RESONATORS

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Analytical description of a Gaussian beam in a ring resonator with a nonplanar axial contour and an even number of mirrors

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***Abstract*. Stability conditions for a ring resonator with an even number of mirrors and a nonplanar axial contour are studied analytically. New explicit expressions are derived to describe the transverse field distribution of the Gaussian mode with general astigmatism produced in this resonator. Field characteristics for a resonator with the specified parameters are calculated.**

Keywords: *ring resonator, nonplanar contour, Gaussian beam with general astigmatism.*

1. Ring resonators with a nonplanar axial contour (see, for example, [1 – 11]) providing spatial rotation of the image around its optical axis, produce the fundamental mode in the form of a Gaussian beam with general-type astigmatism. In this case, the expression for the function, which describes (in the scalar interpretation without allowance for polarisation) the transverse field distribution of the fundamental mode at some resonator cross section in the zero approximation with respect to wave number k, has the form

$$u(r) = c\exp\left(\mathrm{i}\,\frac{kr^{\mathrm{t}}Hr}{2}\right),$$

where

$$r = \begin{pmatrix} x \\ y \end{pmatrix}; \quad r^{\mathrm{t}} = (\,x \quad y\,);$$

$$H = \begin{pmatrix} 1/q_x & 1/q_{xy} \\ 1/q_{xy} & 1/q_y \end{pmatrix}$$

is the square matrix; c is the constant depending on the longitudinal coordinate z. The matrix H is symmetric and has a positive definite imaginary part for a beam concentrated in the vicinity of the resonator axis. If the beam is axially symmetric, the matrix H is proportional to a unit matrix ($q_x = q_y = q$, $1/q_{xy} = 0$). Otherwise, the beam is said to have astigmatism. Astigmatism is called simple if in some coordinate system, the matrix H has a diagonal shape: in this case, the eigenvectors of matrices $\mathrm{Re}\,H$ and $\mathrm{Im}\,H$ have the same direction, i.e. the major axes of the phase and intensity ellipses coincide and can be chosen as the coordinate axes. It is important that when such a beam propagates along the optical axis, the matrix H remains diagonal for all values of the longitudinal coordinate z. In the case of general astigmatism, the major axes of the intensity and phase ellipses are directed at some angle to each other, and $\mathrm{Re}\,H$ and $\mathrm{Im}\,H$ cannot simultaneously have a diagonal form no matter what coordinate axes are chosen. Besides, these axes have different directions for different values of z (see, for example, [10 – 13]), which gave a reason to call such a beam rotating. If $\omega_{1,2}$ are the semi-major and semi-minor axes of the intensity ellipse at the boundary of which the field amplitude [7] or beam energy density that is proportional to the square of the amplitude [10], decreases by e times compared to its value at the axis, then the eigenvalues of the matrix $\mathrm{Im}\,H$ are $2/(k\omega_{1,2}^2)$ or $1/(k\omega_{1,2}^2)$, respectively. We will use the first definition. The eigenvalues of $\mathrm{Re}\,H$ are the major curvatures of the beam wavefront and equal to $R_{1,2}^{-1}$, where $R_{1,2}$ are the major radii of curvature taking a sign 'plus' for a diverging beam or a sign 'minus' for a converging beam; if the wavefront is hyperbolic, these quantities have different signs.

The square matrix H complies with the equation

$$HBH + HA = DH + C, \tag{1}$$

where A, B, C, D are real (for a passive lossless resonator) 2×2 matrices, which make a 4×4 ray matrix T for the resonator round-trip (monodromy matrix [3]):

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Equation (1) follows from the relationship [9]

$$H_{\mathrm{out}} = (C + DH_{\mathrm{in}})(A + BH_{\mathrm{in}})^{-1}, \tag{2}$$

which describes the transformation of a Gaussian beam when it travels across a system characterised by the matrix T, and from the condition for the beam recovery after the resonator round trip: $H_{\mathrm{out}} = H_{\mathrm{in}} = H$. The matrix T is symplectic [3, 9], which implies fulfillment of the condition

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$$T^{-1} = \begin{pmatrix} D^{\mathrm{t}} & -B^{\mathrm{t}} \\ -C^{\mathrm{t}} & A^{\mathrm{t}} \end{pmatrix}.$$

A resonator is stable with respect to the first approximation if all the eigenvalues of the matrix T are equal to unity in modulus and do not have adjoined vectors [1]. In this case, equation (1) enjoys a symmetric solution with a positive imaginary part. Usually, this solution is constructed with the help of components of the monodromy matrix eigenvectors (the method is based on the research by Babich [14] dealing with the eigenfunctions of a Lapalcian concentrated in the vicinity of a closed geodesic; see also book [3], Chapter 8). Within this concept, the asymptotic of the problem on a multiple-mirror resonator for rather general conditions (an inhomogeneous medium, an arbitrary form of mirrors) was obtained by Popov [1] (a systematic description of the results is given in book [3], Chapter 9, the vector generalisation with respect to polarisation was made by Pankratova [4, 5], resonators with selective elements, absorption and amplification are considered in our papers [15, 16]). It is this approach that was later used in paper [9] for studying a four-mirror resonator with one nonplanar (spherical) mirror in the case of a homogeneous medium.

Note that Bykov et al. [10] offer another approach to solve this problem. It involves the analysis of the evolution of a Gaussian beam with general astigmatism (the method is based on Arnaud and Kogelnik's work [12]; an alternative approach was developed by Belousova [13]). Unfortunately, the authors of [10] proceeded from an erroneous (as we will see below) assumption about the matrix ReH in the beam cross section on a spherical mirror (or rather this assumption is true only for resonators with an odd number of mirrors [17]).

Finally, matrix equation (1) (or a resulting system of algebraic equations in elements of the matrix H) can be solved numerically with a solution axis being chosen later that ensures the concentration of the field in the vicinity of the optical.

