

RESONATORS

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Eigenfrequencies and eigenmodes of a ring optical resonator with thin dielectric plates

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***Abstract*. A ring optical resonator containing arbitrarily placed dielectric plates is considered. The resonator inhomogeneity introduced by the plates is insignificant: the thickness of plates and the excess of their refractive index over the average refractive index of the resonator obey the condition of smallness. It is shown that the eigenfrequency spectrum of this resonator is simple and represents an equidistant sequence of weakly split frequency doublets. Splitting in each doublet is found. Modes in the form of perturbed standing waves are quantitatively described. The formalism of the shift matrices along the trajectories of the differential equation, which makes it possible to obtain the result in the simplest way, is used to solve the above spectral problem.**

Keywords: *ring resonators, eigenfrequencies, eigenmodes.*

1. Introduction

Parameters of laser radiation are mainly determined by the resonator. In a uniform (scattering is absent) reciprocal ring optical resonator (ROR) both a pair of counterpropagating (travelling) and a pair of orthogonal standing waves can be independent. Depending on the operation regime, any of these wave pairs can participate in generation. The presence of the resonator nonreciprocity excludes the generation of standing waves. When the nonreciprocity and scattering act simultaneously (in the nonreciprocal and nonuniform ROR), mixed-type waves are generated, the standing wave coefficient depending on the nonreciprocity and inhomogeneity parameters. This is the essence of the problem of ring lasers, which is known as the 'coupling of counterpropagating waves via scattering'. It is important to find the dependence of the coupling strength and character on the inhomogeneity parameters, because it opens up the possibility for decreasing directly this dependence (for some applications of ring lasers, the standing wave coefficient should be close to zero).

For this reason, numerous attempts have been made to study the influence of the inhomogeneity in the ROR (at least in the reciprocal resonator). Because this problem cannot be solved analytically for the inhomogeneity of the general type, we chose rather simple inhomogeneity models: a small dielectric prism [1], a small dielectric sphere [2], a thin cylinder of finite length with the axis parallel to the resonator axis [3], i.e. lumped scatterers. In paper [4] a simple inhomogeneity model imitating the action of two scattering centres was considered, i.e. the simplest multiple scatterer. For a number of applications of ring lasers it is necessary to give a more general description of inhomogeneities (see, for example, [5]). This can be only done by using the perturbation theory in some or other interpretation. Thus, we can study the ROR in the case of rather complex inhomogeneity distributions (but within the applicability of the perturbation theory).

In this paper we consider one of the variants of such an inhomogeneity in the form of three dielectric plates arbitrarily placed along the ROR perimeter. These plates can, in particular, imitate parasitic scattering centres but can also represent elements of the resonator design. Some assumptions are made about the plate parameters, which allow one to treat them as a small perturbation in a uniform ROR. The ROR frequency spectrum and modes are studied (it is shown that the results can be applied to the case of an arbitrary number of plates). As is known (see, for example, [6]), the solution of the spectral problem for a ROR makes it possible to obtain important laser radiation parameters (amplitudes of stationary oscillations, limiting cycle strength, boundaries of the single-frequency lasing region, etc.).

2. Formulation of the problem

Let us assume that the ROR is one-dimensional, x is the coordinate of a point on the axis, and L is the resonator perimeter ($0 \leqslant x \leqslant L$). The refractive index of the filling medium $n(x) = n_0[1 + s(x)]^{1/2}$ is piecewise constant. The function $s(x) = 0$ for $x_i + \delta x < x < x_{i+1}$ and $s(x) = S > 0$ for $x_i \leqslant x \leqslant x_i + \delta x$, where $i = 1, 2, 3$, $x_4 = L$, $x_1 = 0$. This distribution of the refractive index can simulate the action of three scattering centres inside the ROR and corresponds to three dielectric plates of thickness δx located at arbitrarily points x_i. In the one-dimensional approximation (infinitely extended plates), the stationary waves u with the frequency $\omega = kc$ (c is the speed of light in vacuum) in this structure satisfy the wave equation

$$\frac{d^2u}{dx^2} + k^2n^2(x)u = 0$$

and boundary conditions of the periodic type

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$$u(0) = u(L), \quad \frac{du(0)}{dz} = \frac{du(L)}{dz}.$$

The problem consists in determining the approximate values of eigenfrequencies κ (expressed in inverse centimetres) and real modes corresponding to them (distributions of the standing wave type). The approximation is provided by assuming that the parameter $\kappa_{0p} S \delta x$ is small, where κ_{0p} are eigenfrequencies of the uniform ROR (for $S = 0$), which are used as a zero approximation for the eigenfrequencies of the nonuniform ROR.

