Quantum Electronics **38** (4) 359 – 364 (2008)

PACS numbers: 42.15.Dp; 42.60.Jf
DOI: 10.1070/QE2008v038n04ABEH013632

Algorithm for the field-phase distribution reconstruction in Hartmann sensor measurements

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***Abstract*. A new method for the wavefront reconstruction is described which is based on measurements of the beam intensity distributions in two cross sections and of wavefront tilts by a Hartmann sensor in one of these cross sections and makes it possible to find the field phase distribution taking into account possible phase jumps. The iteration algorithm for finding this distribution is developed and its mathematical description based on the calculation of the field in the Fresnel approximation is presented. The algorithm is studied numerically. Relations are obtained to estimate the optimal parameters of a measurement system, in particular, the distance between the cross sections and the number of subapertures of the Hartmann sensor depending on the mode composition of laser radiation.**

Keywords: *field phase distribution in a laser beam, Hartmann sensor, wavefront, radiation intensity distribution.*

1. Introduction

The properties of a Hartmann sensor (HS) were studied in papers [1 – 4] and the problem of the most efficient reconstruction of the spatial distribution of the laser radiation phase by using this sensor was analysed in [5 – 9]. The main attention in these papers was devoted to the minimisation of errors in the determination of the centres of gravity of light spots [1, 4, 8] and the development of an accurate and fast algorithm for the recon-struction of the phase distribution of radiation [1, 5, 9].

The accuracy of the wavefront reconstruction in HS measurements is limited by the discreteness of subapertures. A Hartmann sensor can be used to obtain information on the wavefront tilts, i.e. on the gradient of the phase distribution function. However, information on possible phase jumps (ordinary discontinuities) is lost in this case. The corresponding errors also remain when the number of HS subapertures is increased. As a result, the error of HS measurements of the laser beam divergence can be as high as 20 % – 60 %.

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Received 28 May 2007; revision received 29 October 2007
Kvantovaya Elektronika **38** (4) 359 – 364 (2008)
Translated by M.N. Sapozhnikov

Phase jumps are inherent in laser radiation even in the case of an ideal optical system (with plano-spherical optics) [10]. For example, all the modes of an open resonator, except the TEM_{00} mode, have phase jumps by π rad. This is explained by the fact that the field of these modes can be both positive and negative. The change of the field sign is equivalent to the phase jump by π rad.

Note that to reconstruct the wavefront in HS measurements, additional information on laser radiation is required because phase reconstruction only by two intensity distribution is ambiguous in the general case. For example, let the intensity distribution in the first cross section along the beam path be an even function $I_0(\boldsymbol{\rho}) = I_0(-\boldsymbol{\rho})$ of coordinates ($\boldsymbol{\rho}$ is the coordinate of points in the first cross section). We will describe the beam propagation in the Fresnel – Kirchhoff approximation and will take into account that phase conjugation does not change the field intensity distribution in both cross sections. Then, it can be easily shown that beams with phase distributions $\varphi_0(\boldsymbol{\rho})$ and $-\varphi_0(-\boldsymbol{\rho}) - \rho^2 k/z$ (k is the wave number and z is the distance to the second cross section along the beam path) form the same intensity distribution $I_z(\boldsymbol{\rho})$ in the second cross section along the beam path. The ambiguity can be excluded by using additional information on the phase distribution in one of the cross sections obtained from HS measurements.

In this paper, we propose a new method for wavefront reconstruction based on HS measurements by using additional information on the beam intensity distribution in two cross sections. The algorithm allows us to find the phase distribution taking into account its possible jumps.

2. Foundation of the algorithm and basic calculation relations

Let us compare the calculated intensity distribution in the focus of an optical scheme obtained from the real field distribution at the input of the optical system with the distribution obtained from HS measurements by using the modal algorithm [6, 7]. Figures 1 and 2 present the far-field intensity distributions for modes TEM_{11} and TEM_{22}. One can see that the far-field intensity distribution calculated by reconstructing the phase distribution by using the modal algorithm (Figs 1b and 2b) considerably differs from the real distribution (Figs 1a and 2a).