For the case of a resonator with an even number of mirrors one of which is nonplanar (e.g. spherical), we propose an alternative solution which describes the field analytically, with the aid of explicit expressions. We think that such a description allows us to understand most fully the dependence of the light field characteristics on the resonator parameters. In this case, it is unnecessary to find the eigenvectors of the matrix T.

2. Consider a multiple-mirror ring resonator with an even number of mirrors providing a spatial rotation of the image through the angle ϕ, which, following the notation of [10], we will call the Berry angle, though this angle, as applied to the problem under study, was used long before the publication of well-known paper [18], in particular, in paper [2] (in this connection it is also sensible to mention such researchers as Ignatovsky, Rytov, Vladimirsky [19]). Let L denote the length of the axial contour. The resonator contains a focusing element, which can be, for example, a lens or one of the mirrors (spherical or elliptical); the other mirrors are considered flat. Propagation along the contour is described by the matrix

$$T_L = \begin{pmatrix} E & LE \\ O & E \end{pmatrix},$$

where O and E are the zero and unit 2×2 matrices, and the rotation through the angle ϕ is described by the matrix

$$T_\phi = \begin{pmatrix} U_\phi & O \\ O & U_\phi \end{pmatrix},$$

where

$$U_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

is the matrix of the rotation operator by the angle $+\phi$ (the same matrix describes the transformation of coordinates when the coordinate axes are turned by the angle $-\phi$). Matrices T_L and T_ϕ commute with each other. With a properly chosen coordinate system, the matrix responsible for reflection from a flat mirror is a unit matrix; upon each reflection the orientation of the coordinate system changes and so does the direction of the angle readout (clockwise or counterclockwise). In this paper, we confine our consideration to the case of an even number of mirrors which means that after the resonator round trip the orientation of the coordinate system coincides with the original one.

Passing through the focusing element (quadratic phase corrector [10]) is described by the matrix

$$T_\Psi = \begin{pmatrix} E & O \\ -\Psi & E \end{pmatrix},$$

where Ψ is a symmetric 2×2 matrix which is considered, without loss of generality, as a diagonal matrix: $\Psi = \mathrm{diag}\,[\psi_x, \psi_y]$, otherwise it can be reduced to a diagonal shape by turning the coordinate axes. For a focusing lens $\psi_{x,y} = 1/f_{x,y}$, where $f_{x,y}$ are the focal lengths, $f_x = f_y = f$ for an astigmatism-free lens. For an elliptical mirror one axis of which lies in the plane of incidence (xz plane) and the other is perpendicular to it (directed along the y axis), $\psi_x = 2(R_x \cos\alpha')^{-1}$, $\psi_y = 2R_y^{-1}\cos\alpha'$, where $\alpha' = \alpha/2$ is the angle of incidence; α is the angle between the incident and reflected axial rays; $R_{x,y}$ are the radii of curvature. $R_x = R_y = R$ for a spherical mirror; a special attention will be paid to this case (in particular, to the resonator considered in paper [9]). We should point out now that the comparison of our results with those given in [9] revealed both agreements and disagreements.

Monodromy matrices T_- and T_+ in the beam cross sections located immediately in front of and behind the mirror (lens) are calculated by the expressions

$$T_- = T_\phi T_L T_\Psi = \begin{pmatrix} U_\phi(E - L\Psi) & LU_\phi \\ -U_\phi\Psi & U_\phi \end{pmatrix},$$

$$T_+ = T_\Psi T_\phi T_L = \begin{pmatrix} U_\phi & LU_\phi \\ -\Psi U_\phi & (E - L\Psi)U_\phi \end{pmatrix}.$$

The characteristic equations for the eigenvalues of such matrices λ are reduced to the form

$$v^2 - (2\gamma\cos\phi)v + \left(\gamma^2 - \delta^2 - \sin^2\phi\right) = 0, \tag{3}$$

where $v = (\lambda + \lambda^{-1})/2$; $\gamma = 1 - (\psi_x + \psi_y)L/4$; $\delta = (\psi_y - \psi_x)L/4$ characterise astigmatism. The matrix Ψ expressed via γ and δ has the form $\Psi = 2L^{-1}\,\mathrm{diag}\,[1 - \gamma - \delta, 1 - \gamma + \delta]$.

The sufficient conditions for matrices $T_\pm$ to be stable [values of λ lie at the boundary of a unit circle and are different, values of v lie within the range $(-1, 1)$ and are also different] have the form [19]

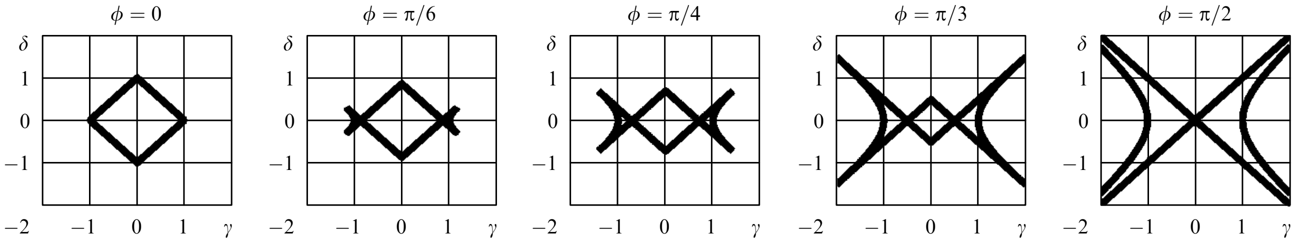

Figure 1. Stability regions in the $\gamma\delta$ plane for different ϕ.

$$|\delta| < ||\gamma| - |\cos\phi||,\ \delta^2 > (\gamma^2 - 1)\sin^2\phi,\ |\gamma| < |\cos\phi|^{-1}. \quad (4)$$

We can obtain the necessary conditions ($|\lambda| = 1$) by replacing in (4) the strict inequalities with nonstrict ones. It is interesting to note that a nonplanar resonator may be stable even when the mirror has a cylindrical or hyperbolic form (one of the eigenvalues of the matrix Ψ is not positive, i.e. $\gamma + |\delta| \geqslant 1$).

