Quantum Electronics **38** (7) 673 – 677 (2008)

PACS numbers: 42.68.Sq; 42.15.Dp
DOI: 10.1070/QE2008v038n07ABEH013652

Algorithms for the reconstruction of the singular wave front of laser radiation: analysis and improvement of accuracy

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Abstract. **The possibility of reconstructing a singular wave front of laser beams by the local tilts of the wave front measured with a Hartmann sensor is considered. The accuracy of the reconstruction algorithm described by Fried is estimated and its modification is proposed, which allows one to improve the reliability of the phase reconstruction. Based on the Fried algorithm and its modification, a combined algorithm is constructed whose advantages are demonstrated in numerical experiments.**

Keywords: *wave-front reconstruction, phase reconstruction, singular wave front*.

1. Introduction

One of the key trends in the development of modern adaptive optical systems is related to the correction of scintillation effects appearing in a light beam propagating in an inhomogeneous medium or reflecting from a rough surface. The scintillation effects lead either to a decrease in the efficiency of the light energy transfer or to a distortion of information carried by a light beam. Because one of the main elements of adaptive systems is a wave-front sensor, researchers and research groups attempt to design sensors capable of providing the necessary spatial resolution and minimal measurement error.

One of the problems in the fabrication of such devices is the development of accurate, efficient from the point of view of computational resources and noise-resistant reconstruction algorithms of a wave front with screw dislocations.

The peculiarities of the development of reconstruction algorithms are reported in a number of theoretical papers [1 – 4]; however, papers, devoted to the experimental results of studies of singular phase distribution, are relatively rare. Thus, authors of [5] studied the phase distribution in different diffraction orders of a specially synthesised hologram intended for the generation of the highest orders of Laguerre – Gaussian modes of a laser beam. To measure the transverse phase distribution and localisation of phase singularities, they used an interferometer with a high spatial resolution. The interferometric wave-front sensor was also used in a high-speed adaptive optical system with a liquid-crystal modulator intended for the compensation for phase distortions under conditions of strong scintillations of coherent radiation in a turbulent atmosphere [6].

Despite the fact that the Shack – Hartmann sensor has found the widest application in adaptive optical systems, the results of experimental studies associated with the use of this device to measure the singular phase distribution have not been published so far. Apart from papers of the authors of the present work, recall paper [7] in which the directions of the Poynting vector were determined in a light beam carrying optical vortices (phase singularities) of different orders.

This paper presents the results of numerical experiments on the singular phase reconstruction with the help of the algorithm developed by Fried. We also propose two new reconstruction algorithms (modification of the Fried algorithm and combine algorithm), which allow the accuracy of the method to be improved. To estimate the reconstruction quality, the input parameters were used such as the gradients of the phase profile obtained by differentiating the specified functions with the known position of vortices (see section 4) and tilts of the wave front detected with the help of the model of the Hartmann sensor (section 6).

The developed algorithms were included in the software of a real sensor, which experimentally measured the phase of radiation carrying optical vortices [8].

2. Fried algorithm and its modification

The performed analysis of the reconstruction algorithms of a singular wave front by local tilts has shown [8] that the algorithm proposed by Fried has the maximum accuracy among the algorithms developed by now. The algorithm is also efficient from the point of view of computational burden and has high noise immunity [4].

The Fried algorithm (complex exponential reconstructor with a weighted dispersion) consists of three stages: reduction or simplification, solution and reconstruction [4]. It is intended for operation on a rectangular network of size $(2^K + 1) \times (2^K + 1)$ (K is an arbitrary natural number) in the Hadgin geometry. In this case, the reconstruction problem is reduced to reconstructing the distribution of

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Received 19 June 2007; revision received 17 September 2007
Kvantovaya Elektronika **38** (7) 673 – 677 (2008)
Translated by I.A. Ulitkin

'phasors' in the cross section xy of the beam – complex numbers with the unit modulus and argument equal to the phase $S(x, y)$ of the optical field. To solve the problem, transformation of differential 'phasors' $\exp[\mathrm{i}(S_{i-1,j} - S_{i,j})]$ and $\exp[\mathrm{i}(S_{i,j-1} - S_{i,j})]$ is performed, where $S_{i,j}$ is the phase field corresponding to the node i, j of the computational network or to a point with coordinates x_i, y_i.