3. Shift matrices and matrices of the ROR round trip

The wave equation along the entire length of the resonator has the form

$$\frac{d^2 u}{dx^2} + \chi^2 u(x) = 0,$$

where $\chi^2 = \bar{\kappa}^2 = k^2 n_0^2 (1 + S)$ in intervals $(x_i, x_i + \delta x)$ and $\chi^2 = \kappa^2 = k^2 n_0^2$ in intervals $(x_i + \delta x, x_{i+1})$. In the phase space ($u$, $y = du/dx$), the vector equation

$$\frac{d}{dx}\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\chi^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ y \end{pmatrix}$$

corresponds to this equation. We will solve the vector equation by using the shift matrix $U(x'', x')$ along the phase trajectories (see, for example, [7]). In the case of constant coefficients χ, this expression has the form:

$$U(x'', x') = \begin{pmatrix} \cos[\chi(x'' - x')] & \dfrac{1}{\chi}\sin[\chi(x'' - x')] \\ -\chi \sin[\chi(x'' - x')] & \cos[\chi(x'' - x')] \end{pmatrix}.$$

The shift matrix allows one to shift the solution from point $x = x'$ to the poin $x = x''$:

$$\begin{pmatrix} u(x'') \\ y(x'') \end{pmatrix} = U(x'', x')\begin{pmatrix} u(x') \\ y(x') \end{pmatrix}.$$

Starting from point $x = x_1 = 0$, we will successively pass to points $x_i + \delta x$, x_{i+1}. The shift in intervals $(x_i, x_i + \delta x)$ for $\chi = \bar{\kappa}$ can be written in the form:

$$\begin{pmatrix} u(x_i + \delta x) \\ y(x_i + \delta x) \end{pmatrix} = \bar{U}(x_i + \delta x, x_i)\begin{pmatrix} u(x_i) \\ y(x_i) \end{pmatrix},$$

and the shift in intervals $(x_i + \delta x,\ x_{i+1})$ for $\chi = \kappa$ has the form:

$$\begin{pmatrix} u(x_{i+1}) \\ y(x_{i+1}) \end{pmatrix} = U(x_{i+1}, x_i + \delta x)\begin{pmatrix} u(x_i + \delta x) \\ y(x_i + \delta x) \end{pmatrix}.$$

In the matrix U, we set $\chi = \kappa$, and in the matrix $\bar{U}$, $\chi = \bar{\kappa}$.

The matrix of the ROR round trip is a shift matrix from point $x = x_1 = 0$ to the same point with the coordinate $x = x_4 = L$:

$$U(L, 0) = \prod_{i=1}^{3} U(x_{i+1}, x_i + \delta x)\bar{U}(x_i + \delta x, x_i). \quad (1)$$

We will simplify this matrix taking into account the following first-order approximations (E is the unit matrix):

$$\bar{U}(x_i + \delta x, x_i) \approx \bar{U}(x_i, x_i) + \frac{d\bar{U}(x_i, x_i)}{dx}\delta x$$

$$= E + \frac{d\bar{U}(x_i, x_i)}{dx}\delta x, \quad (2)$$

$$U(x_{i+1}, x_i + \delta x) \approx U(x_{i+1}, x_i) - \frac{dU(x_{i+1}, x_i)}{dx}\delta x. \quad (3)$$

Because

$$\frac{d\bar{U}(x_i, x_i)}{dx} \neq \begin{pmatrix} 0 & 1 \\ -\bar{\kappa}^2 & 0 \end{pmatrix} = A,$$

all $\bar{U}(x_i + \delta x, x_i)$ are identical and equal to $E + A\delta x$. By using this expression and Eqns (2), (3), we can rewrite expression (1) in the form:

$$U(L, 0) = U_0(L, 0) + \left[\sum_{i=1}^{3} U(L, x_i) A U(x_i, 0)\right.$$

$$\left. - \sum_{i=1}^{3} U(L, x_{i+1}) \frac{dU(x_{i+1}, x_i)}{dx} U(x_i, 0)\right]\delta x. \quad (4)$$

$$U_0(L, 0) = U(L, x_3)U(x_3, x_2)U(x_2, x_1)$$

$$= \begin{pmatrix} \cos\kappa L & \dfrac{1}{\kappa}\sin\kappa L \\ -\kappa\sin\kappa L & \cos\kappa L \end{pmatrix}. \quad (5)$$

