle\tilde{\varepsilon}(\boldsymbol{x}';z)\tilde{\varepsilon}(\boldsymbol{x}'+\boldsymbol{x};z+\xi)\rangle = 2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\kappa_x\mathrm{d}\kappa_y\exp(\mathrm{i}\boldsymbol{\kappa}\boldsymbol{x})\Phi_\varepsilon(\boldsymbol{\kappa}), \quad (11)$$

where $\delta(\boldsymbol{x})$ is the delta function; $\Phi_\varepsilon(\boldsymbol{\kappa})$ is the three-dimensional spectrum of the field $\tilde{\varepsilon}$; and $\boldsymbol{\kappa}$ is the two-dimensional vector. Then, by using the Furuzu–Novikov–Donsker formula [14], we calculate the mean of the product of the random filed and its functional:

$$\langle\tilde{\varepsilon}(\boldsymbol{r};z)R[\tilde{\varepsilon}]\rangle = \int_0^z\mathrm{d}\xi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}y'\langle\tilde{\varepsilon}(\boldsymbol{r};z)\tilde{\varepsilon}(\boldsymbol{r}';\xi)\rangle \times\left\langle\frac{\delta R[\tilde{\varepsilon}]}{\delta\tilde{\varepsilon}(\boldsymbol{r}';\xi)}\right\rangle. \quad (12)$$

Because in the calculation of the intensity variations in the filed $\tilde{\varepsilon}$ the equality

$$\left.\frac{\delta I(\boldsymbol{r};\xi)}{\delta\tilde{\varepsilon}(\boldsymbol{r}';\xi')}\right|_{\xi'=\xi}=0 \tag{13}$$

is fulfilled, the mean OAM in the Markov random process approximation proves to be equal to the initial momentum:

$$\langle L_z(z)\rangle = L_{z0}. \tag{14}$$

Therefore, the angular momentum dispersion B_l is described by the expression

$$B_l=\frac{1}{4P_0^2c^2}\int_0^z \mathrm{d}\xi_1\int_0^z \mathrm{d}\xi_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x_1\mathrm{d}y_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x_2\mathrm{d}y_2$$

$$\times\langle I(\boldsymbol{r}_1;\xi_1)I(\boldsymbol{r}_2;\xi_2)[\boldsymbol{r}_1\times\nabla_{\perp\boldsymbol{r}_1}\tilde{\varepsilon}(\boldsymbol{r}_1;\xi_1)][\boldsymbol{r}_2\times\nabla_{\perp\boldsymbol{r}_2}\tilde{\varepsilon}(\boldsymbol{r}_2;\xi_2)]\rangle. \tag{15}$$

By using relation (12) for calculating correlations in (15), after the calculation of a number of variational derivatives, we obtain in the Markov approximation the expression

$$B_l=\frac{1}{4P_0^2c^2}\int_0^z \mathrm{d}\xi_1\int_0^z \mathrm{d}\xi_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x_1\mathrm{d}y_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x_2\mathrm{d}y_2$$

$$\times\langle I(\boldsymbol{r}_1;\xi_1)I(\boldsymbol{r}_2;\xi_2)\rangle\langle[\boldsymbol{r}_1\times\nabla_{\perp\boldsymbol{r}_1}\tilde{\varepsilon}(\boldsymbol{r}_1;\xi_1)][\boldsymbol{r}_2\times\nabla_{\perp\boldsymbol{r}_2}\tilde{\varepsilon}(\boldsymbol{r}_2;\xi_2)]\rangle \tag{16}$$

with the 'splitting' of the correlation of $\tilde{\varepsilon}$ and I, which, after representing the vector products in terms of coordinates, can be written in the form

$$B_l=\frac{1}{4P_0^2c^2}\int_0^z \mathrm{d}\xi_1\int_0^z \mathrm{d}\xi_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x_1\mathrm{d}y_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x_2\mathrm{d}y_2$$

$$\times\langle I(\boldsymbol{r}_1;\xi_1)I(\boldsymbol{r}_2;\xi_2)\rangle\bigg(x_1x_2\frac{\partial^2}{\partial y_1\partial y_2}-x_1y_2\frac{\partial^2}{\partial y_1\partial x_2}-y_1x_2\frac{\partial^2}{\partial x_1\partial y_2}$$

$$+y_1y_2\frac{\partial^2}{\partial x_1\partial x_2}\bigg)\langle\tilde{\varepsilon}(\boldsymbol{r}_1;\xi_1)\tilde{\varepsilon}(\boldsymbol{r}_2;\xi_2)\rangle. \tag{17}$$