Recall that the network representation of functions, where the input data $\nabla_\perp S$ ($\nabla_\perp = \partial/\partial x + \partial/\partial y$) are specified in the same nodes in which the phase S is reconstructed, is called the Fried geometry. If the network nodes for the projections $\partial S/\partial x$ and $\partial S/\partial y$ of the phase gradient on the beam cross section are displaced along coordinates x and y with respect to the network nodes for S, data are treated to be specified in the Hadgin geometry.

The phrase 'with a weighted dispersion', which enters the name of the algorithm means that it takes into account the differences in dispersions in measurements of individual differential 'phasors', namely, that the degree of influence of differential 'phasors' on the result of reconstruction is inversely proportional to their dispersion. This peculiarity of the Fried algorithm allows one to:

(i) apply it on a network of an arbitrary size and not only on a network of size $(2^K + 1) \times (2^K + 1)$ [8];

(ii) take into account the average inequality of measurement errors of the phase gradient in different regions of the beam (for example, corresponding to the subapertures of the Shack–Hartmann sensor), if the same beam is multiply measured;

(iii) take into account the *a priori* representation about the inequality of measurement errors of the phase gradient in these regions, if the beam parameters are measured only once.

The differential 'phasors' in the Fried algorithm are unit vectors which are obtained by the normalisation of complex numbers. It can be assumed that the amplitudes of differential 'phasors' and 'phasors' obtained after reduction and reconstruction contain information on measurement errors of phase differences in the experiment. Taking this assumption into account, an attempt was made to modify this algorithm, which involved the exclusion of the operation of complex vector normalisation. The numerical experiments showed that the modified algorithm has a number of advantages compared to the initial one [8]. The results illustrating the properties of the algorithms are presented below.

3. Combined algorithm: algorithm of search for vortices and phase reconstruction

It is known that the quality of the phase reconstruction algorithm decreases with increasing the number N of vortices contained in the radiation wave front. In this case, the parameter affecting the accuracy is the surface density N/N_g of vortices (where N_g is the number of cells in the computational network). As the density is increased, the modulus of the phase gradient $\nabla_\perp S$ average over the cross section increases and integration errors appear [8].

The algorithm proposed in this paper employs information about the coordinates of singularities, and hence, it is necessary to determine the dislocation coordinates in the phase distribution. To solve this problem, a number of methods (algorithms) were developed for vortex localisation:

(i) by the distribution of the phase gradient $\nabla_\perp S$ in the beam cross section – $A(-\nabla_\perp S)$ [9];

(ii) by the distribution of the phase S in the transverse plane of the beam – $A(S) \equiv A(S_0)$, where S_0 is the exact phase distribution in this plane;

(iii) by the distribution of the phase S_1 or S_2 obtained from data $\nabla_\perp S$ by using the Fried algorithm or its modification, – $A(S_1)$ or $A(S_2)$ [10]. In fact, $A(S_1)$ and $A(S_2)$ are the variants of the $A(S_0)$ algorithm, which involve the phase reconstruction algorithms.

By using algorithms $A(S)$, $A(S_1)$ and $A(S_2)$ for $N/N_\mathrm{g} = 9\,\%$, no less than 90 % of vortices are found on average and by using the algorithm $A(-\nabla_\perp S)$ under the same conditions – 45 % of vortices. Relatively low accuracy in the latter case is explained by the fact that when the saddle and the focus of the vector field fall into one cell of the discrete network, the algebraic number of singularities of the vector field $\nabla_\perp S$ in this cell becomes equal to zero and it is impossible to determine their presence with the help of the $A(-\nabla_\perp S)$ algorithm. These realisations of the random field appear, for example, when vortices with the same sign are located in neighbouring cells [the saddle is between the vortices (Fig. 1a)]. It is obvious that the probability of such situations to appear increases with increasing the surface