Algorithms for reconstructing phase distributions [1, 6 – 8], based on measurements of local wavefront tilts, cannot take into account jumps in the phase distribution. The beam diameter calculated in the focus of the optical

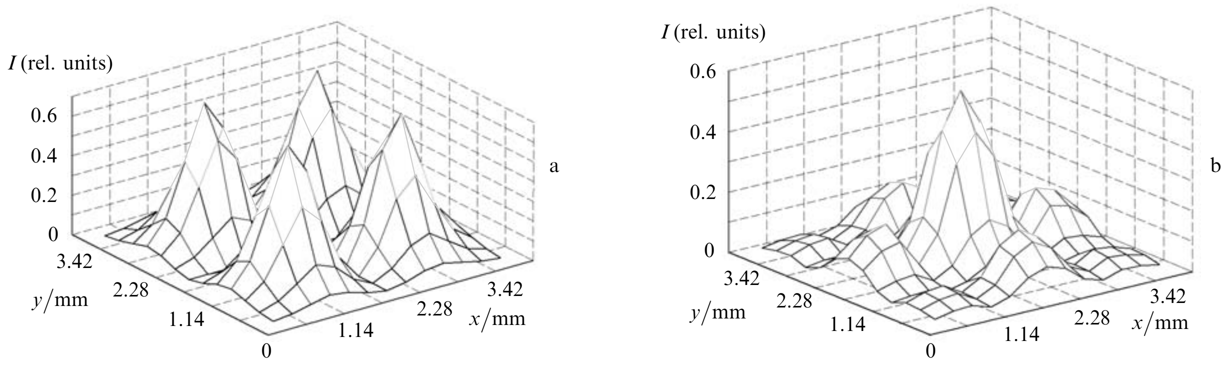

Figure 1. Intensity distribution of a focused TEM_{11} beam (a) and this distribution obtained by using the traditional phase reconstruction algorithm (b).

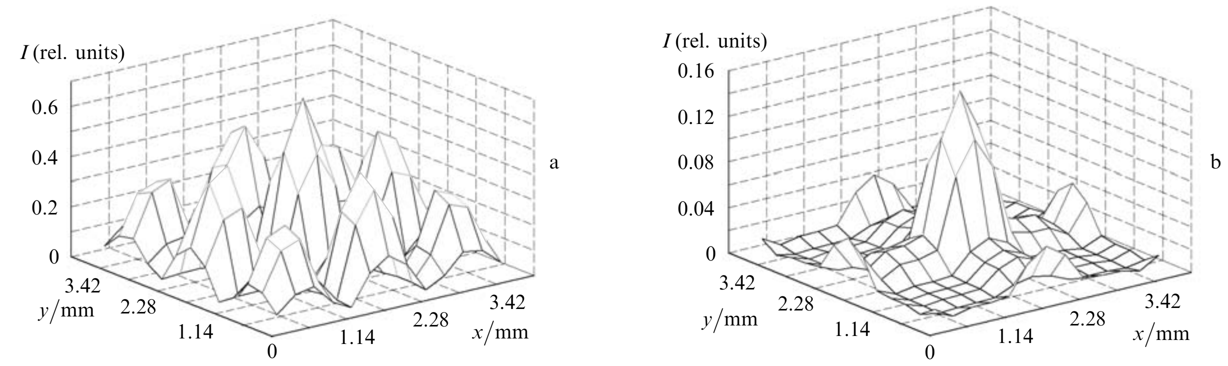

Figure 2. Intensity distribution of a focused TEM_{22} beam (a) and this distribution obtained by using the traditional phase reconstruction algorithm (b).

system by using such reconstruction algorithms is 1.5 – 2 times smaller that the real beam diameter.

Consider the possibility of the radiation wavefront reconstruction by measuring field characteristics in the two cross sections of the beam separated by distance z.

Let only the intensity distribution $I_0(\boldsymbol{\rho})$ be measured in cross section 1. The intensity distribution $I_z(\boldsymbol{r})$ and angular tilts $\boldsymbol{\Theta}_z(\boldsymbol{r})$ of the wavefront are measured simultaneously with a HS in cross section 2. The relation between these fields in the Fresnel approximation [11] can be written in the form

$$\iint [I_0(\boldsymbol{\rho})]^{1/2} \exp[\mathrm{i}\varphi_0(\boldsymbol{\rho})] K(\boldsymbol{\rho},\boldsymbol{r}) \mathrm{d}^2\boldsymbol{\rho} = [I_z(\boldsymbol{r})]^{1/2} \exp[\mathrm{i}\varphi_z(\boldsymbol{r})], \tag{1}$$