Let us calculate the terms in (4) for the round-trip transit matrix. The direct calculation proves that

$$U(L, x_{i+1})\frac{dU(x_{i+1}, x_i)}{dx} U(x_i, 0)$$

$$= \kappa\begin{pmatrix} -\sin\kappa L & \dfrac{1}{\kappa}\kappa L \\ -\kappa\cos\kappa L & -\sin\kappa L \end{pmatrix}. \quad (6)$$

The product of the matrices can be represented in a more convenient form:

$$U(L, x_i) A U(x_i, 0) = \begin{pmatrix} -\sin\kappa L - S\cos\kappa x_i \sin[\kappa(L - x_i)] & \dfrac{1}{\kappa}\{[\cos\kappa L - S\sin\kappa x_i \sin[\kappa(L - x_i)]\} \\ -\kappa\{\cos\kappa L + S\cos\kappa x_i \cos[\kappa(L - x_i)]\} & -\sin\kappa L - S\sin\kappa x_i \cos[\kappa(L - x_i)] \end{pmatrix}. \quad (7)$$

By substituting (5)–(7) into expression (4) we can represent it in the form:

$$U(L,0) = \begin{pmatrix} \cos\kappa L & \frac{1}{\kappa}\sin\kappa L \\ -\kappa\sin\kappa L & \cos\kappa L \end{pmatrix} + 3\kappa \begin{pmatrix} -\sin\kappa L & \frac{1}{\kappa}\cos\kappa L \\ -\kappa\cos\kappa L & -\sin\kappa L \end{pmatrix}\delta x$$

$$-\kappa\sum_{i=1}^{3}\begin{pmatrix} -\sin\kappa L - S\cos\kappa x_i\sin[\kappa(L-x_i)] & \frac{1}{\kappa}\{\cos\kappa L - S\sin\kappa x_i\sin[\kappa(L-x_i)]\} \\ -\kappa\{\cos\kappa L + S\cos\kappa x_i\cos[\kappa(L-x_i)]\} & -\sin\kappa L - S\sin\kappa x_i\cos[\kappa(L-x_i)] \end{pmatrix}\delta x. \quad (8)$$

4. Eigenfrequencies of a nonuniform ROR

The round trip matrix allows one to determine the eigenfrequencies of a nonuniform ring resonator. The eigenfrequencies κ should be so that

$$U(0,L)\begin{pmatrix} u(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} u(0) \\ y(0) \end{pmatrix}.$$

Because the determinant of any shift matrix is 1, the eigenfrequencies should be so that $\mathrm{Sp}U(L,0)=2$. Due to (5), (8), the matrix trace of the ring resonator round trip is

$$\mathrm{Sp}U(L,0) = \mathrm{Sp}U_0(L,0) - 3S\kappa\delta x\sin\kappa L$$

$$= 2\cos\kappa L - 3S\kappa\delta x\sin\kappa L.$$

Thus, the eigenfrequencies should be the roots of the equation

$$2\cos\kappa L - 3S\kappa\delta x\sin\kappa L = 2. \quad (9)$$

The eigenfrequencies of a uniform ROR are the roots of (9) for $S=0$, i.e. $\kappa_{0p}=2\pi p/L$, where $p \gg 0$ is a large integer. The eigenfrequencies $\kappa_p^{\pm}$ of a nonuniform ROR for a small perturbation $S\kappa_{0p}\delta x \ll 1$ are close to eigenfrequencies of a uniform ROR and produce a doublet: $\kappa_p^{\pm} = \kappa_{0p}(1\mp\varepsilon)$, where $0<\varepsilon\ll 1$. From (9) we obtain the expression

$$(\kappa_{0p}\varepsilon L)^2 \mp 3S\kappa_{0p}\delta x(\kappa_{0p}\varepsilon L) = 0,$$

to determine ε, i.e. we have $0<\varepsilon = 3S\delta x/L$. Thus, the range of eigenfrequencies of a nonuniform ROR consists of a sequence of doublets with frequencies

$$\kappa_p^{\pm} = \kappa_{0p}\left(1 \mp \frac{3S\delta x}{L}\right) \quad (10)$$

and a relative frequency range in the doublet $(\kappa_p^{-} - \kappa_p^{+})/\kappa_{0p} = 2(3S\delta x/L)$, which are independent of its central frequency.