The area (4) in Fig. 1 is given in coordinates γ, δ for different ϕ. When $\phi = n\pi$ (n is an integer), this area is a square $|\gamma| + |\delta| < 1$, or $0 < \psi_{x,y}L < 4$. When $0 < |\cos\phi| < 1$, the stability region splits into three subzones: the square $|\gamma| + |\delta| < |\cos\phi|$ [or $2(1 - |\cos\phi|) < \psi_{x,y}L < 2\times(1 + |\cos\phi|)$] and two figures bounded by straight lines $|\gamma| - |\delta| = |\cos\phi|$ and hyperbolic segments $\gamma^2 - (\delta/\sin\phi)^2 = 1$ which are tangent to the straights at their end points with coordinates $\gamma = \pm|\cos\phi|^{-1}$, $\delta = \pm\sin^2\phi|\cos\phi|^{-1}$. When $\phi = \pi/2 + n\pi$, the central square disappears, and the subzones between the straights and hyperbolic segments become unbounded: straights $\gamma = \pm\delta$ become asymptotes of hyperbola $\gamma^2 - \delta^2 = 1$ rather than its tangent.

The instances of multiples of λ, when inequalities in (4) turn into equalities, require a separate consideration because of the possible appearance of adjoined vectors. The analysis shows that of all the boundary points only those belonging to curves $\gamma = \pm\cos\phi$, $\delta = 0$ connecting the subzones of (4) ensure the fulfillment of the stability condition. Thus, with these points taken into account, the set of parameters providing the resonator stability proves bound.

Figure 2 shows the region in the $\gamma\delta$ plane which combines stability regions for all possible values of ϕ. In the interval $\gamma \in [-1/2, 1/2]$ the boundaries of the region meets the equation $\delta = \pm(1 - |\gamma|)$ and coincide with the boundary of the central square at $\phi = 0$. Then, the upper and lower boundaries have the form $\delta = \pm|\gamma|$ as in the case with $\phi = \pi/2$. Finally, if $|\gamma| \geqslant 1$, the region is bound by curves $\delta = \pm(\gamma^2 - 1)/|\gamma|$ which hold the end points of the hyperbolic segments (Fig. 1) – points $\gamma = \pm|\cos\phi|^{-1}$, $\delta = \pm\sin^2\phi|\cos\phi|^{-1}$ for all possible ϕ.

Let us state the stability conditions for different subsets of the region (Fig. 2) with respect to $|\cos\phi|$. The sufficient condition of stability for the points belonging to the square $|\gamma| + |\delta| < 1$ is the inequality $|\cos\phi| > |\gamma| + |\delta|$. For the points located between the bisectrixes of the coordinate angles and the wings of the hyperbola $\gamma^2 - \delta^2 = 1$ and satisfying the inequalities $|\delta| < |\gamma| < (1 + \delta^2)^{1/2}$, the sufficient condition for the stability is the inequality $|\cos\phi| < |\gamma| - |\delta|$. For the points belonging only to one of the sets, these conditions are necessary as well. As to the common part of these sets consisting of two squares and meeting inequalities $|\delta| < |\gamma| < 1 - |\delta|$, satisfaction of any of these

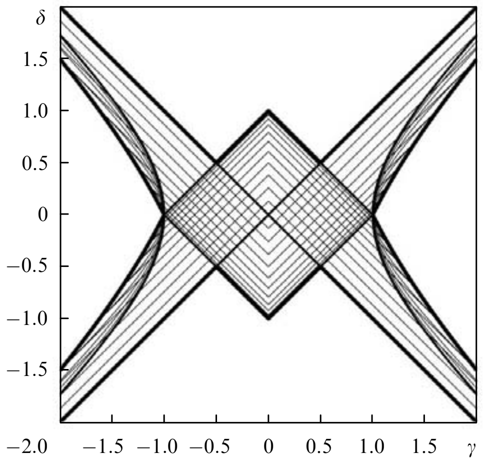

Figure 2. Combined stability region in the $\gamma\delta$ plane. For certain values of ϕ the points of this region correspond to a stable resonator. Thin lines denote the boundaries of stability regions for different ϕ spaced by 5°.

inequalities is the sufficient and necessary condition of stability. As to the segment $|\gamma| < 1$, $\delta = 0$, by virtue of the above remark about the boundary points, the stability takes place at any ϕ, including those at $|\cos\phi| = |\gamma|$. Finally, for the region lying between the hyperbolas and meeting the inequalities $(\gamma^2 - 1)/|\gamma| < |\delta| \leqslant (\gamma^2 - 1)^{1/2}$, the stability condition is $[\delta^2/(\gamma^2 - 1) - 1]^{1/2} < |\cos\phi| < |\gamma| - |\delta|$. Note that the condition $|\cos\phi| < |\gamma|^{-1}$ does not need testing because in this region $|\gamma| - |\delta| < |\gamma|^{-1}$ (in fact this condition enters the equation of the boundary $|\delta| = (\gamma^2 - 1)/|\gamma|$).

In the space of three variables (ϕ, γ, δ) region (4) is symmetric with respect to the planes $\phi = n\pi/2$, $\gamma = 0$, $\delta = 0$, with respect to the points $(n\pi/2, 0, 0)$ and periodic in ϕ with a period π. The general shape of region (4) for $\phi \in (-\pi/2, \pi/2)$ is given in Fig. 3 (the trails going to infinity are cut off). Figure 1 shows cross sections of this figure at different ϕ, and Fig. 2 gives its projection on the $\gamma\delta$ plane.

Note that all this is true if the number of mirrors is even. When a resonator has an odd number of mirrors, another, quite different from (4), system of inequalities follows from the stability conditions and the geometry of the stability region in this case bears no resemblance to that shown in Figs 1 – 3.