where $\varphi_0(\boldsymbol{\rho})$ and $\varphi_z(\boldsymbol{r})$ are field phase distributions in cross sections 1 and 2, respectively; $k = 2\pi/\lambda$ is the wave number; λ is the radiation wavelength; and

$$K(\boldsymbol{\rho},\boldsymbol{r}) = \frac{k}{2\mathrm{i}\pi z} \exp\left[\frac{\mathrm{i}k}{2z}(\boldsymbol{\rho}-\boldsymbol{r})^2 + \mathrm{i}kz\right]$$

is the kernel of the Fresnel integral. It is assumed in (1) that $\varphi_z(\boldsymbol{r})$ in cross section 2 is reconstructed from the tilt angles $\boldsymbol{\Theta}_z(\boldsymbol{r}) = \mathrm{grad}(\varphi_z(\boldsymbol{r}))/k$ of the radiation wavefront.

The relation

$$\iint [I_z(\boldsymbol{r})]^{1/2} \exp[\mathrm{i}\varphi_z(\boldsymbol{r})] K^*(\boldsymbol{r},\boldsymbol{\rho}) \mathrm{d}^2 r = [I_0(\boldsymbol{\rho})]^{1/2} \exp[\mathrm{i}\varphi_0(\boldsymbol{\rho})] \tag{2}$$

is also valid. Here, the asterisk means complex conjugation. Expression (2) takes into account the principle of reversibility of beam paths. It follows from (2) that

$$\left| \iint [I_z(\boldsymbol{r})]^{1/2} \exp[\mathrm{i}\varphi_z(\boldsymbol{r})] K^*(\boldsymbol{r},\boldsymbol{\rho}) \mathrm{d}^2 r \right| = [I_0(\boldsymbol{\rho})]^{1/2}. \tag{3}$$

Relation (3) is valid in the case of ideal measurements. As mentioned above, under real conditions of discrete measurements, the function $\varphi_z(\boldsymbol{r})$ cannot be always reconstructed from $\boldsymbol{\Theta}_z(\boldsymbol{r})$. Therefore, (3) can be considered as the equation for refining the partially known field phase distribution. In this case, it is reasonable to represent the required phase distribution as the sum

$$\varphi_z(\boldsymbol{r}) = \varphi_c(\boldsymbol{r}) + \varphi_s(\boldsymbol{r}), \tag{4}$$

where $\varphi_c(\boldsymbol{r})$ is a continuous function, which can be reconstructed from wavefront tilts by using known algorithms [6, 7]; $\varphi_s(\boldsymbol{r})$ is a discontinuous function taking into account possible phase jumps and calculated taking into account intensity distributions $I_z(\boldsymbol{r})$ and $I_0(\boldsymbol{\rho})$ (3).

The problem of reconstruction of the phase $\varphi_z(\boldsymbol{r})$ in cross section 2 is solved by assuming that intensity distributions $I_z(\boldsymbol{\rho})$ and $I_0(\boldsymbol{r})$ are known from experiments, while the phase $\varphi_z(\boldsymbol{\rho})$ is known partially, with an accuracy to possible jumps.

Equation (2) is solved by using the iteration algorithm realised by several steps:

(i) The field distribution $[I_z(\boldsymbol{r})]^{1/2}\exp[\mathrm{i}\varphi_z(\boldsymbol{r})]$ in cross section 2 is corrected by using the phase addition $\varphi_s(\boldsymbol{r})$ taking into account possible phase jumps.

(ii) The field $[I_0(\boldsymbol{\rho})]^{1/2}\exp[\mathrm{i}\varphi_0(\boldsymbol{\rho})]$ in section 1 is calculated from (2).

(iii) The field calculated in step 2 is corrected so that a new filed receives the measured intensity distribution, while the phase distribution remains as in step 2.

(iv) The field in section 2 is calculated from (1).

(v) The parameter estimating the degree of similarity of the fields calculated in step 4 and used in step 1 is calculated.

(vi) If the compared field distributions are similar enough, the search for solutions is terminated, and the field obtained in step 4 is considered as the reconstructed phase distribution.

(vii) If the compared fields are substantially different, step 1 is repeated. In this case, a new field in section 2 is used in calculations, which has the phase distribution calculated in step 4; the intensity of this field is equal to the measured intensity $I_z(\boldsymbol{r})$.

