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# Multicomponent cnoidal waves in cascade parametric frequency conversion

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***Abstract*.** **It is shown that four-mode interaction in quasi-synchronous cascade frequency conversion on a quadratic nonlinearity can be described in terms of an effective cubic nonlinearity, which reduces the problem to solving the system of two coupled nonlinear Schrödinger equations (NSEs) with respect to the amplitudes of waves involved in both nonlinear processes. Analytic solutions of a new type found for this system have the form of cnoidal waves with components representing the sum and difference of the identical fundamental solutions of the NSE with shifted arguments. The obtained solutions cover the entire range of variation of boundary conditions, allowing the optimisation of the conversion efficiency in any particular situation.**

***Keywords***: *quadratic nonlinearity, cascade frequency conversion, effective cubic nonlinearity, stationary nonlinear Schrödinger equation, multicomponent cnoidal wave.*

## 1. Introduction

Cnoidal waves (CWs) are self-consistent periodic solutions of nonlinear differential equations of the second and higher orders {nonlinear Schrödinger equation (NSE), Korteweg-de Vries (KdV), sine-Gordon (SG), and other equations [1 – 7]} and are in fact the modes of the corresponding nonlinear problems. When CWs contain several components, we are dealing with multicomponent CWs (MCWs). The term MCW is used in nonlinear hydrodynamics [1, 8] and plasma physics [2, 9], in the description of the packets of electronic wave functions (excitons, biexcitons, superconducting pairs, etc.), and in the physics of one-dimensional chains (conjugated polymers) [10] and two-dimensional planes (ferromagnetics and high-temperature semiconductors) [11]. The concept of MCWs in optics is also quite universal because equations of this type usually appear when the lowest terms in the expansion of a nonlinear polarisation wave are taken into account. Multicomponent CWs are the solutions of one-dimensional problems on the dispersionless propagation of pulse trains in optical fibres [3 – 6, 12] and on the parametric generation in the synchronous pumping regime [13], and of two-dimensional problems on the diffractionless propagation of wave fronts with a special periodic transverse structure in photorefractive crystals [7, 14] and crystals with quadratic nonlinearity [15].

It was shown in [16] that the NSE solutions in the form of MCWs play a key role in a classical problem of nonlinear optics – the description of parametric up and down frequency conversion in quadratically nonlinear media [17]. It was found that the exact analytic solution of the problem of stationary interaction between three modes with frequencies $\omega_{1-3}$ can be obtained by using a new approach of increasing the order of a system of truncated nonlinear equations. In this case, the problem is reduced to the solution of three independent NSEs, each of them being coupled with two others only via boundary conditions and describing a complex CW formed from quadrature components. The possibility of such a reduction of the initial problem was interpreted as the passage to the description of the result of competition of processes of merging ($\omega_1 + \omega_2 \rightarrow \omega_3$) and decay ($\omega_3 \rightarrow \omega_1 + \omega_2$) of photons proceeding on a quadratic nonlinearity by means of the effective cascade cubic Kerr nonlinearity [18].

By using the approach similar to [16], we show below that, when wave mismatches can be neglected (quasi-phase matching), the parametric interaction of four modes during cascade frequency conversion on a quadratic nonlinearity also can be described in terms of the effective cubic nonlinearity. In this case, the initial problem is reduced to a standard system of two coupled NSEs with respect to the complex amplitudes of the waves involved in two nonlinear processes [14, 19]. It is also shown that this system can be transformed to two identical independent equations, which determines its solution in the unusual form of the sum and difference of two identical solutions of the same NSE with shifted arguments. Due to a complete overlap of the range of possible variations in the boundary conditions, the analytic solutions obtained in this way provide the possibility of optimisation of the conversion efficiency in any particular situation.

## 2. Cascade frequency conversion and effective cubic nonlinearity

Consider the parametric interaction of four (the subscript $i = 1 - 4$) plane collinear monochromatic waves – modes in a quadratically nonlinear medium. Similarly to [16], we assume that the modes have frequencies $\omega_1$, $\omega_2 = \omega_1$,

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$\omega_3 = \omega_1 + \omega_2 = 2\omega_1$ and $\omega_4 = \omega_1 + \omega_3 = 3\omega_1$, wave vectors $\boldsymbol{k}_{1-4}$, and complex amplitudes $A_{1-4}$. Let us assume that conditions for nonlinear processes of two types $\omega_1 + \omega_{2,3} \to \omega_{3,4}$ with the wave mismatches $\Delta\boldsymbol{k}_{1,2} = \boldsymbol{k}_1 + \boldsymbol{k}_{2,3} - \boldsymbol{k}_{3,4}$ and nonlinear coupling constants $\beta_{1,2}$, respectively, are realised in the medium. By assuming that the nonlinearity is not of the resonance type and directing the $z$ axis along $\boldsymbol{k}_{1-4}$, we write the system of truncated equations describing the interaction of the modes in the form

$$\frac{\partial A_1}{\partial z} = -\mathrm{i}\beta_1 A_2^* A_3 \exp(-\mathrm{i}\Delta k_1 z) - \mathrm{i}\beta_2 A_3^* A_4 \exp(-\mathrm{i}\Delta k_2 z), \quad (1a)$$

$$\frac{\partial A_2}{\partial z} = -\mathrm{i}\beta_1 A_1^* A_3 \exp(-\mathrm{i}\Delta k_1 z), \quad (1b)$$

$$\frac{\partial A_3}{\partial z} = -\mathrm{i}2\beta_1 A_1 A_2 \exp(\mathrm{i}\Delta k_1 z) - \mathrm{i}2\beta_2 A_1^* A_4 \exp(-\mathrm{i}\Delta k_2 z), \quad (1c)$$

$$\frac{\partial A_4}{\partial z} = -\mathrm{i}3\beta_2 A_1 A_3 \exp(\mathrm{i}\Delta k_2 z). \quad (1d)$$

It is easy to verify that, although system (1) has five second-order integrals $J_{0-4} = \mathrm{const}$, which correspond to the law of conservation of energy flux

$$J_0 = I_1 + I_2 + I_3 + I_4 \quad (2)$$

and the Manley–Rowe relations

$$J_1 = I_1 - 2I_2 - \tfrac{1}{2}I_3, \quad J_2 = I_1 - I_2 + \tfrac{1}{3}I_4,$$
$$J_3 = I_1 + \tfrac{1}{2}I_3 + \tfrac{2}{3}I_4, \quad J_4 = I_2 + \tfrac{1}{2}I_3 + \tfrac{1}{3}I_4, \quad (3)$$

only two of them are independent (here, $I_i = A_i A_i^*$ are proportional to the intensities of waves). Therefore, we can write, for example, that

$$I_2 - I_{20} = \tfrac{1}{2}(I_1 - I_{10}) - \tfrac{1}{4}(I_3 - I_{30}),$$
$$I_4 - I_{40} = -\tfrac{3}{2}(I_1 - I_{10}) - \tfrac{3}{4}(I_3 - I_{30}) \quad (4)$$

($I_{i0} = A_i A_i^*|_{z=0}$).