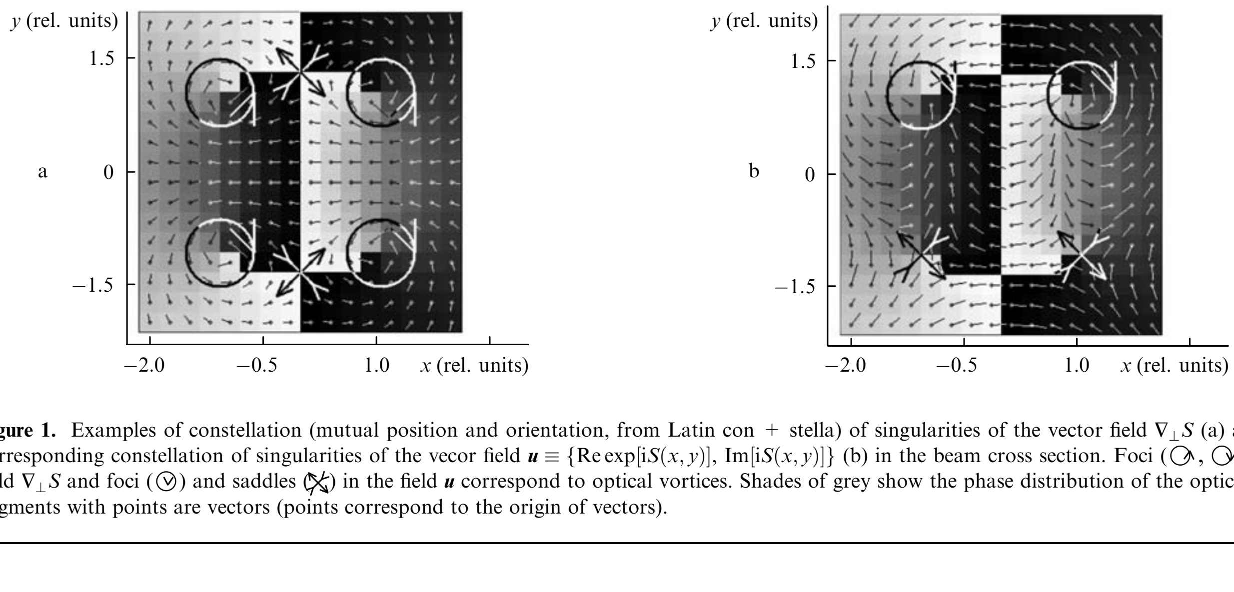

Figure 1. Examples of constellation (mutual position and orientation, from Latin con + stella) of singularities of the vector field $\nabla_\perp S$ (a) and the corresponding constellation of singularities of the vecor field $\boldsymbol{u} \equiv \{\mathrm{Re}\exp[\mathrm{i}S(x, y)], \mathrm{Im}[\mathrm{i}S(x, y)]\}$ (b) in the beam cross section. Foci (◔, ◕) in the field $\nabla_\perp S$ and foci (⊙) and saddles (✕) in the field $\boldsymbol{u}$ correspond to optical vortices. Shades of grey show the phase distribution of the optical field. Segments with points are vectors (points correspond to the origin of vectors).

density of vortices. At the same time, when the distribution $\exp(iS)$ (Fig. 1b) is used as an input parameter of the algorithm, vortices with the same time are detected even if they are located in one and the same cell, which provides a higher percent of vortex detection.

The analysis of the determination accuracy of coordinates of vortices with the help of algorithms $A(-\nabla_{\perp}S)$ and $A(S)$ shows that the error sometimes exceeds half the cell size of the computational network. This indicates the possibility of erroneous localisation of a singularity not in the right cell where it is located but in the neighbouring cell. In this respect, the algorithm $A(-\nabla_{\perp}S)$ has an advantage over the algorithm $A(S)$: the error averaged over the cross section for one realisation does not exceed here 0.45 of the cell size, while for $A(S)$ this error can achieve 0.90. Thus, the probability of placing a vortex in the neighbouring cell of the computational network is noticeably higher when the algorithm $A(S)$ is used. In addition, in some cases algorithms $A(S)$, $A(S_1)$, $A(S_2)$ detect nonexistent vortices.