3. The stability conditions for the case when the focusing element is a spherical mirror are given in paper [8] in an implicit form, namely, as a condition for coefficients of a quadratic equation, that is similar to (3), for which the roots lie within a particular interval. We assume that it will be useful to write explicit expressions. (Such expresion are obtained in [6] only for $\phi = n\pi/2$, and the expressions given

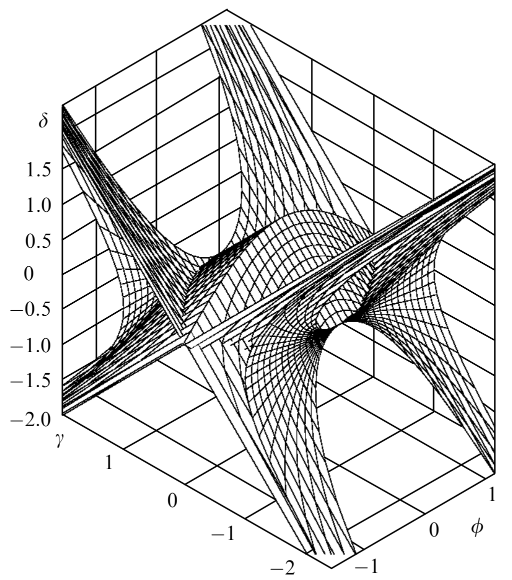

Figure 3. Stability region in the ϕ, γ, δ space for $\phi \in (-\pi/2,\ \pi/2)$. The figure repeats itself along the ϕ axis.

in [5] are untrue because of an erroneous position of one of the cosines in the reflection matrix of the spherical mirror.) For the case under study $\gamma = 1 - (\cos\alpha' + \cos^{-1}\alpha') \times L/(2R)$, $\delta = (\cos\alpha' - \cos^{-1}\alpha')L/(2R)$. It follows from these equalities that the sector $\gamma < 1$, $\gamma - 1 < \delta < 0$ in the $\gamma\delta$ plane

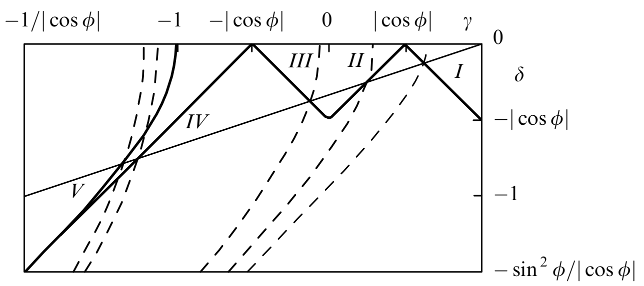

Figure 4. Stability region in the $\gamma\delta$ plane for a resonator with a spherical mirror. The straight $\alpha = \text{const}$ (thin solid line) intersects all three subzones, hyperbolas $R = \text{const}$ (dashed curves) go through the intersections of this straight with the boundaries of the stability region $L/R = (1 - |\cos\phi|)\cos\alpha'$ (*I*), $(1 - |\cos\phi|)(\cos\alpha')^{-1}$ (*II*), $(1 + |\cos\phi|) \times \cos\alpha'$ (*III*), $(1 + |\cos\phi|)/\cos\alpha'$ (*IV*) and $4\cos\alpha'(1 + \cos^2\alpha')\sin^2\phi \times [4\cos^2\alpha' - \cos^2\phi(1 + \cos 2\alpha')^2]^{-1}$ (*V*).

corresponds to this particular case; for any other points entering the stability region one has to use elliptic, cylindrical or hyperbolic mirrors (or lenses).

The hyperbolic segments $\delta = -[(1-\gamma)^2 - (L/R)^2]^{1/2}$ are the curves that host points of equal R for all possible angles of incidence, and rays $\delta = (\gamma - 1)\sin^2\alpha'/(1 + \cos^2\alpha')$ (for $\gamma < 1$) are the lines that host points of different R for a fixed angle of incidence. These rays intersect the right subzone of the stability region for any α and ϕ; the resultant segment corresponds to $L/R < (1 - |\cos\phi|)\cos\alpha'$. If $\sin^2\alpha'(1 + \cos^2\alpha')^{-1} < |\cos\phi|$, the ray intersects the central zone at

$$(1 - |\cos\phi|)/\cos\alpha' < L/R < (1 + |\cos\phi|)\cos\alpha'.$$

Finally, if $\sin^2\alpha'/(1 + \cos^2\alpha') < 1 - |\cos\phi|$, the ray intersect the left subzone at

$$(1 + |\cos\phi|)/\cos\alpha' < L/R < 4\cos\alpha'(1 + \cos^2\alpha')\sin^2\phi$$

$$\times [4\cos^2\alpha' - \cos^2\phi(1 + \cos^2\alpha')^2]^{-1}.$$

Thus, depending on values α and ϕ, one to three intervals of R exist in which the resonator is stable. The inclined ray in Fig. 4 corresponds to the last case; in Fig. 5 the values α and ϕ for which the vertical lines intersect the three subzones of the stability region at once correspond to it.

4. Consider in detail a four-mirror resonator described in paper [9]. Figure 6 shows the scheme of its ray path that follows the edges of the ABCD tetrahedron. The path lies in the planes ABD and BCD, the angle between them is β. DAB and BCD are isosceles triangles, α is the angle between AB and AD, $\alpha' = \alpha/2$ is the angle between AB and altitude AE. A spherical mirror of radius R is positioned at point A, flat mirrors – at points B, C, D.