Then, we pass to the discrete representation of the algorithm in relations (1)–(4) taking into account the parameters of the measurement system. Let us introduce coordinates ρ_m and r_n of the centres of subapertures within which laser beam parameters vary in cross sections 1 and 2, respectively; $n = 1, 2, \ldots, N$; $m = 1, 2, \ldots, M$ is the number of subapertures.

After iterations $p = 1, 2, \ldots, P$ (p is the number of a current iteration, P is the number of the last iteration), the field distribution $U_z^{(p)}(\boldsymbol{r})$ becomes known. If the iteration process is terminated, the reconstructed phase distribution has the form

$$\varphi_z(\boldsymbol{r}) = \arg U_z^{(p)}(\boldsymbol{r}). \tag{5}$$

If the iteration process continues, the correcting phase addition

$$\varphi_s^{(p+1)}(\boldsymbol{r}_n) = \arg U_z^{(p+1)}(\boldsymbol{r}_n) \tag{6}$$

is calculated in step $p+1$ and the field $U_0^{(p+1)}$

$$U_0^{(p+1)}(\boldsymbol{\rho}_m) = \sum_{n=1}^{N} f_n^{(p+1)}[I_z(\boldsymbol{r}_n)]^{1/2} K_{nm}(\boldsymbol{r}_n, \boldsymbol{\rho}_m) \tag{7}$$

in cross section 1, where

$$f_n^{(p+1)} = \exp\left[\mathrm{i}\varphi_s^{(p+1)}\right](\boldsymbol{r}_n);$$

$$K_{nm}(\boldsymbol{r}_n, \boldsymbol{\rho}_m) = -\frac{\mathrm{i}k}{2\pi z}\int_{S_z(n)} \exp(-\mathrm{i}kz)\times$$

$$\times\exp\left[-\mathrm{i}k\boldsymbol{\Theta}_z(\boldsymbol{r}_n)(\boldsymbol{r}-\boldsymbol{r}_n) + \frac{\mathrm{i}k}{2\pi z}(\boldsymbol{r}+\boldsymbol{r}_n-\boldsymbol{\rho}_m)^2\right]\mathrm{d}^2\boldsymbol{r};$$

$\boldsymbol{\Theta}_z(\boldsymbol{r}_n)$ is the measured vector of tilt angles and $S_z(n)$ is the subaperture area with the centre ρ_n in cross section 2. It is assumed in (7) that the approximation of the phase $\varphi_z^{(p+1)}$ distribution within the nth subaperture is linear:

$$\varphi_z^{(p+1)}(\boldsymbol{r}) = \varphi_s^{(p+1)}(\boldsymbol{r}_n) + k\boldsymbol{\Theta}_z(\boldsymbol{r}_n)(\boldsymbol{r}-\boldsymbol{r}_n). \tag{8}$$

At the next step, the field $U_z^{(p+1)}$ in section 2 is calculated:

$$U_z^{(p+1)}(\boldsymbol{r}_m) = \sum_{m=1}^{M} g_m^{(p+1)}[I_0(\boldsymbol{\rho}_m)]^{1/2} L_{nm}(\boldsymbol{r}_n, \boldsymbol{\rho}_m), \tag{9}$$

where

$$g_m^{(p+1)} = \arg U_0^{(p+1)}(\boldsymbol{\rho}_m);$$

$$L_{nm}(\boldsymbol{r}_n, \boldsymbol{\rho}_m) = \frac{\mathrm{i}k}{2\pi z}\int_{S_0(m)} \exp(\mathrm{i}kz)\exp\left[\frac{\mathrm{i}k}{2z}(\boldsymbol{\rho}+\boldsymbol{\rho}_m-\boldsymbol{r}_n)\right]^2\mathrm{d}^2\boldsymbol{r};$$

and $S_0(m)$ is the subaperture area with the centre r_m in cross section 1. Then, the iteration process can be terminated or continued taking into account (5) and (6), respectively.

It is necessary to specify the first step of the iteration process. The phase addition $\varphi_s^{(1)}(\boldsymbol{r}_n)$ at this step is specified as a random quantity with the width of the region of values no less than π rad. Further calculations are performed by expressions (7)–(9) for $p = 0$.