Therefore, we can conclude that each of the above algorithms has certain disadvantages, which does not allow one to perform unambiguously one of them.

To increase the fraction of detected dislocations, improve the determination accuracy of their coordinates and improve the quality of singular phase reconstruction, we developed a combined algorithm in which the effect of the above negative factors was reduced by subtracting the component of the phase gradient S_v caused by vortices, whose position and the topological charge had been already found, from $\nabla_{\perp}S$. The phase calculated with the help of the combined algorithm will be denoted by S_3.

4. Phase reconstruction in Hadgin and Fried geometries

Figure 2 presents the results of testing the combined algorithm together with the data obtained by using the Fried algorithm and the modified Fried algorithm. The dependence of the average value of the Strehl parameter $\langle \mathrm{Sh} \rangle$, which was found by reconstructing the phase, and of the error $\sigma_{1-3} = \min_{\varphi}(\langle |[S_{1-3}(\boldsymbol{r}) + \varphi - S_0(\boldsymbol{r})]| \rangle_r)$ ($\boldsymbol{r} = \{x, y\}$ and φ varies from $-\pi$ to π) of this reconstruction on the number of dislocations introduced into the beam wave front, is shown. The square brackets mean the reduction operation of the involved values in the interval $[-\pi; \pi)$ by adding or subtracting the values of $2\pi n$, where n is an integer.

It follows from Figs 2a, b that for the average surface density N/N_g of vortices (5 %, $N = 55$), the modified Fried algorithm allows one to obtain the Strehl parameter and an error by 2 % higher and 30 % lower, respectively, than the Fried algorithm. For low and high (above 9 %, which corresponds to $N = 100$) density of vortices, the averaged