The length of the path is $L = 2(L_1 + L_2)$, where $L_1 = |AB| = |AD|$, $L_2 = |BC| = |CD|$. Let $h_1 = |AE|$ and $h_2 = |CE|$ be the altitudes of ABD and BCD triangles, then $L_1 = h_1/\cos\alpha'$, $|BE| = h_1\tan\alpha'$, and $L_2 = (h_1^2\tan^2\alpha' + h_2^2)^{1/2}$. In particular, the resonator from [9] has $h_1 = 3$ mm, $h_2 = 2$ mm, $\alpha = \pi/3$, $R = 50$ mm (Nd : YAG monolithic nonplanar ring resonator), $L = 2(2\sqrt{3} + \sqrt{7})$ mm $\approx$ ≈ 12.22 mm, and $L/R \approx 0.2444$ (in this case, $\gamma \approx 0.7531$, $\delta \approx -0.0353$). One can see from Fig. 5 that for this value of the ordinate the stability region encompasses all values of $|\cos\phi|$ except for a small region between lines *I* and *II*. Then, the stability condition takes the form

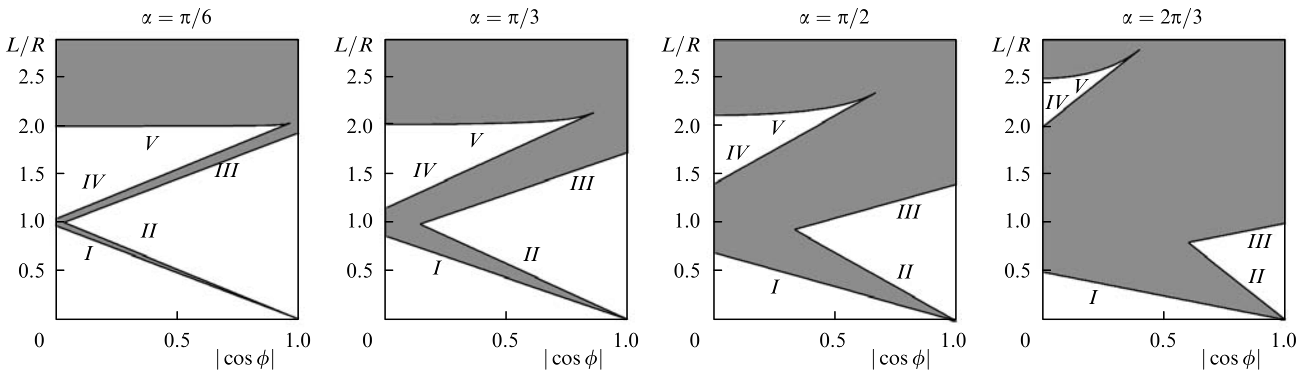

Figure 5. Regions of stability (white) and instability (black) for different angles α. The equations for the boundaries $I - V$ are the same as in Fig. 4.

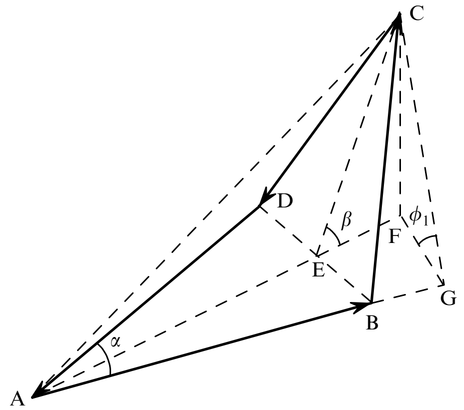

Figure 6. Scheme of the ray path in the resonator. The arrows show the propagation direction.

$|\cos\phi| \notin (1 - L/(R\cos\alpha'), 1 - (L\cos\alpha')/R)$, which gives $|\cos\phi| \notin (0.7178,\ 0.7883$ (the boundary of the instability region coincides with $\gamma+\delta, \gamma-\delta$) for the given parameters.

Let us tie the Berry angle to the geometric characteristics of the resonator. Let H_+ be the matrix corresponding to a cross section at point A immediately behind the mirror. The z axis is directed along the edge AB, the x axis lies in the ABD plane and is directed outwards, the y axis is perpendicular to this plane; the directional vectors form the right-hand triple. The matrix $H(z)$ describing the field distribution in the cross section at a distance z away from point A is connected to the matrix H_+ by relation (2) in which $A = D = E$, $B = zE$, $C = O$:

$$H(z) = H_+(E + zH_+)^{-1}. \tag{5}$$

(Note that the longitudinal-coordinate dependence of the field is determined also by the multiplier c, the expression for which in the jth arm of the resonator has, in the zero approximation, the form $c(z) = c_j[\det H(z)]^{1/2}\exp(ikz)$, and the relation between c_j is determined by the boundary conditions on the mirrors.)

Passing to the coordinate system related to the ABC plane necessitates its clockwise rotation around the z axis through an angle equal to $\phi(|\mathrm{AB}|)$ – the angle between the ABD and ABC planes. In the new coordinate system, the square matrix takes the form $U_{\phi(|\mathrm{AB}|)}H(z)U_{-\phi(|\mathrm{AB}|)}$. Reflection from a flat mirror at point B does not affect the form of the matrix H, but changes the orientation of the coordinate axes: the x axis becomes directed inwards the ABC triangle. Another transformation involves passing to the coordinate system related to the plane BCD. To this effect it is necessary to make a rotation around a new z axis, which is directed along the BC edge, through an angle equal to $\phi(|\mathrm{BC}|)$ – the angle between the ABC and BCD planes. Although it is a counterclockwise rotation, the directional vectors form a left-hand triple after reflection in this event and the sign of the rotation angle remains the same as in the first rotation. As a result, the square matrix takes the form $U_{\phi(|\mathrm{AB}|)+\phi(|\mathrm{BC}|)}H(z)U_{-\phi(|\mathrm{AB}|)-\phi(|\mathrm{BC}|)}$ and remains the same after reflection from a flat mirror at point C (the orientation of the coordinate axes changes again in this case). Following the line of reasoning, we come to a conclusion that in the coordinate systems related to the ACD plane (CD and DA edges) the matrix has the form

$$U_{\phi(|\mathrm{AB}|)+\phi(|\mathrm{BC}|)+\phi(|\mathrm{CD}|)}H(z)U_{-\phi(|\mathrm{AB}|)-\phi(|\mathrm{BC}|)-\phi(|\mathrm{CD}|)},$$

and in the coordinate system related to the ABD plane (the DA edge) it takes the form

$$U_{\phi(|\mathrm{AB}|)+\phi(|\mathrm{BC}|)+\phi(|\mathrm{CD}|)+\phi(|\mathrm{DA}|)}H(z)$$

$$\times U_{-\phi(|\mathrm{AB}|)-\phi(|\mathrm{BC}|)-\phi(|\mathrm{CD}|)-\phi(|\mathrm{DA}|)},$$

where $\phi(|\mathrm{CD}|)$ is the angle between BCD and ACD planes, and $\phi(|\mathrm{DA}|)$ is the angle between ACD and ABD planes. Supposing that $z = L$, we obtain

$$H_- = U_\phi H(L)U_{-\phi}, \tag{6}$$

where H_- is the square matrix in the beam cross section located at point A immediately in front of the mirror, and

$$\phi = \phi(|\mathrm{AB}|) + \phi(|\mathrm{BC}|) + \phi(|\mathrm{CD}|) + \phi(|\mathrm{DA}|) \tag{7}$$

is the angle through which the image rotates after the resonator round trip. Matrices $H_\pm$ are related as $H_+ = H_- - \Psi$.