The criterion for the wavefront reconstruction quality is the smallness of the least square of the phase difference calculated in two successive iterations p and $p+1$:

$$\sigma_\mathrm{p}^2 = \sum_{n=1}^{N}[I_z(\boldsymbol{r}_n)]^{1/2}\left[\varphi_z^{(p+1)}(\boldsymbol{r}_n) - \varphi_z^{(p)}(\boldsymbol{r}_n)\right]^2 \Big/ \sum_{n=1}^{N}[I_z(\boldsymbol{r}_n)]^{1/2}. \tag{10}$$

As the number of iterations is increased, the value of σ_p^2 should approach zero.

The error of the phase distribution reconstruction by using the iteration algorithm considered here is related to the passage from integration in (1)–(3) to summation in (6)–(9). In this case, no less than three measurement points should correspond to each local maximum of the intensity distributions $I_0(\boldsymbol{\rho})$ and $I_z(\boldsymbol{r})$. To reconstruct the phase distribution of the TEM_{kl} laser mode, the condition

$$\min\left(\sqrt{N};\sqrt{M}\right) \geqslant 3[(k+1)(l+1)]^{1/2} \tag{11}$$

should be fulfilled, where N and M are the number of subapertures in cross sections 1 and 2, respectively. Relation (10) takes into account that the number of local maxima of the intensity distribution for the TEM_{kl} mode is $(k+1)(l+1)$.

In addition, the phase can be correctly reconstructed only if subbeams (corresponding to subapertures in cross sections 1 and 2) exchange their energy during propagation from one cross section to the other. This requirement is satisfied if the broadening of subbeams during diffraction from subaperture edges exceeds the beam half-width, i.e.

$$\frac{\lambda}{a\sqrt{N}}z \geqslant 2a, \quad \text{or} \quad z \geqslant \frac{a^2}{\lambda}\frac{2}{\sqrt{N}}, \tag{12}$$

where a is the subaperture size.

Relations (5)–(10) completely describe the iteration algorithm considered here. The accuracy and efficiency of the algorithm were studied numerically.

3. Results of the computing experiment

The computing experiment was performed in the following way. First, the field intensity distributions $I_z(\boldsymbol{r})$ and $\varphi_z(\boldsymbol{r})$ were specified in cross section 2. Then, the field distribution

$U_0(\boldsymbol{\rho})$ and intensity $I_0(\boldsymbol{\rho}) = |U_0(\boldsymbol{\rho})|^2$ were calculated in cross section 1. We next assumed that, as in the real measurement and phase reconstruction process, the discrete values of the beam intensity $I_0(\boldsymbol{\rho})$ and $I_z(\boldsymbol{r})$ in two cross sections and discrete values of the wavefront tilts $\boldsymbol{\Theta}_z(t) = k^{-1} \times \mathrm{d}[\varphi_{\mathrm{c}}(\boldsymbol{r})]/\mathrm{d}\boldsymbol{r}$ in one of the cross sections are known. The wavefront reconstruction quality was estimated from the quantity characterising the deviation of the reconstructed wavefront from the real one.

In the computing experiment, we varied $I_z(\boldsymbol{r})$, $\varphi_z(\boldsymbol{r})$, the distance z between cross sections, the number of HS subapertures, and the radiation intensity distribution in cross section 1. The centres of subapertures were in the nodes of a square network, the number of measurements in both cross sections being the same ($N = M$). The field distribution in cross section 2 was specified in the form

$$U_z(\boldsymbol{r}) = A_{kl}(\boldsymbol{r}) \exp[\mathrm{i}\varphi_{\mathrm{c}}(\boldsymbol{r})], \tag{13}$$

$$A_{kl}(\boldsymbol{r}) = \exp\left(-2\,\frac{x^2+y^2}{b^2}\right) H_k\left(\frac{x}{b}\right) H_l\left(\frac{y}{b}\right), \tag{14}$$

where $H_{k(l)}(x, y/b)$ are Hermitian polynomials of the order k, $l = 0, 1, 2\ldots$; $\boldsymbol{r} = (x, y)$; $2a = 4$ mm is the size of the square HS aperture; $b = (0.7-1)a$; and $\lambda = 1064$ nm. The phase distribution function $\varphi_{\mathrm{c}}(\boldsymbol{r})$ was written in the form of a polynomial of (x, y) with the amplitude varying up to ~ 10 rad. In this case, the jumps of the phase $\varphi_{\mathrm{s}}(\boldsymbol{r})$ are related only to a change in the sign of the function $A_{kl}(\boldsymbol{r})$ depending on $\boldsymbol{r}$, $\varphi_{\mathrm{s}}(\boldsymbol{r}) = \arg(A_{kl}(\boldsymbol{r}))$. Figures 3–5 present some results of calculations. The reconstructed intensity distributions for the TEM_{11} and TEM_{22} modes coincide with those presented in Figs 1a and 2a, respectively. Figures 3c, 4b, and 5b present the reconstructed function $\varphi_{\mathrm{s}}(\boldsymbol{r})$. As follows from numerical calculations and Figs 3–5, the reconstructed parameters of the laser beam agree with the specified parameters.