Following the approach used in [16], we make the change of variables

$$A_i(z) = \tilde{A}_i(z)\exp(-\mathrm{i}\alpha_i z) \quad (5)$$

and choose constants $\alpha_i$ providing the fulfilment of conditions

$$\Delta\alpha_{1,2} = \alpha_1 + \alpha_{2,3} - \alpha_{3,4} = \Delta k_{1,2}. \quad (6)$$

By substituting (5) into (1) and taking (6) into account, we obtain

$$\frac{\partial \tilde{A}_1}{\partial z} - \mathrm{i}\alpha_1\tilde{A}_1 = -\mathrm{i}\beta_1\tilde{A}_2^*\tilde{A}_3 - \mathrm{i}\beta_2\tilde{A}_3^*\tilde{A}_4, \quad (7a)$$

$$\frac{\partial \tilde{A}_2}{\partial z} - \mathrm{i}\alpha_2\tilde{A}_2 = -\mathrm{i}\beta_1\tilde{A}_1^*\tilde{A}_3, \quad (7b)$$

$$\frac{\partial \tilde{A}_3}{\partial z} - \mathrm{i}\alpha_3\tilde{A}_3 = -\mathrm{i}2\beta_1\tilde{A}_1\tilde{A}_2 - \mathrm{i}2\beta_2\tilde{A}_1^*\tilde{A}_4, \quad (7c)$$

$$\frac{\partial \tilde{A}_4}{\partial z} - \mathrm{i}\alpha_4\tilde{A}_4 = -\mathrm{i}3\beta_2\tilde{A}_1\tilde{A}_3. \quad (7d)$$

Let us construct now the functional $H$ so that system (7) would follow from relations

$$\frac{\partial \tilde{A}_i}{\partial z} = -\mathrm{i}\frac{\omega_i}{\omega_1}\frac{\partial H}{\partial \tilde{A}_i^*}. \quad (8)$$

This gives the expression

$$H = \beta_1\tilde{A}_1\tilde{A}_2\tilde{A}_3^* + \beta_1\tilde{A}_1^*\tilde{A}_2^*\tilde{A}_3 + \beta_2\tilde{A}_1\tilde{A}_3\tilde{A}_4^* + \beta_2\tilde{A}_1^*\tilde{A}_3^*\tilde{A}_4$$
$$- \alpha_1\tilde{A}_1\tilde{A}_1^* - \alpha_2\tilde{A}_2\tilde{A}_2^* - \tfrac{1}{2}\alpha_3\tilde{A}_3\tilde{A}_3^* - \tfrac{1}{3}\alpha_4\tilde{A}_4\tilde{A}_4^* \quad (9)$$

for the functional, which for $\Delta k_{1,2} = 0$ ($\alpha_{1-4} = 0$) represents the part of the Hamiltonian describing the interaction of the field with the medium, i.e. the time-averaged free energy density [20]. By differentiating (9) and substituting (7) into the result obtained, it is easy to verify that $\partial H/\partial z \equiv 0$ and, therefore, $H = H_0 = \mathrm{const}$ is another integral of system (7). Note that, after passing to real variables, i.e. after introducing phases $\varphi_i$ with the help of expressions

$$\tilde{A}_i = \sqrt{I_i}\exp(\mathrm{i}\varphi_i), \quad (10)$$

functional (9) can be rewritten in the form

$$H = 2\beta_1\sqrt{I_1 I_2 I_3}\cos\Delta\varphi_1 + 2\beta_2\sqrt{I_1 I_3 I_4}\cos\Delta\varphi_2$$
$$- \alpha_1 I_1 - \alpha_2 I_2 - \tfrac{1}{2}\alpha_3 I_3 - \tfrac{1}{3}\alpha_4 I_4, \quad (11)$$

where $\Delta\varphi_{1,2} = \varphi_1 + \varphi_{2,3} - \varphi_{3,4}$. As a result, the equations will take the form known from [20]

$$\frac{\partial I_i}{\partial z} = \frac{\omega_i}{\omega_1}\frac{\partial H}{\partial \varphi_i}, \quad \frac{\partial \varphi_i}{\partial z} = -\frac{\omega_i}{\omega_1}\frac{\partial H}{\partial I_i}. \quad (12)$$

Following the approach used in [16], we pass from (7) to a system of second-order equations. By differentiating (7) and excluding the first derivatives, taking (4) into account, we obtain the system of equations

$$\frac{\partial^2 \tilde{A}_1}{\partial z^2} = -(\beta_1^2 + 3\beta_2^2)|\tilde{A}_1|^2\tilde{A}_1 + \frac{3}{2}(\beta_1^2 - 3\beta_2^2)|\tilde{A}_3|^2\tilde{A}_1$$
$$+ (\beta_1^2 J_1 + 3\beta_2^2 J_3 - \alpha_1^2)\tilde{A}_1 + (\alpha_1 - \alpha_2 + \alpha_3)\beta_1\tilde{A}_2^*\tilde{A}_3$$
$$+ (\alpha_1 - \alpha_3 + \alpha_4)\beta_2\tilde{A}_3^*\tilde{A}_4, \quad (13a)$$

$$\frac{\partial^2 \tilde{A}_2}{\partial z^2} = -4\beta_1^2|\tilde{A}_2|^2\tilde{A}_2 - \beta_1^2(2I_{10} - 4I_{20} - I_{30})\tilde{A}_2$$
$$+ \beta_1\beta_2\tilde{A}_3\tilde{A}_3\tilde{A}_4^* - 2\beta_1\beta_2\tilde{A}_1^*\tilde{A}_1^*\tilde{A}_4$$
$$-(\alpha_1 - \alpha_2 - \alpha_3)\beta_1\tilde{A}_1^*\tilde{A}_3 - \alpha_2^2\tilde{A}_2, \quad (13b)$$