By substituting the correlation function (11) into (17), we obtain the expression

$$B_l=\frac{\pi}{2P_0^2c^2}\int_0^z \mathrm{d}\xi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\kappa_x\mathrm{d}\kappa_y\Phi_\varepsilon(\boldsymbol{\kappa})\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x_1\mathrm{d}y_1$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x_2\mathrm{d}y_2\left(x_1x_2\kappa_x^2-x_1y_2\kappa_x\kappa_y-y_1x_2\kappa_x\kappa_y+y_1y_2\kappa_y^2\right)$$

$$\times\exp[\mathrm{i}\boldsymbol{\kappa}(\boldsymbol{r}_1-\boldsymbol{r}_2)]\langle I(\boldsymbol{r}_1;\xi)I(\boldsymbol{r}_2;\xi)\rangle. \tag{18}$$

The calculation of the dispersion B_l from (18) involves the preliminary determination of the function $\Gamma_4(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,\boldsymbol{r}_4;z)=\langle u(\boldsymbol{r}_1;z)u(\boldsymbol{r}_2;z)u^*(\boldsymbol{r}_3;z)u^*(\boldsymbol{r}_4;z)\rangle$, which can be calculated by asymptotic or numerical methods [14]. Because asymptotic methods can be applied only in the limiting cases of the weak or strong turbulence, we derive the approximate formula to estimate B_l by using the so-called 'mean-intensity' approximation involving the replacement

$$\langle I(\boldsymbol{r}_1;\xi)I(\boldsymbol{r}_2;\xi)\rangle\approx\langle I(\boldsymbol{r}_1;\xi)\rangle\langle I(\boldsymbol{r}_2;\xi)\rangle. \tag{19}$$

This approximation was used earlier to calculate fluctuations of the 'centre of gravity' of laser beams in a random medium and gave good results [12] not only in limiting situations but also in the intermediate case (the region of fluctuation focusing). By using the Fourier transform for the intensity

$$I(\boldsymbol{r};z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}J(\boldsymbol{\kappa},z)\exp(\mathrm{i}\boldsymbol{\kappa}\boldsymbol{r})\mathrm{d}\kappa_x\mathrm{d}\kappa_y, \tag{20}$$

$$J(\boldsymbol{\kappa},z)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}I(\boldsymbol{r};z)\exp(-\mathrm{i}\boldsymbol{\kappa}\boldsymbol{r})\mathrm{d}x\,\mathrm{d}y$$

$$(J^*(\boldsymbol{\kappa},z)=J(-\boldsymbol{\kappa},z)), \tag{21}$$

we obtain from (18)

$$B_l=\frac{8\pi^5}{P_0^2c^2}\int_0^z \mathrm{d}\xi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\kappa_x\mathrm{d}\kappa_y\Phi_\varepsilon(\boldsymbol{\kappa})\bigg[\kappa_x^2\left\langle\frac{\partial}{\partial\kappa_x}J(\boldsymbol{\kappa},\xi)\right\rangle$$

$$\times\left\langle\frac{\partial}{\partial\kappa_x}J^*(\boldsymbol{\kappa},\xi)\right\rangle-\kappa_x\kappa_y\left\langle\frac{\partial}{\partial\kappa_x}J^*(\boldsymbol{\kappa},\xi)\right\rangle\left\langle\frac{\partial}{\partial\kappa_y}J(\boldsymbol{\kappa},\xi)\right\rangle$$

$$-\kappa_x\kappa_y\left\langle\frac{\partial}{\partial\kappa_y}J^*(\boldsymbol{\kappa},\xi)\right\rangle\left\langle\frac{\partial}{\partial\kappa_x}J(\boldsymbol{\kappa},\xi)\right\rangle$$

$$+\kappa_y^2\left\langle\frac{\partial}{\partial\kappa_y}J(\boldsymbol{\kappa},\xi)\right\rangle\left\langle\frac{\partial}{\partial\kappa_y}J^*(\boldsymbol{\kappa},\xi)\right\rangle\bigg]. \tag{22}$$