To avoid misunderstanding we should point out that all the angles in the right side of (7) are positive (a similar expression has the same form in [10], for example). This expression takes such a simple form because the coordinate axes and the direction of rotation change at the same time. Another method of description (and a single possible one for more complicated designs) involves an alternating algebraic sum of the angles between the planes of incidence on successively placed mirrors, where the sign of the terms should correspond to the direction of rotation (see, for example, [2, 5, 6]).

Now we write expressions relating angles $\phi(|\mathrm{AB}|) = \phi(|DA|) = \phi_1$ and $\phi(|\mathrm{BC}|) = \phi(|\mathrm{CD}|) = \phi_2$ with the geometrical characteristics of the contour. Let us drop a perpendicular CF from point C onto the plane ABD, provided $|\mathrm{CF}| = h_2\sin\beta$, $|\mathrm{AF}| = h_1 + h_2\cos\beta$. Let us then drop a perpendicular FG from point F onto the segment AB or its extension, provided $|\mathrm{FG}| = |\mathrm{AF}|\sin\alpha'$. The FGC plane is perpendicular to AB because AB is orthogonal to CF and FG and, therefore, the angle FGC coincides with the angle

$$\phi_1 = \arctan\frac{h_2\sin\beta}{(h_1 + h_2\cos\beta)\sin\alpha'}.$$

A similar procedure is needed to derive the expression for ϕ_2 – it is only necessary to interchange h_1 and h_2 and replace α' by the angle BCE equal to $\arctan[(h_1/h_2)\tan\alpha']$. As a result, we obtain

$$\phi_2 = \arctan\frac{\left(h_1^2 + h_2^2\cot^2\alpha'\right)^{1/2}\sin\beta}{h_1\cos\beta + h_2},$$

and

$$\phi = 2(\phi_1 + \phi_2).$$

When β decreases, angles $\phi_{1,2}$ also decrease and vanish simultaneously with β. If it is α that decreases, angles $\phi_{1,2}$ increase and tend to $\pi/2$ as α approaches zero. When angles α and β approach zero simultaneously, angles $\phi_{1,2}$ are the

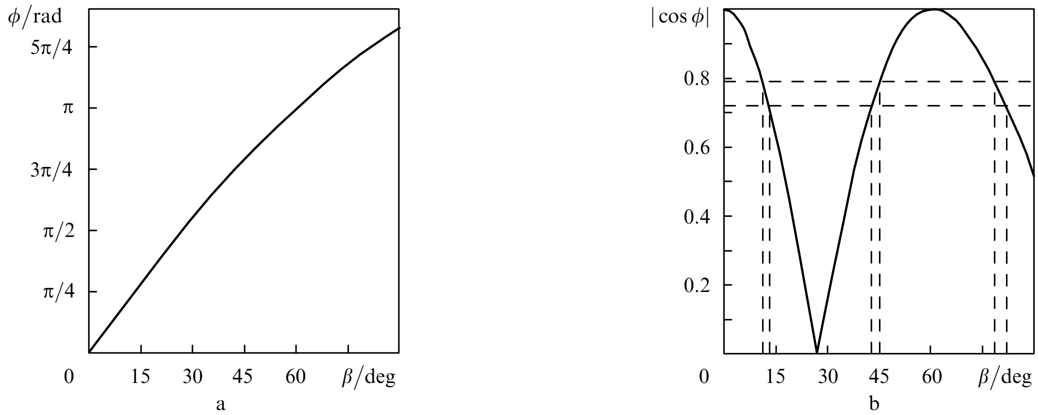

Figure 7. Dependences of the image rotation angle ϕ (a) and $|\cos\phi|$ (b) on the angle between β planes for $\alpha = \pi/3$, $h_1 = 3$ mm, $h_2 = 2$ mm (parameters are borrowed from [9]). The horizontal dashed lines in Fig. 7b are the boundaries of the instability intervals for $|\cos\phi|$ [$|\cos\phi| \in (0.718, 0.788)$], the vertical dashed lines are the boundaries of the instability intervals for β ($\beta = 11°8' - 12°58'$, $42°51' - 45°9'$ and $78°28' - 81°42'$) for $R = 50$ mm.

same in the zero approximation and equal to $\arctan[2h_2\beta/(h_1 + h_2)\alpha]$. The calculation results for a resonator with the chosen parameters are shown in Fig. 7.

By using the expressions from this and previous sections, it is possible to find the stability and instability regions for this resonator in coordinates $\beta, f = R/2$. Comparing Fig. 8, which shows the results of our calculations, and similar Fig. 2 from paper [9], one can make sure that although the general shapes of the regions look the same, in our case these regions are more extended along the β axis and, therefore, the position of instability segments differs from that in [9]. We can assume that the dependence $\phi(\beta)$ used in calculations in [9] differed from that given in Fig. 7.