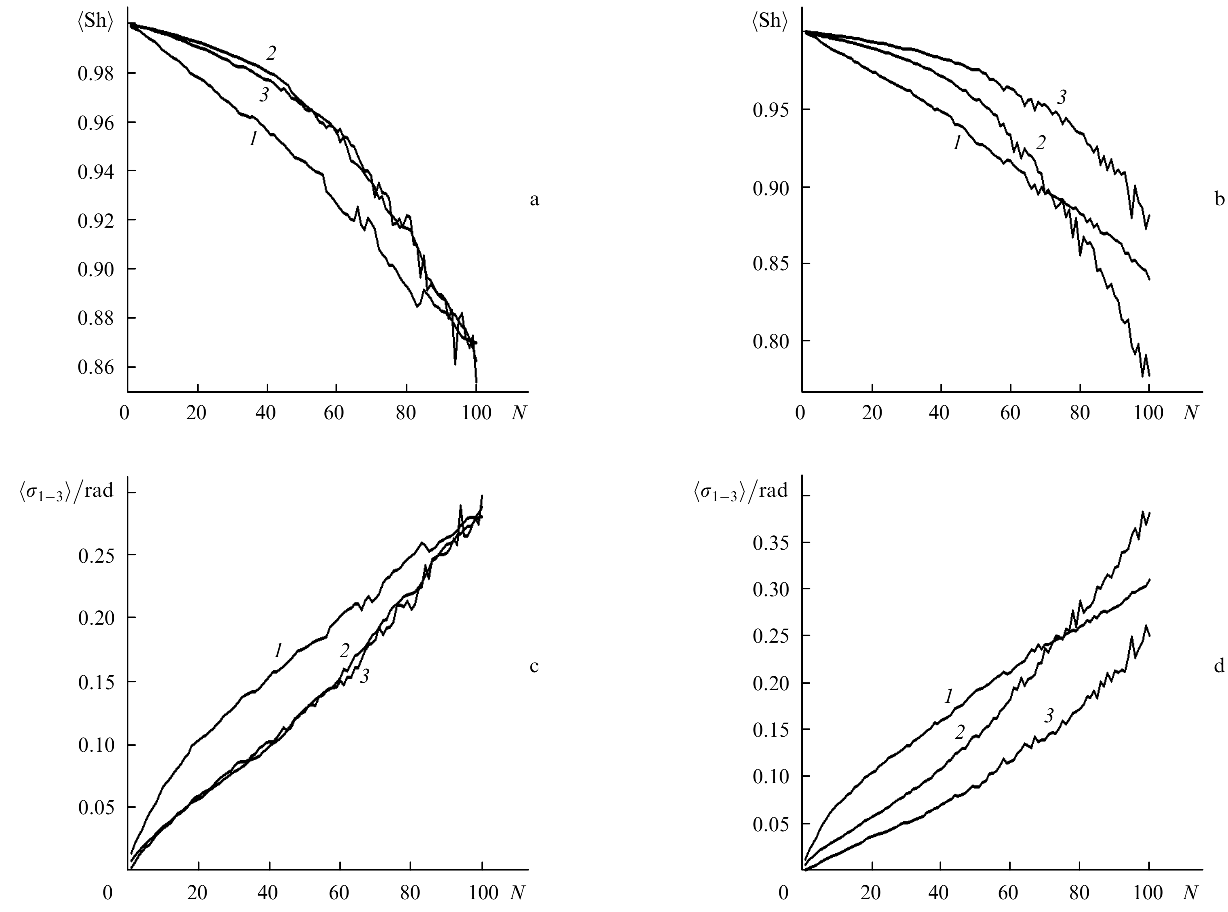

Figure 2. Dependences of the average value of the Strehl parameter (a, b) and averaged error of the phase reconstruction $\langle \sigma_{1-3} \rangle$ (c, d) on the number of vortices in the wave front. The phase reconstructed by data $\nabla_{\perp}S$ with the help of Fried algorithm (1), modified Fried algorithm (2) and combined algorithm (3) in the Hadgin (a, c) and Fried (b, d) geometries. Averaging was performed over 100 realisations of the field on a network of size 33 × 33.

values of the criterion are approximately the same for all algorithms [8].

It is obvious that the combined algorithm preserves the advantages of the modified algorithm over the Fried algorithm. The accuracy improvement (with respect to the modified algorithm) is not observed due to the low efficiency of the vortex search procedure and due to refinement of their position. Indeed, in the Hadgin geometry a cell of the computational network is limited by four points on the centres of its sides. In these points only two values of the arbitrary phase are defined by coordinates x and y; they do not contain enough information about the vortex position. Therefore, to obtain more precise results, the modernisation of this part of the algorithm is required.

It follows from the comparison of Figs 2b, d with Figs 2a, c that the operation in an alien geometry (the Fried geometry) deteriorates the accuracy of the Fried algorithm and its modification. For the surface density of vortices 9 % ($N = 100$), the Strehl parameter decreases from 0.86 (Fig. 2a) to 0.85 and 0.76 [Fig. 2b, curves (*1*) and (*2*)] and the average error of the phase reconstruction increases from 0.275 (Fig. 2c) to 0.275 and 0.375 rad [Fig. 2d, curves (*1*) and (*2*)]. In this case, first, the second algorithm operates better and if the number of vortices is more than 65, – worse than the first algorithm. The combine algorithm allows one to compensate for the adverse effect of the Fried geometry on the phase reconstruction. It improves the accuracy of the phase reconstruction by 1.36 times compared to the modified Fried algorithm. Note that the geometry of the data $\nabla_{\perp} S$ under study corresponds to the design of the Shack – Hartmann sensor. Therefore, it is the combined algorithm that should be included into its software.

It has been assumed above that the combined algorithm will allow one to increase the share of detected dislocations and enhance the determination accuracy of their coordinates. Let us check it.