If the condition

$$\langle J(\boldsymbol{\kappa},z)\rangle=\langle J^*(\boldsymbol{\kappa},z)\rangle=\langle J(|\boldsymbol{\kappa}|,z)\rangle \tag{23}$$

is fulfilled, then by using in (22) instead of the Cartesian coordinate system $(\kappa_x\kappa_y)$ the polar system $(\kappa\varphi)$, after integration over the angular variable φ, we obtain

$$B_l=\frac{8\pi^6}{P_0^2c^2}\int_0^z \mathrm{d}\xi\int_0^{\infty}\mathrm{d}\kappa\kappa^3\Phi_\varepsilon(\kappa)\left[\frac{\partial}{\partial\kappa}\langle J(\kappa,\xi)\rangle\right]^2. \tag{24}$$

4. Mean intensity distribution and mean intensity spectrum of Laguerre – Gaussian beams

To perform calculations by expression (24), we calculate the mean intensity spectrum for beams with the initial intensity distribution (3) by using the well-known solution [14]

$$\Gamma_2(\boldsymbol{r},\boldsymbol{\rho};z)=\langle U_2(\boldsymbol{r},\boldsymbol{\rho};z)\rangle=\frac{k^2}{4\pi^2z^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}y'$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\rho_x'\mathrm{d}\rho_y'U_2(\boldsymbol{r}',\boldsymbol{\rho}';0)\exp\bigg[\frac{\mathrm{i}k}{z}(\boldsymbol{\rho}-\boldsymbol{\rho}')(\boldsymbol{r}-\boldsymbol{r}')$$

$$-\frac{\pi k^2}{4}\int_0^z H\left(\boldsymbol{\rho}\frac{\zeta}{z}+\boldsymbol{\rho}'\left(1-\frac{\zeta}{z}\right)\right)\mathrm{d}\zeta\bigg], \tag{25}$$

where

$$H(\boldsymbol{x})=2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1-\cos\boldsymbol{\kappa}\boldsymbol{x})\Phi_\varepsilon(\boldsymbol{\kappa})\mathrm{d}\kappa_x\mathrm{d}\kappa_y.$$

We will study the beam OAM with $l=1$, 2, 0. By setting $\boldsymbol{\rho}=0$ in (25) and using (3) with the parameter $l=1$, after integration over the variable $\boldsymbol{r}'$, we obtain

$$\langle I(\boldsymbol{r};z)\rangle = \frac{k^2}{4\pi^2 z^2}\left(\frac{1}{a^2}\sqrt{\frac{8\Phi}{c}}\right)^2 \pi a^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\rho_x' \mathrm{d}\rho_y' \times\left[1-\frac{1}{4a^2}g^2(z)\rho'^2\right]\exp\left[-\frac{1}{4a^2}g^2(z)\rho'^2 - \frac{\mathrm{i}k}{z}\boldsymbol{\rho}'\boldsymbol{r} - \frac{\pi k^2}{4}\int_0^z H\left(\boldsymbol{\rho}'\left(1-\frac{\zeta}{z}\right)\right)\mathrm{d}\zeta\right],\ l=1, \quad (26)$$

where $g^2(z) = 1 + \Omega^2$; $\Omega = ka^2/z$ is the diffraction Fresnel parameter. By integrating (26) in the xy plane, we find P_0 for $l = 1$:

$$P_0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x'\mathrm{d}y' \langle I(\boldsymbol{r}';z)\rangle = 8\pi\frac{\Phi}{c}. \quad (27)$$