As any other perturbation, the inhomogeneity under study causes splitting of the double-degenerate eigenfrequencies of a uniform ROR. As a result, the frequency spectrum of the nonuniform ROR proves to be simple. In this case, the splitting is proportional to the density of the total 'area' of the refractive index inhomogeneity in thin plates, which is uniformly distributed along the resonator perimeter. Therefore, the inhomogeneity under study does not produce conditions for the appearance of internal resonators (equivalent high Q linear resonators do not appear in the ROR). The splitting is independent of the location of inhomogeneities. By summarising the obtained result, we can ascertain that no matter how many thin inhomogeneities are there inside the ring resonator and how they are located with respect to each other, their effect on the frequency spectrum is additive and equivalent to the uniformly distributed inhomogeneity (which is not, however, reduced to the change in the refractive index upon uniform filling). In the terms of phenomenological coupling coefficients of counterpropagating waves, 'thin' inhomogeneities correspond to the complex conjugate coupling coefficients (this follows from the doublet splitting), which is natural because the inhomogeneity of the dielectric constant is considered.

5. Eigenmodes of a nonuniform ROR

In a uniform ROR, two orthogonal modes, which can be chosen arbitrarily, for example $\cos\kappa_{p0}x$ and $\sin\kappa_{p0}x$, correspond to each frequency. In a nonuniform ROR, a single mode correspond to each eigenfrequency. Let us determine modes for each of the doublet frequency. To simplify calculations, we assume that the inhomogeneities are distributed uniformly: $x_{i+1}-x_i=\Delta X = L/3$. The round trip matrix $U(L,0)$ can be approximately represented in the form

$$U(L,0) = [B + (BA - C)\delta x]^3,$$

where

$$B = \begin{pmatrix} \cos(\kappa\Delta X) & \frac{1}{\kappa}\sin(\kappa\Delta X) \\ -\kappa\sin(\kappa\Delta X) & \cos(\kappa\Delta X) \end{pmatrix}$$

$$C = \kappa\begin{pmatrix} -\sin(\kappa\Delta X) & \frac{1}{\kappa}\cos(\kappa\Delta X) \\ -\kappa\cos(\kappa\Delta X) & -\sin(\kappa\Delta X) \end{pmatrix}. \quad (11)$$

The periodic boundary conditions $u_0 = u(0)$, $y_0 = \mathrm{d}u(0)/\mathrm{d}x$ should satisfy the equality

$$[B + (BA - C)\delta x]\begin{pmatrix} u_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} u_0 \\ y_0 \end{pmatrix}. \quad (12)$$

Thus, the vector

$$\begin{pmatrix} u_0 \\ y_0 \end{pmatrix}$$

is the eigenvector for the above matrix and its components should satisfy the system of equations

$$\begin{pmatrix} \cos(\kappa\Delta X) - S\kappa\delta x\sin(\kappa\Delta X) - 1 & \frac{1}{\kappa}\sin(\kappa\Delta X) \\ -\kappa[\sin(\kappa\Delta X) + S\kappa\delta x\cos(\kappa\Delta X)] & \cos(\kappa\Delta X) - 1 \end{pmatrix} \times \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} = 0, \quad (13)$$

which can be easily derived if we use explicit expressions (11) and (12). The zero solution of this system exists at $\kappa = \kappa_p^{\pm} = \kappa_{0p}(1 \mp \varepsilon)$, where, as was shown above, $\kappa_{0p}\Delta X = 2\pi p$, $\varepsilon = S\delta x/\Delta X \ll 1$. Under these conditions

$$\cos(\kappa_p^{\pm}\Delta X) \approx \left[1 - \frac{1}{2}(\kappa_{0p}\varepsilon\Delta X)^2\right],$$

$$\sin(\kappa_p^{\pm}\Delta X) \approx \mp\kappa_{0p}\varepsilon\Delta X.$$

We will use the first line of matrix (13) to determine one of eigenvectors. Let

$$u_0 = -\frac{1}{\kappa_p}\sin(\kappa_p^{\pm}\Delta X),$$

$$y_0 = \cos(\kappa_p^{\pm}\Delta X) - S\kappa_p^{\pm}\delta x\sin(\kappa_p^{\pm}\Delta X) - 1.$$

This gives approximately

$$u_0 = \pm\varepsilon\Delta X + \varepsilon^2\Delta X,$$
$$y_0 = \left(-\frac{1}{2} \pm 1 - \varepsilon\right)(\kappa_{0p}\varepsilon\Delta X)^2. \quad (14)$$