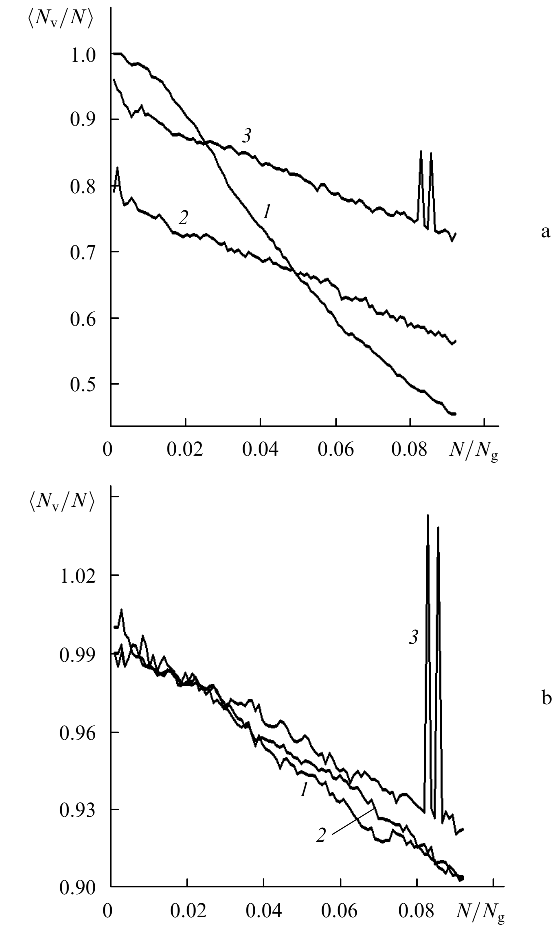

Figure 3. Number N_v of vortices (normalised to the specified number N of vortices and averaged over 100 realisations) determined by using the algorithms $A(-\nabla_{\perp} S)$, $A(S_2)$, $A(S_3)$ and combined algorithm as a function of N/N_g for different geometries of input data $\nabla_{\perp} S$. Curves correspond to the algorithm $A(-\nabla_{\perp} S)$ in the Fried geometry (*1*), the combined algorithm in the Hadgin (*2*) and Fried (*3*) geometries (a) and the algorithms $A(S_2)$ in the Hadgin (*2*) geometry, $A(S_3)$ in the Hadgin and Fried (*3*) geometries (b).

5. Results of measuring the number of vortices

The accuracy of the localisation procedure of singularities is illustrated in Fig. 3, which presents the dependence of the number N_v of dislocations found with the help of algorithms on the number N of dislocations introduced into the phase. The presented data show a significant advantage of the combined algorithm over the algorithm $A(-\nabla_{\perp} S)$ for large densities N/N_g (Fig. 3a). However, at small values of N/N_g due to inconsistent operation of algorithms $A(-\nabla_{\perp} S)$ and $A(S_2)$ in the combined algorithm, the situation is opposite. Apparently, for a small number of vortices the comparison of the results of the operation of the algorithms $A(-\nabla_{\perp} S)$ and $A(S_2)$ performed in the combined algorithm [the vortex registration only under the condition that it is found with the help of both the algorithm $A(-\nabla_{\perp} S)$ and $A(S_2)$] is unreasonable.

Differences in the quality of the operation of the algorithms $A(S_2)$ in the Hadgin geometry and $A(S_3)$ in the Hadgin and Fried geometries (Fig. 3b) are explained by a higher accuracy of the phase reconstruction with the help of the combined algorithm (see Fig. 2). Note also the presence of two tall peaks on curves (*3*) in Fig. 3. Their appearance is caused by self-excitation of the combined algorithm on two of 1000 processed realisations of the field. The self-excitation of the combined algorithm is expressed by filling each cell of the computational network with vortices. The reasons of this unstable behaviour require a separate study and its possibility restrains the application of the combined algorithm in practice.

6. Study of algorithms in the model of the Hartmann sensor

After presenting above the theoretical estimates, consider the possibility of using the algorithms under conditions when the tilts of the wave front are their input parameters. To perform these studies, the Fried algorithm and its modification realised in the form of a dynamic link library were introduced into the software of the experimental sensor prototype, which reconstructed the phase surface consisting of one singularity. The possibility of using the algorithm under real conditions has been proved in the laboratory experiments. At the same time, the accuracy of the phase reconstruction is convenient to study by using numerical methods, when the parameters of the problem vary in a broad range and the measurement of criteria characterising the accuracy is simple enough.

For this purpose, we developed a numerical model completely corresponding to the real device. In the model radiation propagates through an array of microlenses, then the intensity cross section is measured and the local tilts of the wave front are calculated; based on these measurements the phase profile is reconstructed.