By using (21) and (26), we obtain the mean intensity spectrum for the beam with $l = 1$ in the form

$$\langle J(\kappa,\xi)\rangle = 2\frac{\Phi}{\pi c}\left[1-\frac{1}{4a^2}g^2(\xi)\left(\frac{\xi}{k}\right)^2\kappa^2\right] \times \exp\left[-\frac{1}{4a^2}g^2(\xi)\left(\frac{\xi}{k}\right)^2\kappa^2 - \frac{\pi k^2}{4}\int_0^\xi H\left(-\boldsymbol{\kappa}\frac{\xi}{k}\left(1-\frac{\zeta}{\xi}\right)\right)\mathrm{d}\zeta\right]. \quad (28)$$

For $l = 2$ and 0, we have

$$\langle J(\kappa,\xi)\rangle = \frac{\Phi}{\pi c}\left\{\frac{\kappa^4}{16a^4}\left(\frac{\xi}{k}\right)^4[g^2(\xi)]^2 - \frac{1}{a^2}g^2(\xi)\left(\frac{\xi}{k}\right)^2\kappa^2 + 2\right\} \times \exp\left[-\frac{1}{4a^2}g^2(\xi)\left(\frac{\xi}{k}\right)^2\kappa^2\right] \times \exp\left[-\frac{\pi k^2}{4}\int_0^\xi H\left(-\boldsymbol{\kappa}\frac{\xi}{k}\left(1-\frac{\zeta}{\xi}\right)\right)\mathrm{d}\zeta\right],\ l=2, \quad (29)$$

$$\langle J(\kappa,\xi)\rangle = 2\frac{\Phi}{\pi c}\exp\left[-\frac{1}{a^2}g^2(\xi)\left(\frac{\xi}{k}\right)^2\kappa^2 - \frac{\pi k^2}{4}\int_0^\xi H\left(-\boldsymbol{\kappa}\frac{\xi}{k}\left(1-\frac{\zeta}{\xi}\right)\right)\mathrm{d}\zeta\right],\ l=0. \quad (30)$$

For P_0 and $l = 2$ and $l = 0$, the same equality (27) will be fulfilled.

5. Orbital angular momentum dispersion in the turbulent atmosphere

Expressions (28)–(30) are valid for any shape of the spectrum $\Phi_\varepsilon(\boldsymbol{\kappa})$ and, therefore, for any function $H(\boldsymbol{x})$. We will assume below that fluctuations of the permittivity of the medium are caused by temperature pulsations and set

$$\Phi_\varepsilon(\boldsymbol{\kappa}) = 0.033C_\varepsilon^2\kappa^{-11/3}, \quad (31)$$

where C_ε^2 is the structural characteristic of fluctuations of the air permittivity. In this case, we have $H(\boldsymbol{x}) = 0.465C_\varepsilon^2|\boldsymbol{x}|^{5/3}$. By substituting this expression into (28), we obtain

$$\langle J(\kappa,\xi)\rangle = 2\frac{\Phi}{\pi c}\left[1-\frac{1}{4a^2}g^2(\xi)\left(\frac{\xi}{k}\right)^2\kappa^2\right]\exp\left[-\frac{1}{4a^2}\times g^2(\xi)\left(\frac{\xi}{k}\right)^2\kappa^2 - 1.1824\beta_{0\xi}^2\frac{3}{8}\left(\frac{\xi}{k}\right)^{5/6}\kappa^{5/3}\right],\ l=1, \quad (32)$$

where $\beta_{0\xi}^2 = 0.307C_\varepsilon^2 k^{7/6}\xi^{11/6}$ is the mean square of fluctuations of the plane-wave intensity at a distance ξ, found in the continuous perturbation approximation. It follows from (32) that the condition of applicability of expression (24) is fulfilled.