The mode distribution in the range under study can be obtained by shifting the derived eigenvector:

$$\begin{pmatrix} u(x) \\ y(x) \end{pmatrix} = \begin{pmatrix} \cos\kappa x & \frac{1}{\kappa}\sin\kappa x \\ -\kappa\sin\kappa x & \cos\kappa x \end{pmatrix}\begin{pmatrix} u_0 \\ y_0 \end{pmatrix}.$$

It follows that

$$u(x) = u_0\cos\kappa_p x + \frac{1}{\kappa_p}y_0\sin\kappa_p x. \quad (15)$$

By substituting expressions (14) into (15) and selecting the upper sign, we obtain for the corresponding normalisation an expression for the mode, which corresponds to the eigenfrequency $\kappa_p^{+} = (2\pi p/\Delta X)(1 - \varepsilon)$:

$$u(x) \equiv \cos(\kappa_p^{+}x) + \frac{1}{2}\kappa_{0p}\varepsilon\Delta X\sin(\kappa_p^{+}x). \quad (16)$$

Let us use the second line of the matrix in expression (13) to determine the second eigenvector. Its components can be selected in the form:

$$u_0 = \cos(\kappa_p^{\pm}\Delta X) - 1 \approx -\frac{1}{2}(\kappa_{0p}\varepsilon\Delta X)^2, \quad (17)$$

$$y_0 = -\kappa_p^{\pm}[\sin(\kappa_p^{\pm}\Delta X) + S\kappa_p^{\pm}\delta x\cos(\kappa_p^{\pm}\Delta X)]$$
$$\approx -\kappa_{0p}(\kappa_{0p}\varepsilon\Delta X)\left\{1 \mp (1 \mp \varepsilon) \times\left[1 - \frac{1}{2}(\kappa_{0p}\varepsilon\Delta X)^2\right]\right\}. \quad (18)$$

The mode distribution for the eigenfrequency κ_p^{-} is then determined as

$$u(x) = u_0\cos(\kappa_p^{-}x) + \frac{1}{\kappa_p^{-}}y_0\sin(\kappa_p^{-}x)$$
$$\approx u_0\cos(\kappa_{0p}x) + \frac{1}{\kappa_{0p}}y_0\sin(\kappa_{0p}x). \quad (19)$$

If in (18) we take the lower sign, we obtain the mode

$$u(x) = -2(\kappa_{0p}\varepsilon\Delta X)\left[\left(1 + \frac{1}{2}\varepsilon\right)\sin(\kappa_p^{-}x) + \frac{1}{4}(\kappa_{0p}\varepsilon\Delta X) \times\cos(\kappa_p^{-}x)\right] \equiv \sin(\kappa_p^{-}x) + \frac{1}{4}(\kappa_{0p}\varepsilon\Delta X)\cos(\kappa_p^{-}x). \quad (20)$$

Here the mode is normalised so that, as in (16), the coefficient at the first term be equal to unity.

6. Conclusions

In the case of a weak intracavity inhomogeneity in the form of three arbitrarily located thin dielectric plates, the spectrum of the ROR eigenfrequencies represents an equidistant sequence of doublets. The doublet centres coincide with eigenfrequencies of a uniform (without plates) resonator $\kappa_{0p} = 2\pi p/L$. The splitting within each doublet is proportional to κ_{0p} and depends on the degree of the inhomogeneity: $\kappa_p^{-} - \kappa_p^{+} = 2\kappa_{0p}\varepsilon = 2\kappa_{0p}(3S\delta x/L)$. The splitting is independent of the mutual location of thin plates and is determined only by the average density of the total inhomogeneity, which is distributed uniformly along the resonator perimeter. This regularity is preserved in the case of an arbitrary (not equal to three) number of similar inhomogeneities. The modes of each doublet are close to real modes corresponding to the double degenerate eigenfrequency of a uniform initial ROR. One can see from expressions (16), (20) that along with the main component (of the same type as that in the uniform resonator), a small additional standing wave orthogonal to it is present in the normalised modes of each doublet. The relative contribution of these small components is of the same order as that of the splitting in the corresponding doublet of eigenfrequencies. The mode properties in the intervals between thin inhomogeneities are also independent of the mutual location and the number of these inhomogeneities.

An alternative description of modes is their representation in the form of travelling waves. In a uniform ROR two modes exist at each eigenfrequency in the form of a pair of independent travelling waves or a pair of orthogonal standing waves. Modes of a nonuniform reciprocal ROR can be only standing waves but they can always be expanded in the counterpropagating travelling waves. However, such waves are not independent. If a laser with the resonator under study is used, we could observe waves propagating in the same direction (or counterpropagating waves) and corresponding to one doublet of eigenfrequencies.

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