$$\frac{\partial^2 \tilde{A}_3}{\partial z^2} = -3(\beta_1^2 + 3\beta_2^2)|\tilde{A}_1|^2\tilde{A}_3 + \frac{1}{2}(\beta_1^2 - 3\beta_2^2)|\tilde{A}_3|^2\tilde{A}_3$$
$$+ (\beta_1^2 J_1 + 3\beta_2^2 J_3 - \alpha_3^2)\tilde{A}_3 + 2(\alpha_1 + \alpha_2 + \alpha_3)\beta_1\tilde{A}_1\tilde{A}_2$$
$$- 2(\alpha_1 - \alpha_3 - \alpha_4)\beta_2\tilde{A}_1^*\tilde{A}_4, \quad (13c)$$

$$\frac{\partial^2 \tilde{A}_4}{\partial z^2} = 4\beta_2^2|\tilde{A}_4|^2\tilde{A}_4 - \beta_2^2(6I_{10} + 3I_{30} + 4I_{40})\tilde{A}_4$$
$$- 6\beta_1\beta_2\tilde{A}_1\tilde{A}_1\tilde{A}_2 - 3\beta_1\beta_2\tilde{A}_2^*\tilde{A}_3\tilde{A}_3$$

$$+3(\alpha_1+\alpha_2+\alpha_4)\beta_2\tilde{A}_1\tilde{A}_3-\alpha_4^2\tilde{A}_4. \qquad (13d)$$

One can easily see that only for $\Delta k_{1,2}=0$ Eqns (13a) and (13c) can be reduced to a closed system of two coupled NSEs for the field amplitudes $A_{1,3}$, by describing their interaction via the effective cubic nonlinearity. For $\Delta k_{1,2}\neq 0$, the right-hand side of equations contains terms proportional to the products $\tilde{A}_i\tilde{A}_j$ and $\tilde{A}_i\tilde{A}_j^*$, and, therefore, such a reduced description cannot be used. Note that Eqns (13b) and (13d) also can be transformed to a similar closed system, however, under more severe restrictions. Apart from the condition $\Delta k_{1,2}=0$, it is necessary to require that $A_{1,3}$ would be real.

## 3. Quasi-synchronous interaction

The condition $\Delta k_{1,2}=0$ cannot be fulfilled in the general case due to dispersion [17]. Because of this, the so-called phase-matching conditions are provided to realise cascade processes [21]. This can be achieved by producing, for example, a periodic structure in a nonlinear medium, in which the sign of coupling constants $\beta_{1,2}$ periodically changes [22], $\beta_{1,2}\to\beta_{1,2}g(z)$. Here, $g(z)$ is a sign alternating function with a spatial period $\Lambda=(2m_{1,2}+1)(2\pi/\Delta k_{1,2}$ specified by the coherence lengths of two nonlinear processes, and $m_{1,2}$ are positive integers. By expanding

$$g(z)=\sum_{m=-\infty}^{m=+\infty} g_m\exp\left(\mathrm{i}2\pi m\frac{z}{\Lambda}\right)$$

to a Fourier series and taking into account the amplitudes of four synchronous modes, we obtain after averaging (1) the system

$$\frac{\partial A_1}{\partial z}=-\mathrm{i}\gamma_1A_2^*A_3-\mathrm{i}\gamma_2A_3^*A_4, \qquad (14a)$$

$$\frac{\partial A_2}{\partial z}=-\mathrm{i}\gamma_1A_1^*A_3, \qquad (14b)$$

$$\frac{\partial A_3}{\partial z}=-\mathrm{i}2\gamma_1^*A_1A_2-\mathrm{i}2\gamma_2A_1^*A_4, \qquad (14c)$$

$$\frac{\partial A_4}{\partial z}=-\mathrm{i}3\gamma_2^*A_1A_3. \qquad (14d)$$

Here, $\gamma_{1,2}=\langle\beta_{1,2}\exp(-\mathrm{i}\Delta k_{1,2}z)\rangle_z$ are averaged and (in the general case) complex nonlinear coupling constants for processes $\omega_1+\omega_{2,3}\to\omega_{3,4}$, respectively.

After averaging, the passage from (14) to second-order equations gives the required closed system of two nonlinear equations for the amplitudes $A_{1,3}$ waves in the form

$$\frac{\partial^2 A_1}{\partial z^2}=-G_+|A_1|^2A_1+\frac{3}{2}G_-|A_3|^2A_1$$
$$+(|\gamma_1|^2J_1+3|\gamma_2|^2J_3)A_1, \qquad (15a)$$

$$\frac{\partial^2 A_3}{\partial z^2}=-3G_+|A_1|^2A_3+\frac{1}{2}G_-|A_3|^2A_3$$
$$+(|\gamma_1|^2J_1+3|\gamma_2|^2J_3)A_3, \qquad (15b)$$

with the boundary conditions

$$A_1|_{z=0}=A_{10},\ \left.\frac{\partial A_1}{\partial z}\right|_{z=0}=-\mathrm{i}\gamma_1A_{20}^*A_{30}-\mathrm{i}\gamma_2A_{30}^*A_{40}, \qquad (16a)$$

$$A_3|_{z=0}=A_{30},\ \left.\frac{\partial A_3}{\partial z}\right|_{z=0}=-\mathrm{i}2\gamma_1^*A_{10}A_{20}-\mathrm{i}2\gamma_2A_{10}^*A_{40}, \qquad (16b)$$

where $G_\pm=|\gamma_1|^2\pm 3|\gamma_2|^2$. In this case, although equations for the wave amplitudes $A_{2,4}$ are not reduced to the analogous system [see (13)], their intensities can be found from relations (4). Note that analysis of the solutions of systems of nonlinear equations of this type is a subject of recent extensive studies [12, 19, 23].