Then, we can calculate

$$\frac{\partial}{\partial\kappa}\langle J(\kappa,\xi)\rangle = 2\frac{\Phi}{\pi c}\exp\left[-\frac{1}{4a^2}g^2(\xi)\left(\frac{\xi}{k}\right)^2\kappa^2 - 1.1824\beta_{0\xi}^2\frac{3}{8}\left(\frac{\xi}{k}\right)^{5/6}\kappa^{5/3}\right]\left\{-\frac{1}{a^2}g^2(\xi)\left(\frac{\xi}{k}\right)^2\kappa + \left[\frac{g^2(\xi)}{4a^2}\left(\frac{\xi}{k}\right)^2\right]^2 2\kappa^3 - \left[1-\frac{g^2(\xi)}{4a^2}\left(\frac{\xi}{k}\right)^2\kappa^2\right] \times 1.1824\beta_{0\xi}^2\frac{5}{8}\left(\frac{\xi}{k}\right)^{5/6}\kappa^{2/3}\right\}. \quad (33)$$

By squaring (33) and substituting the result into (24), by using spectrum (31) and the calculated value of P_0, we obtain after simple transformations

$$B_1 = \frac{1}{\omega^2}0.530\beta_0^2\int_0^1 \mathrm{d}\xi\int_0^\infty \mathrm{d}\kappa\kappa^{-2/3}\exp\left[-\frac{1}{2\Omega}q^2(\xi)\kappa^2 - 0.887\beta_0^2\xi^{8/3}\kappa^{5/3}\right]\left\{\frac{1}{\Omega}q^2(\xi)\kappa + 0.739\,\beta_0^2\xi^{8/3}\kappa^{2/3} - 2\left[\frac{1}{4\Omega}q^2(\xi)\right]^2\kappa^3 - 0.739\frac{1}{4\Omega}q^2(\xi)\beta_0^2\xi^{8/3}\kappa^{8/3}\right\}^2, \quad (34)$$

where $q^2(\xi) = \xi^2 + \Omega^2$ and $\beta_0^2 = 0.307C_\varepsilon^2 k^{7/6}z^{11/6}$.

Expression (34) can be used to calculate OAM fluctuations for arbitrary turbulent conditions in the laser beam path. These conditions can be determined based on the turbulence parameter β_0^2 and expression (34) can be simplified for the weak and strong turbulence. Consider the behaviour of the relative OAM dispersion $\sigma_{Ll}^2 = B_l/\langle L_z\rangle^2$. Taking (14) into account, we have for the weak turbulence ($\beta_0^2 \ll 1$)

$$\sigma_{L1}^2 = 0.53\beta_0^2\int_0^1 \mathrm{d}\xi\int_0^\infty \mathrm{d}\kappa\kappa^{-2/3}\exp\left[-\frac{1}{2\Omega}q^2(\xi)\kappa^2\right] \times\left\{\left[\frac{1}{\Omega}q^2(\xi)\right]^2\kappa^2 + 4\left[\frac{1}{4\Omega}q^2(\xi)\right]^4\kappa^6 - 4\frac{1}{\Omega}q^2(\xi)\left[\frac{1}{4\Omega}q^2(\xi)\right]^2\kappa^4\right\}. \quad (35)$$

After the calculation of integrals entering (35), we obtain

$$\sigma_{L1}^2 = 0.53\frac{331\cdot 2^{1/6}\Gamma(7/6)}{576}\beta_0^2\Omega^{-5/6}\int_0^1 \mathrm{d}\xi\left[q^2(\xi)\right]^{5/6} = 0.317\beta_0^2\Omega^{5/6}\,{}_2F_1\left(-\frac{5}{6},\frac{1}{2};\frac{3}{2};-\frac{1}{\Omega^2}\right). \quad (36)$$

Here, $\Gamma(x)$ is the gamma function and ${}_2F_1(a,b;c;x)$ is the hypergeometric Gaussian function [15]. In the near-field diffraction zone ($\Omega \gg 1$), we find from (36):

$$\sigma_{L1}^2 \approx 0.317\Omega^{5/6}\beta_0^2 \quad (\beta_0^2 \ll 1). \tag{37}$$

In the far-field zone ($\Omega \ll 1$), we have

$$\sigma_{L1}^2 \approx 0.119\Omega^{-5/6}\beta_0^2 \quad (\beta_0^2 \gg 1). \tag{38}$$

In the case of strong turbulence, we will also use (34), but consider only the far-field zone, when the relative OAM dispersion can be asymptotically estimated as

$$\sigma_{L1}^2 = 0.0534\beta_0^4 \quad (\beta_0^{-3/5} \ll \Omega \ll 1,\ \beta_0^2 \gg 1). \tag{39}$$

Note that in the case of strong turbulence, σ_{L1}^2 becomes independent of the beam size. It is obvious that the values of β_0^2 for intermediate values of the parameter σ_{L1}^2 can be found by numerical integration in expression (34).