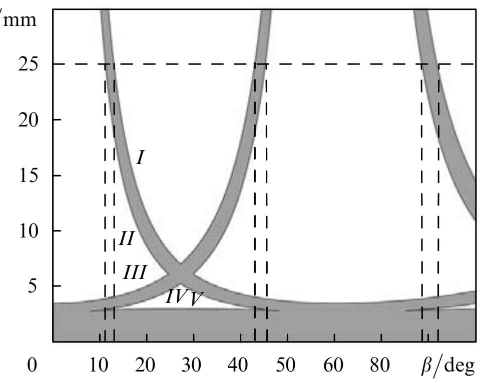

Figure 8. Regions of stability (white) and instability (black) in coordinates β, f for a resonator with $\alpha = \pi/3$, $h_1 = 3$ mm, $h_2 = 2$ mm. The dashed lines are the boundaries of the instability intervals for $f = 25$ mm, curves $I - V$ are the boundary curves corresponding to the curves presented in Figs 4 and 5.

5. Let us return to the general case and write the expressions for matrices $H_\mp$ corresponding to the cross sections situated immediately in front of and behind the focusing element:

$$H_\mp = H_0 \pm \frac{\Psi}{2}, \tag{8}$$

where H_0 is the complex symmetric matrix corresponding to an equivalent resonator [7] whose nonplanar mirror is replaced by a flat mirror with an adjacent astigmatic lens. The imaginary part of the matrix has the diagonal form

$$\mathrm{Im} H_0 = \frac{t}{4L\gamma\delta|\cos\phi|}$$

$$\times \mathrm{diag}\left[(\gamma+\delta)^2 - \cos^2\phi + d, -(\gamma-\delta)^2 + \cos^2\phi - d\right], \tag{9}$$

and the real part is

$$\mathrm{Re} H_0 = \frac{\left(\gamma^2 - \delta^2 - \cos^2\phi + d\right)\tan\phi}{2L\delta}\,\sigma \equiv K\sigma. \tag{10}$$

Here,

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

quantities d and t are determined by equalities

$$d = \left\{\left[(\gamma+\delta)^2 - \cos^2\phi\right]\left[(\gamma-\delta)^2 - \cos^2\phi\right]\right\}^{1/2}$$

$$\times\,\mathrm{sign}(|\cos\phi| - |\gamma|), \tag{11}$$

$$t = \left[2\left(\gamma^2 - 2\gamma^2\cos^2\phi + \cos^2\phi - \delta^2 + d\right)\right]^{1/2}. \tag{12}$$

The quantity d vanishes on the surface $|\delta| = ||\gamma| - |\cos\phi||$, and t vanishes on the surface $\delta^2 = (\gamma^2 - 1)\sin^2\phi$, lines $\gamma = 0$, $|\delta| = |\cos\phi|$ and also when $\cos\phi = 0$.

The directions of the eigenvectors of the $\mathrm{Im}\,H_0$ matrix coincide with the coordinate axes, and its eigenvalues (diagonal elements) are equal to $2/(k\omega_{x,y}^2)$, where $\omega_{x,y}$ are the semi-axes of the intensity ellipse; when the eigenvalue tends to zero, the transverse beam size along the appropriate axis grows infinitely. The eigenvectors of the $\mathrm{Re}\,H_0$ matrix coincide with the bisectors of the coordinate angles, the eigenvalues have different signs and coincide in modulus with the coefficient K at σ in (10).

Matrices H_+ and H_- meet equation (1), where A, B, C, D are blocks of matrices T_+ and T_-, respectively, and their imaginary part, which coincides with $\mathrm{Im}\,H_0$, is positive. The eigenvalues of $\mathrm{Im}\,H_0$ (the diagonal elements of the matrix) are positive in region (4). One of them vanishes on the boundary surface $|\gamma+\delta| = |\cos\phi|$, the other – on the surface $|\gamma-\delta| = |\cos\phi|$; when $\delta^2 = (\gamma^2-1)\sin^2\phi$, the matrix $\operatorname{Im} H_0$ turns into a zero matrix (the case of $\gamma\delta\cos\phi = 0$ requires special consideration). The analysis shows that all the elements of matrices $\operatorname{Re} H_0$ and $\operatorname{Im} H_0$ change dramatically near the boundary of (4).

Here we do not give derivation of expressions (9)–(12). A method permitting explicit solutions of equations of type (1) (the simplest way is to consider the equation with respect to H_0) is described in detail in [20] (see also [21]), where the expression for the matrix H in the beam cross section situated at distance $L/2$ away from the mirror has been obtained.

Consider briefly the concept of 'equivalent' linear resonator. According to this concept, the field distribution in the fundamental mode of a ring resonator is the same as that of a linear resonator consisting of two identical elliptical mirrors spaced by distance L apart. The major radii of the mirrors are equal to the effective radii of a spherical mirror, and the major axes of curvature of one of the mirrors make the Berry angle with the axes of curvature of the other. It is this concept (not formulated explicitly, but accepted as an obvious fact) that underlies an algorithm offered in [10] to solve this problem. It is asserted that in propagation along the resonator axis the major axes of curvature of the wavefront rotate by the Berry angle (and should evidently coincide with one another and the coordinate axes on the spherical mirror) and that in the beam cross section corresponding to the spherical mirror the major radii of curvature of the beam are equal to half the effective radii of curvature of this mirror. However, it turns out that in a ring resonator the major axes of the intensity ellipse rather than the phase ellipse (which is the case with a linear resonator) coincide with the coordinate axes: the matrix $\operatorname{Im} H_\pm = \operatorname{Im} H_0$ is diagonal unlike the matrix $\operatorname{Re} H_\pm$. The exception is the simplest case of $\phi = n\pi/2$ considered in [6] when, according to (10) and (14) (see below), $\operatorname{Re} H_0 = O$ and $\operatorname{Re} H_\mp = \pm\Psi/2$. Then matrices $H_\pm$ are diagonal matrices, and a beam with simple astigmatism (without rotation of the axes of the amplitude and phase distributions) develops in the resonator.