Following [16], we consider now the moduli and phases of the required solutions:

$$A_j(z)=X_j(z)\exp[\mathrm{i}\varphi_j(z)]. \qquad (17)$$

By substituting the result of differentiating of (17) into (15) and separating the real and imaginary parts, we obtain the system of equations

$$\frac{\partial^2 X_1}{\partial z^2}-X_1\left(\frac{\partial\varphi_1}{\partial z}\right)^2=-G_+X_1^3+\frac{3}{2}G_-X_3^2X_1$$
$$+(|\gamma_1|^2J_1+3|\gamma_2|^2J_3)X_1, \qquad (18a)$$

$$2\frac{\partial X_1}{\partial z}\frac{\partial\varphi_1}{\partial z}+X_1\frac{\partial^2\varphi_1}{\partial z^2}=0, \qquad (18b)$$

$$\frac{\partial^2 X_3}{\partial z^2}-X_3\left(\frac{\partial\varphi_3}{\partial z}\right)^2=-3G_+X_1^2X_3+\frac{1}{2}G_-X_3^3$$
$$+(|\gamma_1|^2J_1+3|\gamma_2|^2J_3)X_3, \qquad (18c)$$

$$2\frac{\partial X_3}{\partial z}\frac{\partial\varphi_3}{\partial z}+X_3\frac{\partial^2\varphi_3}{\partial z^2}=0. \qquad (18d)$$

Because the solutions for which $X_{1,3}(z)=0$ are not of interest for us, two known integrals [16] for phases $\varphi_{1,3}$ follow from (18b) and (18d). Moreover, it is easy to show that these integrals are also not independent and can be expressed in terms of $H$:

$$X_1^2\frac{\partial\varphi_1}{\partial z}=I_{10}\varphi_{10}'=-\frac{1}{2}H,\ \ X_3^2\frac{\partial\varphi_3}{\partial z}=I_{30}\varphi_{30}'=-H, \qquad (19)$$

where

$$H=\gamma_1^*A_1A_2A_3^*+\gamma_1A_1^*A_2^*A_3+\gamma_2^*A_1A_3A_4^*$$
$$+\gamma_2A_1^*A_3^*A_4=\text{const}. \qquad (20)$$

Hereafter, the notations $\varphi_{1,3}|_{z=0}=\varphi_{10,30}$, $\partial\varphi_{1,3}/\partial z|_{z=0}=\varphi_{10,30}'$, and $X_i^2|_{z=0}=I_{i0}$ are used.

As in the interaction of three modes [16], it follows from (18b), (18d), and (19) that, if at least one point $z_0$ exists on the $z$ axis at which $X_{1,3}|_{z=z_0}=0$ and $\partial X_{1,3}/\partial z|_{z=z_0}\neq 0$, then $\partial\varphi_{1,3}/\partial z\equiv 0$ at all points for which $X_{1,3}(z)\neq 0$. In these situations the phases $\varphi_{1,3}$ can change on the $z$ axis only abruptly, which can be taken into account by assuming that $X_{1,3}(z)\neq|A_{1,3}(z)|$ and can be negative. If this is not the case, $\varphi_{1,3}(z)$ can be found by integrating (19):

$$\varphi_1(z)=\varphi_{10}-\frac{1}{2}H\int_0^z X_1^{-2}(z')\mathrm{d}z',$$
$$\varphi_3(z)=\varphi_{30}-H\int_0^z X_3^{-2}(z')\mathrm{d}z'. \qquad (21)$$

## 4. Analytic solutions of the problem

Thus, we have shown that the initial problem is reduced to the solution of a closed system of two ordinary differential equations describing the interaction of the waves $A_{1,3}$ in terms of the effective cubic nonlinearity and after the substitution

$$z = \tilde{z}/\sqrt{G_+} \tag{22}$$

can be represented in the form

$$\frac{\partial^2 X_1}{\partial \tilde{z}^2} + \frac{1}{4G_+}\frac{H^2}{X_1^3} = -X_1^3 + \frac{3}{2}\frac{G_-}{G_+}X_3^2 X_1 + J_{13}X_1, \tag{23a}$$

$$\frac{\partial^2 X_3}{\partial \tilde{z}^2} + \frac{1}{G_+}\frac{H^2}{X_3^3} = -3X_1^2 X_3 + \frac{1}{2}\frac{G_-}{G_+}X_3^3 + J_{13}X_3, \tag{23b}$$

where

$$J_{13} = \frac{|\gamma_1|^2 J_1 + 3|\gamma_2|^2 J_3}{|\gamma_1|^2 + 3|\gamma_2|^2}.$$

Consider below only situations when at least one point $z_0$ exists on the $z$ axis at which the amplitude $A_{1,3}$ of at least one of the waves vanishes (one of these two waves is completely depleted or is absent in the input plane $z = 0$), and therefore, $H = 0$ and $\varphi_{1,3}(z) = \varphi_{10,30}$ (see above).

Note at once that in a particular case $|\gamma_1|^2 = 3|\gamma_2|^2$, the obtained system is reduced to

$$\frac{\partial^2 X_1}{\partial \tilde{z}^2} = -X_1^3 + \frac{1}{2}(J_1 + J_3)X_1, \tag{24a}$$

$$\frac{\partial^2 X_3}{\partial \tilde{z}^2} = -3X_1^2 X_3 + \frac{1}{2}(J_1 + J_3)X_3, \tag{24b}$$

i.e. to the well-known problem of the independent periodic variation of the amplitude $X_1$ in a medium with the Kerr-type nonlinearity [14]. Nevertheless, the oscillation period of $X_1(\tilde{z})$ depends on the initial intensities of all other waves (on the sum of integrals $J_1 + J_3$), while the dependence $X_3(\tilde{z})$ is determined by solving the second-order Lame equation [24].

The solutions of (24a) in the standard form [14] for the nonlinearity of this type are described by the expressions

$$X_1 = \sqrt{I_{10}}\,\mathrm{cn}(\beta\tilde{z}, k), \tag{25a}$$

$$X_3 = \sqrt{I_{3M}}\,\mathrm{sn}(\beta\tilde{z}, k)\mathrm{dn}(\beta\tilde{z}, k) \tag{25b}$$

for $\beta^2 = I_{20} - \frac{1}{3}I_{40}$, $k^2 = \frac{1}{2}I_{10}(I_{20} - \frac{1}{3}I_{40})^{-1}$, $2(I_{20} - \frac{1}{3}I_{40}) \geqslant I_{10} \geqslant 0$, and

$$X_1 = \sqrt{I_{10}}\,\mathrm{dn}(\beta\tilde{z}, k), \tag{26a}$$

$$X_3 = \sqrt{I_{3M}}\,\mathrm{sn}(\beta\tilde{z}, k)\mathrm{cn}(\beta\tilde{z}, k) \tag{26b}$$

for $\beta^2 = \frac{1}{2}I_{10}$, $k^2 = 2I_{10}^{-1}(I_{20} - \frac{1}{3}I_{40})$, and $I_{10} \geqslant 2(I_{20} - \frac{1}{3}I_{40})$. Here, $k$ is the modulus of the elliptic Jacoby functions $\mathrm{sn}(z, k)$, $\mathrm{cn}(z, k)$, and $\mathrm{dn}(z, k)$ [25], and the parameter $I_{3M}$ is determined by the boundary conditions and depends not only on the initial intensities $I_{i0}$ of all the waves but also on relation between their phases $\varphi_{i0}$ [see (16b)]. Note that all the other solutions of system (24), including situations when $I_{20} - \frac{1}{3}I_{40} \leqslant 0$, are reduced to a simple translation of solutions (25) and (26) along the $\tilde{z}$ axis. Here and below, the expressions for dependences $I_{2,4}(z)$ are not written because they can be determined from relations (4).