For a Laguerre–Gaussian beam with $l = 2$, by substituting (29) into (24), for ther turbulence spectrum of type (31), we obtain

$$B_2 = \frac{1}{\omega^2}0.530\beta_0^2\int_0^1 \mathrm{d}\xi\int_0^\infty \mathrm{d}\kappa\kappa^{-2/3}\exp\left[-\frac{1}{2\Omega}q^2(\xi)\kappa^2 - 0.887\beta_0^2\xi^{8/3}\kappa^{5/3}\right]\left\{\frac{3}{2}\left[\frac{1}{2\Omega}q^2(\xi)\right]^2\kappa^3 - 0.739\beta_0^2\xi^{8/3}\kappa^{2/3} - 3\frac{1}{2\Omega}q^2(\xi)\kappa - \frac{1}{8}\left[\frac{1}{2\Omega}q^2(\xi)\right]^3\kappa^5 + 0.739\,\frac{1}{2\Omega^2}\beta_0^2\xi^{8/3}\kappa^{8/3} - 0.739\,\frac{1}{8}\beta_0^2\xi^{8/3}\left[\frac{1}{2\Omega}q^2(\xi)\right]^2\kappa^{4/3}\right\}^2. \tag{40}$$

For weak and strong turbulence, we find from (40) the following expressions:

$$\sigma_{L2}^2 = 0.124\beta_0^2\Omega^{5/6}{}_2F_1\left(-\frac{5}{6},\frac{1}{2};\frac{3}{2};-\frac{1}{\Omega^2}\right) \quad (\beta_0^2 \ll 1), \tag{41}$$

$$\sigma_{L2}^2 = 0.0133\beta_0^4 \quad (\beta_0^{-3/5} \ll \Omega \ll 1,\ \beta_0^2 \gg 1). \tag{42}$$

Figure 1 presents root-mean-square deviations of OAM fluctuations calculated for a laser beam propagating under conditions of arbitrary turbulence. Figure 1b shows the initial parts of curves (*1*) and (*2*) at the enlarged scale. It follows from the curves in Fig. 1 and expressions (36), (39), and (41) that three characteristic regions can be distinguished in the dependence of the dispersion of OAM fluctuations on the parameter β_0^2. In the first region (weak turbulence), the OAM dispersion increases with β_0^2 and the inequality $\sigma_{L1}^2 > \sigma_{L2}^2$ is satisfied. This inequality is also fulfilled in the third region (strong turbulence), however, the OAM dispersion increases with β_0^2 much faster. It seems that the increase in σ_{L1}^2 for $\beta_0^2 \ll 1$ is caused by the increase in the 'mechanical' component of the OAM [8], while a faster increase in σ_{L1}^2 is caused by the creation of new optical vortices due to the atmospheric turbulence and the increase in the vortex component of the OAM. In the second region (intermediate), the inequality $\sigma_{L1}^2 < \sigma_{L2}^2$ is fulfilled. We can assume that the change in the inequality in passing from the first region to the second one and the enhancement of OAM fluctuations in the laser beam with $l = 2$ compared to the OAM fluctuations in the beam with $l = 1$ are related to the instability of the optical vortex with the topological charge $l = 2$, carried by the beam, with respect to small perturbations [16] and its decomposition into a system of two randomly arranged vortices of the same sign, each of them having the unit topological sign.