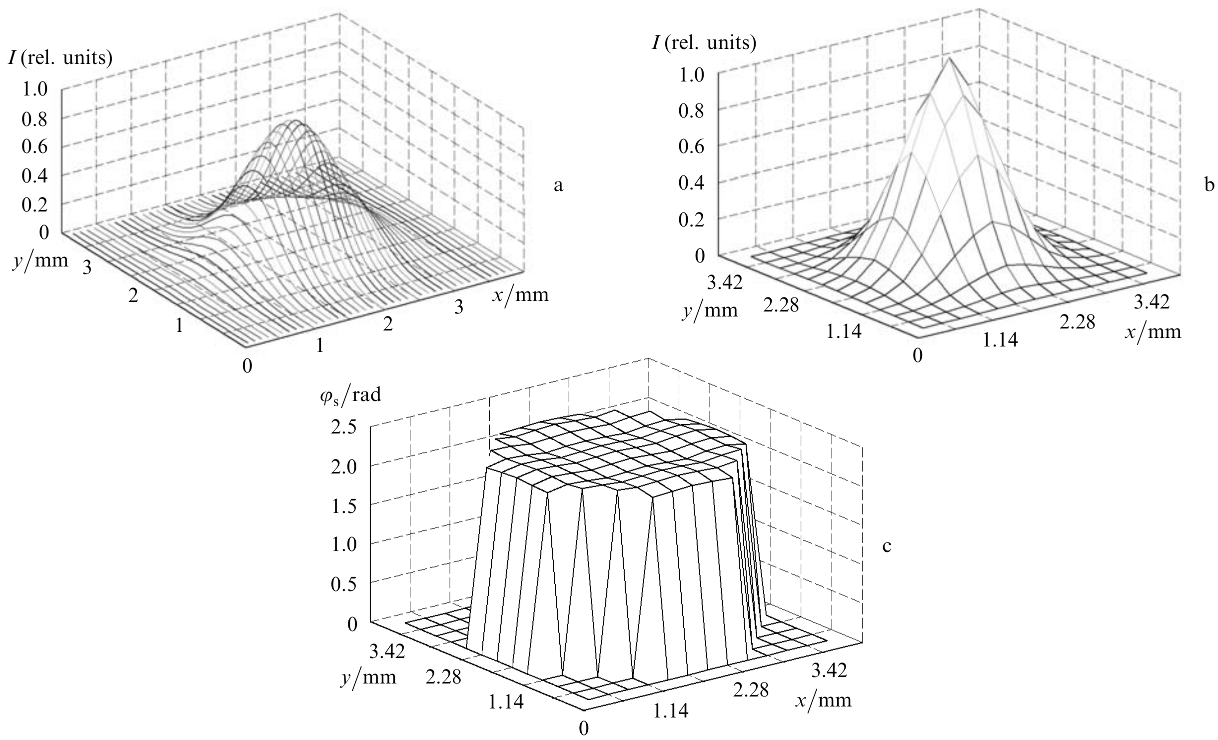

Figure 3. Specified (a) and reconstructed (b) TEM_{00} mode intensity distributions and the reconstructed distribution of the field phase φ_{s} for the TEM_{00} mode (c).