To analyse situations when $|\gamma_1|^2 \neq 3|\gamma_2|^2$, we will pass to the normalised variables

$$X_1 = \tilde{X}_1, \quad X_3 = \sqrt{2|G_+/G_-|}\,\tilde{X}_3, \tag{27}$$

in which system (23) has the form

$$\frac{\partial^2 \tilde{X}_1}{\partial \tilde{z}^2} = -\tilde{X}_1^3 \pm 3\tilde{X}_3^2\tilde{X}_1 + J_{13}\tilde{X}_1, \tag{28a}$$

$$\frac{\partial^2 \tilde{X}_3}{\partial \tilde{z}^2} = -3\tilde{X}_1^2\tilde{X}_3 \pm \tilde{X}_3^3 + J_{13}\tilde{X}_3. \tag{28b}$$

Here, the signs '$\pm$' correspond to cases $|\gamma_1|^2 > 3|\gamma_2|^2$ and $|\gamma_1|^2 < 3|\gamma_2|^2$, respectively. Note that the integrability and the type of solutions of systems of this type are determined by the relation between coefficients at nonlinear terms [25].

The case $|\gamma_1|^2 < 3|\gamma_2|^2$ is simple to analyse because it is known [25] that the next change of variables

$$\tilde{Y}_\pm = \tilde{X}_1 \pm \tilde{X}_3 \quad \text{or} \quad \tilde{Y}_\pm = \tilde{X}_3 \pm \tilde{X}_1 \tag{29}$$

in this situation separates the variables, which reduces the system of equations (28) to two independent NSEs with the nonlinearity of the focusing type

$$\frac{\partial^2 \tilde{Y}_\pm}{\partial \tilde{z}^2} = -\tilde{Y}_\pm^3 + J_{13}\tilde{Y}_\pm. \tag{30}$$

It is easy to verify that both equations are related to each other only via the boundary conditions and have the same proportionality coefficients in linear terms. The identity of these coefficients excludes the use of standard solutions for systems of two NSEs, in which $Y_\pm$ are proportional to the different fundamental solutions $\mathrm{cn}(z, k)$ and $\mathrm{dn}(z, k)$ of the Lame equation [14]. Because both these solutions become degenerate only for $k = 1$, when both functions pass to $\cosh z$, we obtain that either $\tilde{X}_1 \equiv 0$ or $\tilde{X}_3 \equiv 0$, which corresponds to the parametric bleaching regime, when $I_{1-4} = \mathrm{const}$.

However, there also exist two other possibilities. First, the solutions of two equations in (30) can be proportional to the same elliptic function but shifted with respect to each other along the $\tilde{z}$ axis, i.e.

$$\tilde{Y}_\pm = A\,\mathrm{cn}(\beta\tilde{z} \pm \beta\tilde{z}_0, k) \quad \text{or} \quad \tilde{Y}_\pm = A\,\mathrm{dn}(\beta\tilde{z} \pm \beta\tilde{z}_0, k). \tag{31}$$

Here, $\tilde{z}_0$ is the parameter characterising the shift value, which is assumed symmetrical with respect to the point $\tilde{z}_0 = 0$ for functions $\tilde{Y}_\pm$, respectively. This corresponds to the presence of the extrema of intensities $I_{1,3}$ in the input plane. This possibility determines the four nontrivial solutions of system (28) for $|\gamma_1|^2 < 3|\gamma_2|^2$:

$$\tilde{X}_{1,3} = A\,\mathrm{cn}(\beta\tilde{z}_0, k)\frac{\mathrm{cn}(\beta\tilde{z}, k)}{1 - k^2\mathrm{sn}^2(\beta\tilde{z}_0, k)\mathrm{sn}^2(\beta\tilde{z}, k)}, \tag{32a}$$

$$\tilde{X}_{3,1} = -A\,\mathrm{sn}(\beta\tilde{z}_0, k)\mathrm{dn}(\beta\tilde{z}_0, k)\frac{\mathrm{sn}(\beta\tilde{z}, k)\mathrm{dn}(\beta\tilde{z}, k)}{1 - k^2\mathrm{sn}^2(\beta\tilde{z}_0, k)\mathrm{sn}^2(\beta\tilde{z}, k)} \tag{32b}$$

for $\beta = A^2 - J_{13}$, $k^2 = \frac{1}{2}A^2(A^2 - J_{13})^{-1}$, and $A^2 \geqslant \max(J_{13}, 2J_{13})$ and

$$\tilde{X}_{1,3} = A\,\mathrm{dn}(\beta\tilde{z}_0,k)\frac{\mathrm{dn}(\beta\tilde{z},k)}{1-k^2\mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k)}, \tag{33a}$$

$$\tilde{X}_{3,1} = -k^2A\,\mathrm{sn}(\beta\tilde{z}_0,k)\mathrm{cn}(\beta\tilde{z}_0,k) \times \frac{\mathrm{sn}(\beta\tilde{z},k)\mathrm{cn}(\beta\tilde{z},k)}{1-k^2\mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k)} \tag{33b}$$

for $\beta = \frac{1}{2}A^2$, $k^2 = 2(A^2 - J_{13})A^{-2}$, and $2J_{13} \geqslant A^2 \geqslant J_{13}$. Here, the values of constants $A$ and $\tilde{z}_0$ should be chosen to satisfy boundary conditions (16), which give the domains of existence of solutions (32) and (33). Note that after the return to initial variables $X_{1,3}$, the symmetry of expressions (32) and (33) with respect to the permutation of subscripts 1 and 3 is violated due to the renormalisation of the amplitude $\tilde{X}_3$.