The values of the Strehl parameter obtained during the reconstruction of the wave front of a vortex beam are presented in Figs 4 and 5. The number of dislocations in the radiation incident on the sensor was a variable parameter in numerical experiments. Data in Fig. 4 were detected for the size of the array of microlenses (dimensionality of the sensor raster) 32×32 and data in Fig. 5 – for the size of the array 16×16.

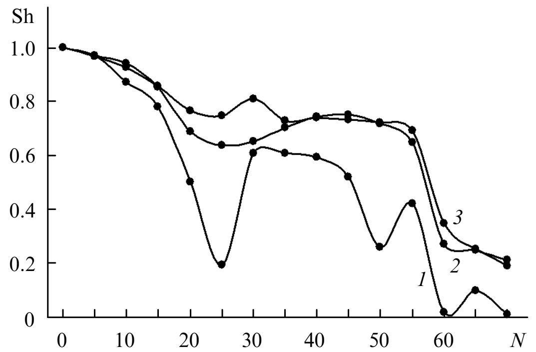

Figure 4. Dependences of the Strehl parameter on the number of dislocations introduced into the phase profile of radiation. The results are obtained for the model of a Hartmann sensor based on the Fried algorithm (*1*), modified Fried algorithm (*2*) and combined algorithm (*3*). The dimensionality of the sensor raster is 32×32.

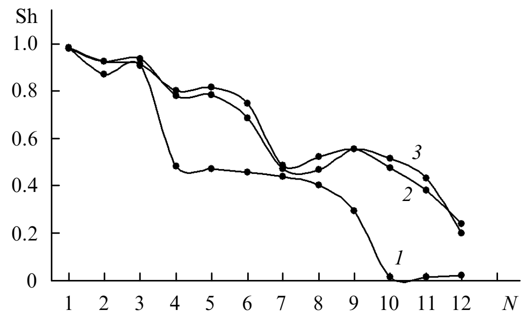

Figure 5. Same as in Fig. 4 for the dimensionality of the sensor raster 16×16.

One can see that for the large dimensionality of the sensor raster (Fig. 4), curves corresponding to the modified Fried algorithm and combined algorithm, virtually coincide. Differences are observed only in the region of variations in the dislocation number 20 – 35, where the combined algorithm provides higher (by 8 % – 10 %) values of Sh. The accuracy of both algorithms decreases with increasing the number of singularities.

A considerably sharper decrease in the accuracy and, hence, lower values of the Strehl parameter are observed when the Fried algorithm is used. For $N = 25$, the difference in the Strehl criteria achieved 50% and in the range $N = 25 - 70$, it is 10 % – 30 %.

The above peculiarities are also observed when a sensor with a raster of smaller dimensionality is used (Fig. 5). Here the modification of the Fried algorithm and the combined algorithm yield approximately equal results, the accuracy of the original Fried algorithm being 15 % – 25 % lower.

7. Conclusions

We have analysed algorithms of singular wave-front reconstruction. The modification of the Fried algorithm and the combined algorithm have been proposed. A comparative study of the accuracy of their operation has been performed, when the phase gradient $-\nabla_{\perp} S$ determined analytically and gradients obtained with the help of a numerical model of the Shack – Hartmann sensor are used as input data.

It has been shown that the modified algorithm operates on average more accurately than the initial one. When the combined algorithm with the input data $-\nabla_{\perp} S$ in the Hadgin geometry is used, the advantages of the Fried algorithm and its modification are preserved. In the case of the Fried geometry, the combined algorithm surpasses the mentioned counterparts. This makes its usage more preferable to process data obtained with the help of the Shack – Hartmann sensor.

Further improvement of the combined algorithm involves perfection of the algorithm of refinement of the vortex position and a change in the strategy of vortex selection (from found ones) to correct the vortex component of the phase gradient as well as adaptation of the combined algorithm for the reconstruction of the phase distribution including high-order vortices.

Acknowledgements. This work was supported by Grant No. MK-4701.2006.9 of the President of the Russian Federation and the International Scientific and Technical Centre (Grant No. 2631).

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