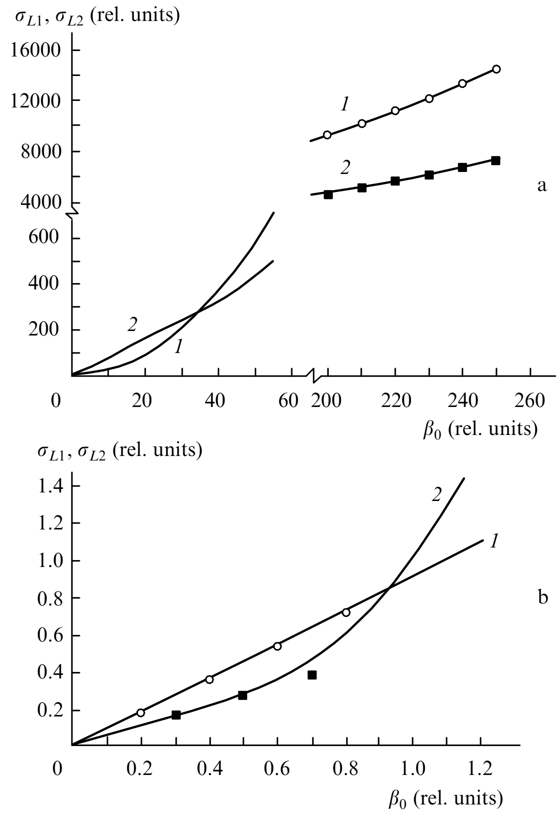

Figure 1. Dependences of the relative dispersions σ_{L1} (*1*) and σ_{L2} (*2*) of the beam OAM fluctuations on the parameter β_0 and their asymptotics (39) (○) and (42) (■) (a), and the initial parts of curves (*1*) and (*2*) at the enlarged scale and their asymptotics (36) (○) and (41) (■) (b) for $\Omega = 1$.

Consider now the 'atmospheric' component of the OAM by calculating the dispersion of fluctuations of the momentum of a Gaussian beam, which had initially the zero OAM. By using (30) and performing calculations according to the scheme presented above, we have in the most general case

$$B_0 = \frac{1}{\omega^2}0.530\beta_0^2\int_0^1 \mathrm{d}\xi\int_0^\infty \mathrm{d}\kappa\kappa^{-2/3}\exp\left[-\frac{1}{2\Omega}q^2(\xi)\kappa^2 - 0.887\beta_0^2\xi^{8/3}\kappa^{5/3}\right]\left\{\left[\frac{1}{2\Omega}q^2(\xi)\right]^2\kappa^2 + \left(0.739\beta_0^2\xi^{8/3}\right)^2\kappa^{4/3} + 2\cdot 0.739\frac{1}{2\Omega}q^2(\xi)\beta_0^2\xi^{8/3}\kappa^{5/3}\right\}, \tag{43}$$

and in the cases of weak and strong turbulence, we obtain

$$B_0 = 0.138\frac{1}{\omega^2}\beta_0^2\Omega^{5/6}{}_2F_1\left(-\frac{5}{6},\frac{1}{2};\frac{3}{2};-\frac{1}{\Omega^2}\right) \quad (\beta_0^2 \ll 1), \tag{44}$$

$$B_0 = 0.0534\beta_0^4\frac{1}{\omega^2} \quad (\beta_0^{-3/5} \ll \Omega \ll 1,\ \beta_0^2 \gg 1), \tag{45}$$

respectively. The ratios of the asymptotic estimates of B_l obtained for the conditions of weak and strong turbulence, depending on the index l have the form

$$\frac{B_1}{B_0} = 2.3,\quad \frac{B_2}{B_0} = 3.6 \quad (\beta_0^2 \ll 1), \tag{46}$$

$$\frac{B_1}{B_0} = 1, \quad \frac{B_2}{B_0} = 1 \ \ (\beta_0^{-3/5} \ll \Omega \ll 1, \ \ \beta_0^2 \gg 1). \qquad (47)$$