Second, the solution of one of the equations in system (30) can be a constant, while the solution of the second one can be proportional to one of the fundamental solutions $\mathrm{cn}(z,k)$ and $\mathrm{dn}(z,k)$ of the first-order Lame equation, i.e.

$$\tilde{Y}_\pm = A = \mathrm{const} \tag{34a}$$

and

$$\tilde{Y}_\mp = B\,\mathrm{cn}(\beta\tilde{z},k) \quad \text{or} \quad \tilde{Y}_\mp = B\,\mathrm{dn}(\beta\tilde{z},k). \tag{34b}$$

This possibility determines the four additional solutions of system (28) for $|\gamma_1|^2 < 3|\gamma_2|^2$:

$$\tilde{X}_{1,3} = \frac{1}{2}\left[\sqrt{J_{13}} \pm B\,\mathrm{cn}(\beta\tilde{z},k)\right] \text{ or } \tilde{X}_{3,1} = \frac{1}{2}\left[\sqrt{J_{13}} \pm B\,\mathrm{cn}(\beta\tilde{z},k)\right] \tag{35}$$

for $\beta^2 = B^2 - J_{13}$, $k^2 = \frac{1}{2}B^2(B^2 - J_{13})^{-1}$, and $B^2 \geqslant 2J_{13} \geqslant 0$ or

$$\tilde{X}_{1,3} = \frac{1}{2}\left[\sqrt{J_{13}} \pm B\,\mathrm{dn}(\beta\tilde{z},k)\right] \text{ or } \tilde{X}_{3,1} = \frac{1}{2}\left[\sqrt{J_{13}} \pm B\,\mathrm{dn}(\beta\tilde{z},k)\right] \tag{36}$$

for $\beta^2 = \frac{1}{2}B^2$, $k^2 = 2(B^2 - J_{13})B^{-2}$, and $2J_{13} \geqslant B^2 \geqslant J_{13} \geqslant 0$. Here, the value of $B$ also should be chosen to satisfy boundary conditions (16). Note that, as before, the return to the initial variables $X_{1,3}$ violates the symmetry of expressions (35) and (36) with respect to the permutation of subscripts due to the renormalisation of the amplitude $\tilde{X}_3$.

In the case when $|\gamma_1|^2 > 3|\gamma_2|^2$, the approach described above also can be applied. To use it, we first make the formal substitution $\tilde{z} = \mathrm{i}\utilde{z}$ [26] and will seek the solution in classes of functions for which either

$$\tilde{X}_1(\mathrm{i}\utilde{z}) = \mathrm{i}\utilde{X}_1(\mathrm{i}\utilde{z}),\ \tilde{X}_3(\mathrm{i}\utilde{z}) = \utilde{X}_3(\utilde{z}) \tag{37a}$$

or

$$\tilde{X}_1(\mathrm{i}\utilde{z}) = \utilde{X}_1(\utilde{z}),\ \ \tilde{X}_3(\mathrm{i}\utilde{z}) = \mathrm{i}\utilde{X}_3(\utilde{z}), \tag{37b}$$

where $\utilde{X}_{1,3}(\utilde{z})$ and $\tilde{X}_{1,3}(\tilde{z})$ are real. Note that elliptic functions $\mathrm{sn}(z,k)$, $\mathrm{cn}(z,k)$, and $\mathrm{dn}(z,k)$, which satisfy well-known relations $\mathrm{sn}(\mathrm{i}z,k) = \mathrm{i}\,\mathrm{sn}(z,k')\,\mathrm{cn}^{-1}(z,k')$, $\mathrm{cn}(\mathrm{i}\,z,k) = \mathrm{cn}^{-1}(z,k')$ and $\mathrm{dn}(\mathrm{i}\,z,k) = \mathrm{dn}(z,k')\,\mathrm{cn}^{-1}(z,k')$, where $k' = (1-k^2)^{1/2}$ [24], belong to these two classes. After this substitution, system (28) can be rewritten in one of the two forms corresponding to the chosen class of solutions:

$$\frac{\partial^2\utilde{X}_1}{\partial\utilde{z}^2} = -\utilde{X}_1^3 - 3\utilde{X}_3^2\utilde{X}_1 - J_{13}\utilde{X}_1, \tag{38a}$$

$$\frac{\partial^2\utilde{X}_3}{\partial\utilde{z}^2} = -3\utilde{X}_1^2\utilde{X}_3 - \utilde{X}_3^3 - J_{13}\utilde{X}_3 \tag{38b}$$

or

$$\frac{\partial^2\utilde{X}_1}{\partial\utilde{z}^2} = \utilde{X}_1^3 + 3\utilde{X}_3^2\utilde{X}_1 - J_{13}\utilde{X}_1, \tag{39a}$$

$$\frac{\partial^2\utilde{X}_3}{\partial\utilde{z}^2} = 3\utilde{X}_1^2\utilde{X}_3 + \utilde{X}_3^3 - J_{13}\utilde{X}_3. \tag{39b}$$

It is easy to see that now, after the substitutions

$$\utilde{Y}_\pm = \utilde{X}_3 \pm \utilde{X}_1, \tag{40}$$

which are analogous to (29), we obtain the two possible pairs of independent NSEs,

$$\frac{\partial^2\utilde{Y}_\pm}{\partial\utilde{z}^2} = -\utilde{Y}_\pm^3 - J_{13}\utilde{Y}_\pm \tag{41a}$$

or

$$\frac{\partial^2\utilde{Y}_\pm}{\partial\utilde{z}^2} = \utilde{Y}_\pm^3 - J_{13}\utilde{Y}_\pm \tag{41b}$$

Equations in pairs (41a) and (41b) are again coupled with each other only via the boundary conditions and have identical proportionality coefficients in linear terms. However, these pairs correspond now to situations with the nonlinearities of the focusing (41a) and defocusing (41b) types. For the same reason, the solutions of equations in each pair should be proportional to the same elliptic function, but now, taking conditions (37) into account, the shift of their arguments should be imaginary (orthogonal to the $\tilde{z}$ axis):

$$\utilde{Y}_\pm = A\,\mathrm{cn}(\beta\utilde{z} \pm \mathrm{i}\beta\utilde{z}_0,k), \tag{42a}$$

$$\utilde{Y}_\pm = A\,\mathrm{dn}(\beta\utilde{z} \pm \mathrm{i}\beta\utilde{z}_0,k) \tag{42b}$$

or

$$\utilde{Y}_\pm = A\,\mathrm{sn}(\beta\utilde{z} \pm \mathrm{i}\beta\utilde{z}_0,k). \tag{42c}$$