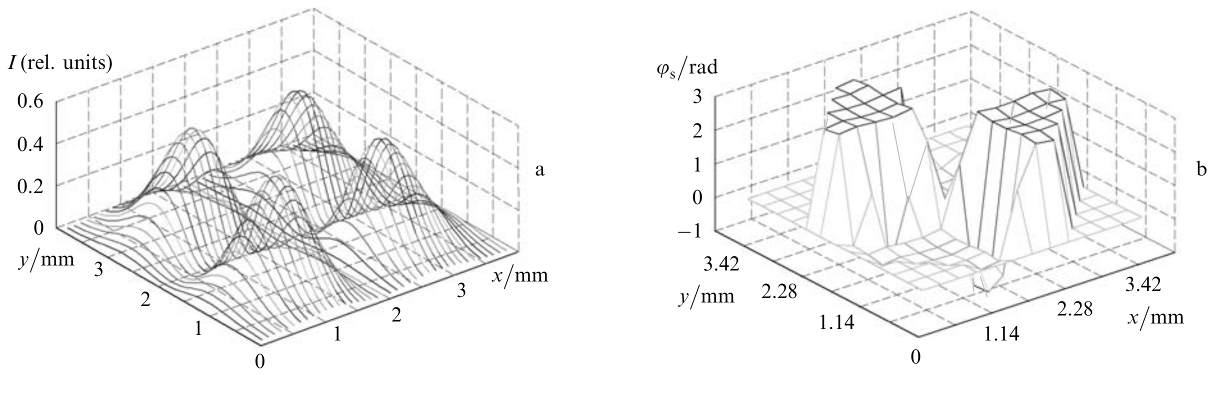

Figure 4. Specified mode intensity distribution (a) and reconstructed field phase distribution (b) for the TEM_{11} mode.

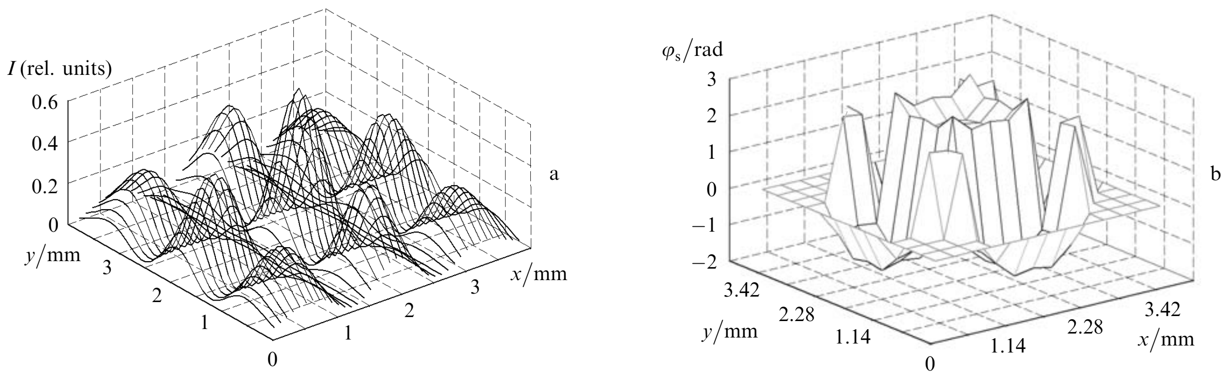

Figure 5. Specified mode intensity distribution (a) and reconstructed field phase distribution (b) for the TEM_{22} mode.

We studied in the computing experiment the dependence of S_p on the number of HS subapertures, the form of the function $U_z(r)$, and the number of iterations for different dependences $\varphi_z(\boldsymbol{r})$. Figure 6 shows that, as the number of iterations is increased, the reconstruction error S_p begins to fluctuate near a residual average level. The number P of iterations required to achieve this level is ~ 100. The residual error also depends on indices k and l: it decreases with increasing the number of iterations and increases with increasing indices k and l. These results confirm the correctness of estimate (11). Note also that the convergence rate of the algorithm strongly depends on the initial distribution $\varphi_z^{(1)}(\boldsymbol{r})$. To increase the convergence rate, it is reasonable to represent $\varphi_z^{(1)}(\boldsymbol{r})$ by a random function with the mean statistical value equal to the 'smoothed' phase distribution function, which can be obtained in phase reconstruction by using one of the known algorithms [1, 6, 7].

We also studied the dependence of the average S_p level on the normalised distance W, where $W = z\lambda/a^2$ (Fig. 7). As expected, the reconstruction error for small $W = 2/\sqrt{N}$ considerably exceeds the residual average level, which is achieved for $W \approx 0.3$. This result is consistent with relation (12).

It is reasonable to demonstrate the reconstruction dynamics of the laser radiation phase distribution. The phase distributions $\varphi_s(\boldsymbol{r})$ for the TEM_{22} laser mode for $n = 13$ are presented in Fig. 8 (the number of iterations is 10 and 100) and Fig. 5b for $P = 1000$.