It follows from (46) and (47) that, as the beam path length or the atmospheric turbulence increase, when $\beta_0 \to \infty$, the initial OAM value does not affect OAM fluctuations, at least for the beams with the diffraction parameter restricted by the condition $\beta_0^{-3/5} \ll \Omega \ll 1$. This is confirmed by the dependences of ratios B_1/B_0 and B_2/B_0 on the parameter Ω calculated for a laser beam propagating under conditions of arbitrary turbulence (Fig. 2). It also follows from Fig. 2 that the dispersion of OAM fluctuations for a beam with $l = 2$ (B_2) comes out on its asymptotics later than for a beam with the dispersion B_1. Note that in the limiting case $\beta_0 \to \infty$, the main contribution to the estimate of B_1, B_2, and B_0 comes from the same term on integrands, which includes the power dependence $\propto \kappa^{2/3}$. It can be shown that the integral representation of B_l will contain the same term in the integrand. Therefore, we can write, without explicit calculations, that

$$\frac{B_l}{B_0} \to 1, \ \text{when} \ \ \beta_0 \to \infty.$$

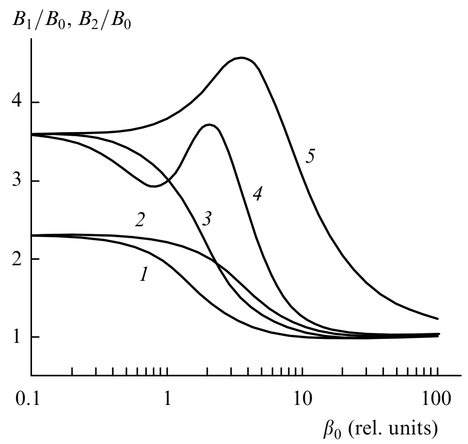

Figure 2. Dependences B_1/B_0 (*1–3*) and B_2/B_0 (*4,5*) on the parameter β_0 for $\Omega = 0.3$ (*1, 4*), 10 (*2, 5*), and 1 (*3*).

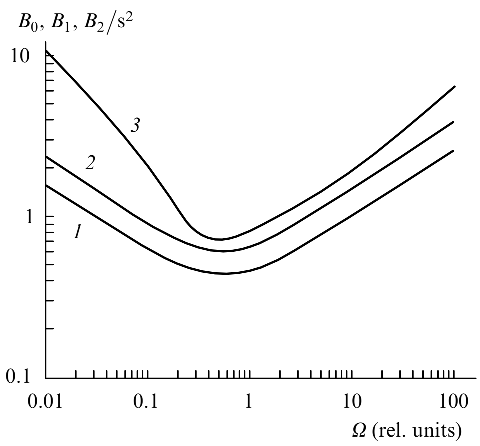

Figure 3. Dependences of dispersion of OAM fluctuations B_0 (*1*), B_1 (*2*), and B_3 (*3*) on the diffraction Fresnel parameter Ω for $\beta_0 = 1$ and a laser wavelength of 1.06 μm.

Figure 3 shows the dependence of B_l on the diffraction parameter Ω calculated for $\beta_0 = 1$. One can see that the minimum of OAM fluctuations is achieved near $\Omega = 1$, i.e. for the narrowest laser beams (in the diffraction sense). The passage to the near- or far-field diffraction zone is accompanied by the increase in fluctuations.

6. Conclusions

We have studied the change in the OAM of a vortex laser beam caused by inhomogeneities of the permittivity of the atmosphere. The integral representation of the laser beam OAM has been obtained. The integral relations for the mean and mean square OAM have been derived. It has been shown that the statistical mean OAM coincides with the OAM in a homogeneous medium. The dependences of the dispersion of OAM fluctuations on the atmospheric turbulence and diffraction parameters of the laser beam have been calculated. It is shown that in passing from the weak to strong turbulence regime during laser beam propagation, the growth rate of OAM fluctuations changes. As the atmospheric turbulence increases, the limiting dependence of the OAM dispersion on the turbulence conditions is unified and dispersion coincides with dispersion of a beam with the zero initial OAM, irrespective of the initial OAM. It has been also found that the relative dispersion of OAM fluctuations in a laser beam with the lower index l exceeds the relative dispersion of OAM fluctuations in a laser beam with the higher index in the weak and strong turbulence regimes, whereas the situation is opposite in the moderate turbulence regime. It is assumed that the change in the inequality sign is related to the instability and decomposition of high-order optical vortices. It has been shown that OAM fluctuations achieve their minimum for narrow diffraction-limited laser beams.

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