Here, $\utilde{z}_0$ is a parameter characterising the shift of the argument of functions $\utilde{Y}_\pm$, which is assumed symmetric with respect to the location of the $\utilde{z}$ axis in the complex plane. The possibilities listed above determine the three nontrivial solutions of system (28) for $|\gamma_1|^2 > 3|\gamma_2|^2$:

$$\tilde{X}_1 = -A\,\mathrm{sn}(\beta\tilde{z}_0,k)\mathrm{dn}(\beta\tilde{z}_0,k) \times \frac{\mathrm{sn}(\beta\tilde{z},k')\mathrm{dn}(\beta\tilde{z},k')}{\mathrm{cn}^2(\beta\tilde{z},k') + k^2\mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k')}, \tag{43a}$$

$$\tilde{X}_3 = A\,\mathrm{cn}(\beta\tilde{z}_0,k)\frac{\mathrm{cn}(\beta\tilde{z},k')}{\mathrm{cn}^2(\beta\tilde{z},k') + k^2\mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k')} \tag{43b}$$

for $\beta^2 = A^2 + J_{13}$, $k^2 = \frac{1}{2}A^2(A^2 + J_{13})^{-1}$, where $A^2 \geqslant \max(-2J_{13}, 0)$;

$$\tilde{X}_1 = -k^2 A\,\mathrm{sn}(\beta\tilde{z}_0,k)\mathrm{cn}(\beta\tilde{z}_0,k) \times \frac{\mathrm{sn}(\beta\tilde{z},k')}{\mathrm{cn}^2(\beta\tilde{z},k') + k^2\mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k')}, \tag{44a}$$

$$\tilde{X}_3 = A\,\mathrm{dn}(\beta\tilde{z}_0,k)\frac{\mathrm{cn}(\beta\tilde{z},k')\mathrm{dn}(\beta\tilde{z},k')}{\mathrm{cn}^2(\beta\tilde{z},k') + k^2\mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k')} \tag{44b}$$

for $\beta^2 = \frac{1}{2}A^2$, $k^2 = 2A^{-2}(A^2 + J_{13})$, and $2|J_{13}| \geqslant A^2 \geqslant |J_{13}|$ for $J_{13} \leqslant 0$;

$$\tilde{X}_1 = A\,\mathrm{sn}(\beta\tilde{z}_0,k)\frac{\mathrm{dn}(\beta\tilde{z},k')}{\mathrm{cn}^2(\beta\tilde{z},k') + k^2\mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k')}, \tag{45a}$$

$$\tilde{X}_3 = -A\,\mathrm{cn}(\beta\tilde{z}_0,k)\mathrm{dn}(\beta\tilde{z}_0,k) \times \frac{\mathrm{sn}(\beta\tilde{z},k')\mathrm{cn}(\beta\tilde{z},k')}{\mathrm{cn}^2(\beta\tilde{z},k') + k^2\mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k')} \tag{45b}$$

for $\beta^2 = J_{13} - \frac{1}{2}A^2$, $k^2 = A^2(2J_{13} - A^2)^{-1}$, and $A^2 \leqslant J_{13}$ for $J_{13} \geqslant 0$.

However, these solutions do not exhaust all possible situations specified by the boundary conditions. The matter is that due to variations in $I_{10-40}$, the coefficient $J_{13}$ in (41) can become negative. In the case of a nonlinearity of a focusing type, solutions (43) and (44) correspond to situations $J_{13} < 0$. In the case of a nonlinearity of a defocusing type, the solution (45) of Eqns (41b) does not exist. At the same time, it is easy to verify that although the function

$$\frac{\mathrm{sn}(\mathrm{i}z,k)\mathrm{dn}(\mathrm{i}z,k)}{\mathrm{cn}(\mathrm{i}z,k)} = \mathrm{i}\frac{\mathrm{sn}(z,k')\mathrm{dn}(z,k')}{\mathrm{cn}(z,k')}$$

is not the fundamental solution of the Lame equation and has singularities on the $z$ axis, it also satisfies each of the equations (41b). In this case, the presence of singularities of this function due to the shifts of its arguments does not prevent a search for solutions in the form

$$\underset{\sim}{Y}_\pm = A\mathrm{sn}(\beta\underset{\sim}{z} \pm \mathrm{i}\beta\underset{\sim}{z}_0,k)\mathrm{dn}(\beta\underset{\sim}{z} \pm \mathrm{i}\beta\underset{\sim}{z}_0,k)[\mathrm{cn}(\beta\underset{\sim}{z} \pm \mathrm{i}\beta\underset{\sim}{z}_0,k)]^{-1}, \tag{46}$$

which leads immediately to the expressions

$$\tilde{X}_1(\tilde{z}) = A\,\mathrm{sn}(\beta\tilde{z}_0,k)\mathrm{cn}(\beta\tilde{z}_0,k)\mathrm{dn}(\beta\tilde{z}_0,k)\big[\mathrm{cn}^2(\beta\tilde{z},k') \times \mathrm{dn}^2(\beta\tilde{z},k') + k^2\mathrm{sn}^2(\beta\tilde{z},k')\big]\big[\mathrm{cn}^2(\beta\tilde{z}_0,k)\mathrm{cn}^2(\beta\tilde{z},k') + \mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{dn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k')\mathrm{dn}^2(\beta\tilde{z},k')\big]^{-1}, \tag{47a}$$

$$\tilde{X}_3(\tilde{z}) = A\big[\mathrm{dn}^2(\beta\tilde{z}_0,k) - k^2\mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{cn}^2(\beta\tilde{z}_0,k)\big] \times \mathrm{sn}(\beta\tilde{z},k')\mathrm{cn}(\beta\tilde{z},k')\mathrm{dn}(\beta\tilde{z},k')\big[\mathrm{cn}^2(\beta\tilde{z}_0,k)\mathrm{cn}^2(\beta\tilde{z},k') + \mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{dn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k')\mathrm{dn}^2(\beta\tilde{z},k')\big]^{-1} \tag{47b}$$

for $\beta^2 = \frac{1}{2}A^2$, $k^2 = \frac{1}{2}(A^2 + J_{13})A^{-2}$, and $A^2 \geqslant |J_{13}|$. As before, the values of constants $A$ and $\tilde{z}_0$ should be chosen to satisfy boundary conditions (16), which determines the domains of existence of solutions in forms (43)–(45) and (47). Note that in the case of $|\gamma_1|^2 > 3|\gamma_2|^2$, the symmetry of expressions for $\tilde{X}_{1,3}$ with respect to the permutation of subscripts 1 and 3 is violated from the outset by requirement (37), while solution (47) itself has no longer singularities on the $\tilde{X}_{1,3}$ axis due to the out-of-phase shifts of arguments $\underset{\sim}{Y}_\pm$.