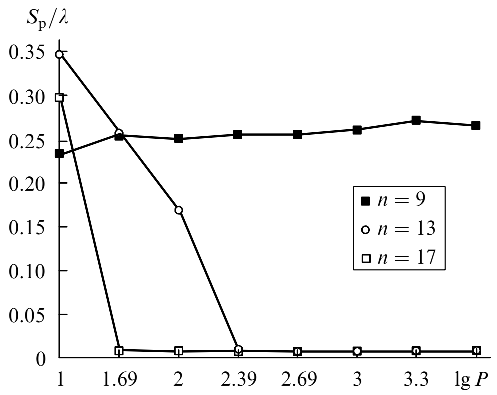

Figure 6. Dependence of S_p on the number P of iterations for the TEM_{22} mode for different numbers n of subapertures ($z = 0.25$).

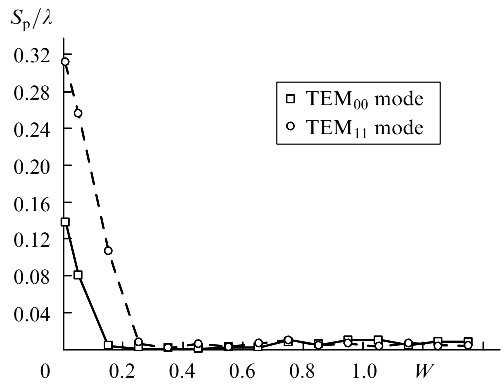

Figure 7. Dependence of S_p on $W = z\lambda/a^2$ for $P = 1000$.

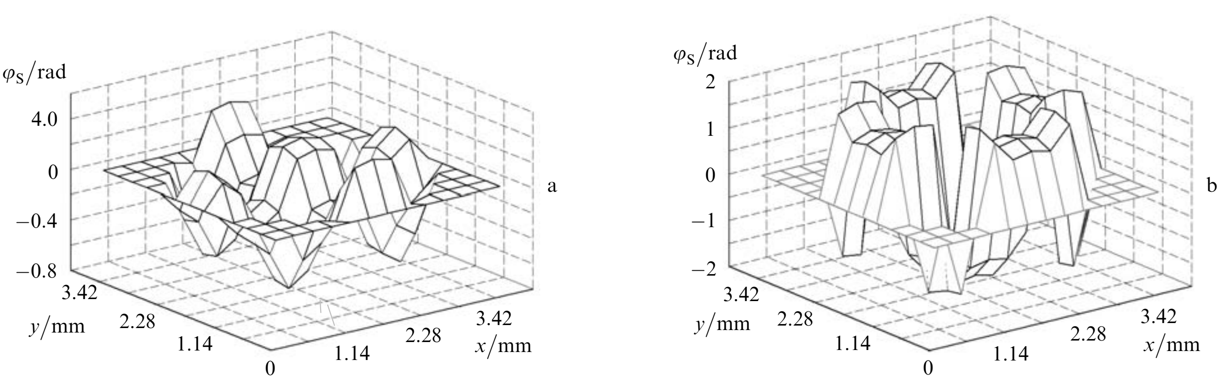

Figure 8. Reconstruction of the field phase φ_s for a TEM_{22} mode of a laser beam for $P = 10$ (a) and 100 (b).

The computing experiments have shown that the algorithm developed in our paper takes into account the features of a laser beam related to possible phase jumps and provides the wavefront reconstruction quality that is sufficient for practical applications.

4. Conclusions

A new method for the wavefront reconstruction has been proposed. The method is based on measurements of the wavefront tilts with a Hartmann sensor by using information on the beam intensity distribution in two cross sections. The main difference of the algorithm proposed in the paper from algorithms developed earlier is that it takes into account the possible phase jumps. We have proposed the iteration algorithm for determining the phase distribution, which is based on the calculation of the field in the Fresnel approximation. The numerical study of the algorithm gave relations for estimating the optimal parameters of the measurement system, in particular, the distance between the beam cross sections and the number of HS subapertures depending on the mode composition of laser radiation. Unlike the algorithms known previously, which allow one to determine the radiation divergence with an accuracy of up 20 % – 60 %, the algorithm proposed in the paper has no systematic errors and provides the reconstruction of the real field with the specified accuracy by increasing the number of HS apertures.

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