The choice of the form in which to seek the solutions of the problem for $J_{13} < 0$ is ambiguous. Thus, it follows from analysis performed in [26] that the function $\mathrm{cn}(\mathrm{i}z,k) = \mathrm{cn}^{-1}(z,k')$ also satisfies Eqns (41b). Therefore, we can seek their solutions in the form

$$\underset{\sim}{Y}_\pm = A\,\mathrm{cn}^{-1}(\beta\underset{\sim}{z} \pm \mathrm{i}\beta\underset{\sim}{z}_0,k), \tag{48}$$

which immediately leads to the expressions,

$$\tilde{X}_1 = \big[\mathrm{cn}^2(\beta\tilde{z},k') + k^2\mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k')\big]A \times \mathrm{cn}(\beta\tilde{z}_0,k)\mathrm{cn}(\beta\tilde{z},k')\big[\mathrm{cn}^2(\beta\tilde{z}_0,k)\mathrm{cn}^2(\beta\tilde{z},k') + \mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{dn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k')\mathrm{dn}^2(\beta\tilde{z},k')\big]^{-1}, \tag{49a}$$

$$\tilde{X}_3 = \big[\mathrm{cn}^2(\beta\tilde{z},k') + k^2\mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{sn}^2(\beta\tilde{z},k')\big]A \times \mathrm{sn}(\beta\tilde{z}_0,k)\mathrm{dn}(\beta\tilde{z}_0,k)\mathrm{sn}(\beta\tilde{z},k')\mathrm{dn}(\beta\tilde{z},k') \times \big[\mathrm{cn}^2(\beta\tilde{z}_0,k)\mathrm{cn}^2(\beta\tilde{z},k') + \mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{dn}^2(\beta\tilde{z}_0,k) \times \mathrm{sn}^2(\beta\tilde{z},k')\mathrm{dn}^2(\beta\tilde{z},k')\big]^{-1} \tag{49b}$$

for $\beta^2 = A^2 - J_{13}$, $k^2 = \frac{1}{2}(A^2 - 2J_{13})(A^2 - J_{13})^{-1}$, $A^2 \geqslant \max(J_{13}, 2J_{13})$. The values of constants $A$ and $\tilde{z}_0$ should be again chosen to satisfy conditions (16). The symmetry of $\tilde{X}_{1,3}$ with respect to the permutation of subscripts 1 and 3 is still violated by condition (37), while solution (49) itself also has no singularities on the $z$ axis due to the shifts of arguments $\underset{\sim}{Y}_\pm$.

## 5. Conclusions

Based on the approach used in [16], we have shown that in the cases when wave mismatches can be neglected (quasi-phase matching), the parametric interaction of four modes during the cascade frequency conversion on a quadratic nonlinearity can be described in terms of the effective cubic nonlinearity. In this case, the initial problem is reduced to the solution of a system of two coupled NSEs for the complex amplitudes of the waves involved in all nonlinear processes [14, 19]. This system is completely integrable and can be split into two identical NSEs by a simple change of variables, which allows one to find its solutions in a quite unusual form as a sum and difference of the two identical solutions of NSEs for focusing or defocusing nonlinearities with shifted arguments. The analytic solutions obtained in this way cover the entire range of variations of boundary conditions, which allows one to analyse in detail the role of the latter and to optimise the conversion efficiency in each particular situation.

The analytic solutions obtained in this paper, could be, of course, also derived by other methods. Thus, we obtained the complete family of solutions similar to (25), (26), (32), (33), (45), (47) and also solutions (43) and (44) shifted by a quarter of the period along the $z$ axis for a particular case $I_{30} = 0$ and $\varphi_{i0} = \mathrm{const}$ in the form

$$\tilde{X}_1 = \frac{1}{k}A\,\mathrm{sn}(\beta\tilde{z}_0,k)\mathrm{dn}(\beta\tilde{z}_0,k) \times \frac{\mathrm{cn}(\beta\tilde{z},k')}{\mathrm{sn}^2(\beta\tilde{z},k') + \mathrm{sn}^2(\beta\tilde{z}_0,k)\mathrm{cn}^2(\beta\tilde{z},k')}, \tag{50a}$$

$$\tilde{X}_3 = \frac{1}{k} A\,\mathrm{cn}(\beta\tilde{z}_0, k) \frac{\mathrm{sn}(\beta\tilde{z}, k')\mathrm{dn}(\beta\tilde{z}, k')}{\mathrm{sn}^2(\beta\tilde{z}, k') + \mathrm{sn}^2(\beta\tilde{z}_0, k)\mathrm{cn}^2(\beta\tilde{z}, k')} \quad (50b)$$

for $\beta^2 = A^2 + J_{13}$, $k^2 = \frac{1}{2}A^2(A^2 + J_{13})^{-1}$, where $A^2 \geqslant \max(-2J_{13}, 0)$, and

$$\tilde{X}_1 = A\,\mathrm{sn}(\beta\tilde{z}_0, k)\mathrm{cn}(\beta\tilde{z}_0, k) \times \frac{\mathrm{cn}(\beta\tilde{z}, k')\mathrm{dn}(\beta\tilde{z}, k')}{\mathrm{sn}^2(\beta\tilde{z}, k') + \mathrm{sn}^2(\beta\tilde{z}_0, k)\mathrm{cn}^2(\beta\tilde{z}, k')}, \quad (51a)$$

$$\tilde{X}_3 = A\,\mathrm{dn}(\beta\tilde{z}_0, k) \frac{\mathrm{sn}(\beta\tilde{z}, k')}{\mathrm{sn}^2(\beta\tilde{z}, k') + \mathrm{sn}^2(\beta\tilde{z}_0, k)\mathrm{cn}^2(\beta\tilde{z}, k')} \quad (51b)$$

for $\beta^2 = \frac{1}{2}A^2$, $k^2 = 2A^{-2}(A^2 + J_{13})$, and $2|J_{13}| \geqslant A^2 \geqslant |J_{13}|$ for $J_{13} \leqslant 0$ by using the cumbersome traditional method [17] of the successive solution of a classical system of truncated first-order differential equations (14).

Note also that the analytic solutions of a system of two NSEs of type (28) have been obtained by us also probably for the first time, while the method for constructing particular solutions of systems of two NSEs in the form of a sum and difference of identical fundamental solutions with shifted arguments is quite universal and, as far as we know, have not been used so far. This method can be used in the cases when the problem to be solved admits the separation of variables [25], which takes place, for example, in the description of the self-consistent dispersionless propagation of the trains of orthogonally polarised laser pulses along single-mode optical fibres [3–6, 12, 19, 27].

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