Note that what has been said applies only to resonators with an even number of mirrors. It can be shown that in the case of an odd number of mirrors the equivalence of a linear and ring resonator really takes place. In the 'equivalent' linear resonator the axes of the intensity ellipse on the elliptical mirrors make angles with the major axes of curvature that are equal in modulus but opposite in sign, which leads, in fact, to violation of the equivalence, unless these angles are multiple of the right angle: in this resonator matrices $\operatorname{Im} H_\pm$ of the beam will not be identical. If the number of mirrors is odd, the above-mentioned difference in sign will be compensated for due to the reorientation of the coordinate axes after a round trip in the resonator under study. These considerations are also confirmed by explicit expressions similar to (9)–(12) according to which with an even number of mirrors, $\operatorname{Re} H_0 = O$ and the matrix $\operatorname{Im} H_0$ is not diagonal, which will be studied elsewhere.

6. Expressions (9), (10) are not applicable if the parameters take values that turn the denominator into zero. To describe the behaviour of the matrices when γ and δ approach zero, we will transform the matrix elements by additionally multiplying the numerators and denominators by conjugate expressions. Then, we come to the expressions

$$\operatorname{Im} H_0 = \frac{t}{L|\cos\phi|}$$

$$\times\operatorname{diag}\left[\frac{(\gamma+\delta)^2-\cos^2\phi}{(\gamma+\delta)^2-\cos^2\phi-d}, \frac{-(\gamma-\delta)^2+\cos^2\phi}{-(\gamma-\delta)^2+\cos^2\phi+d}\right], \quad (13)$$

$$\operatorname{Re} H_0 = \frac{2\delta\sin\phi\cos\phi}{L\left(\gamma 2-\delta^2-\cos^2\phi-d\right)}\,\sigma. \quad (14)$$

At $\gamma = 0$ we obtain

$$H_0(\gamma=0) = \frac{1}{L}\left[\mathrm{i}\,\frac{\left(\cos^2\phi-\delta^2\right)^{1/2}}{|\cos\phi|}\,E - \delta\sigma\tan\phi\right],$$
$$d(\gamma=0) = \cos^2\phi-\delta^2,\ t(\gamma=0) = 2\left(\cos^2\phi-\delta^2\right)^{1/2} \quad (15)$$

at $\delta = 0$ we obtain

$$H_0(\delta=0) = \mathrm{i}\,\frac{\left(1-\gamma^2\right)^{1/2}}{L}\,E,$$
$$d(\delta=0) = \cos^2\phi-\gamma^2,\ t(\delta=0) = 2|\cos\phi|\left(1-\gamma^2\right)^{1/2} \quad (16)$$

and at $\gamma = 0,\ \delta = 0$ we obtain

$$H_0(\gamma=\delta=0) = \mathrm{i}L^{-1}E,$$
$$d(\gamma=\delta=0) = \cos^2\phi,\ t(\gamma=\delta=0) = 2|\cos\phi|. \quad (17)$$

One can see from (15), (16) that at points $\gamma = 0$, $|\delta| = |\cos\phi|$ and $\delta = 0$, $|\gamma| = 1$ lying on the stability region boundary, $\operatorname{Im} H_0$ becomes a zero matrix.

7. When $\delta = 0$, the solution is axially symmetric and independent of ϕ; there is no astigmatism. However, there is a special case of $|\gamma| = |\cos\phi|$ (points on the subzone interfaces) when this solution is not the only one. It is caused by the fact that matrices $T_\pm$ have multiple eigenvalues ($+1$ at $\gamma = \cos\phi$ and -1 at $\gamma = -\cos\phi$) at these points, the corresponding eigenspaces having no fixed sign [22]. The sought-for sets of solutions have the form

$$H_0(\delta=0, \gamma=\pm\cos\phi)$$
$$= L^{-1}\{\mathrm{i}|\sin\phi|\operatorname{diag}[1+\zeta, 1-\zeta] \pm \zeta\sigma\sin\phi\}, \quad (18)$$

where ζ is the complex parameter ($|\zeta| < 1$). Then,

$$\operatorname{Im} H_0(\delta=0, \gamma=\pm\cos\phi)$$
$$= L^{-1}\{|\sin\phi|\operatorname{diag}[1+\operatorname{Re}\zeta, 1-\operatorname{Re}\zeta] \pm \sigma\sin\phi\operatorname{Im}\zeta\},$$

$$\operatorname{Re} H_0(\delta=0, \gamma=\pm\cos\phi)$$
$$= L^{-1}\{|\sin\phi|\operatorname{diag}[-\operatorname{Im}\zeta, \operatorname{Im}\zeta] \pm \sigma\sin\phi\operatorname{Re}\zeta\}.$$

Note the relation of sets (18) with the behaviour of matrix (9), (10) in the vicinity of the points under study. Let $\gamma = \pm\cos\phi + \varepsilon\gamma'$, $\delta = \varepsilon\delta'$ with $|\delta'| < |\gamma'|$. If we let ε approach zero, we obtain in the limit a matrix similar to (18) with $\zeta = \delta'/[\gamma' + (\gamma'^2 - \delta'^2)^{1/2}\operatorname{sign}\gamma']$. However, this does not allow us to determine all set (18), but only its subset with real ζ.

8. Consider now the behaviour of the matrix H_0 when $\cos\phi$ is close to zero. To be exact, the behaviour of only $\operatorname{Im} H_0$ requires a special study because the matrix $\operatorname{Re} H_0$, as seen from (14), can be reduced to the necessary form by additionally multiplying it by a conjugate expression; when $\cos\phi = 0$, $\operatorname{Re} H_0$ becomes a zero matrix. As for $\operatorname{Im} H_0$, the same method can be used to transform expression (12):

$$t = 2\sqrt{2}\,|\gamma||\cos\phi|\left[\frac{(1-\gamma^2)\sin^2\phi + \delta^2}{\gamma^2 - 2\gamma^2\cos^2\phi + \cos^2\phi - \delta^2 - d}\right]^{1/2},$$

which results in the reduction of $|\cos\phi|$ in the numerator and denominator of expressions (9) and (13). Because

$$d(\cos\phi = 0) = -\gamma^2 + \delta^2,\ t|\cos\phi|^{-1}(\cos\phi = 0)$$

$$= 2|\gamma|\left(\frac{1-\gamma^2+\delta^2}{\gamma^2-\delta^2}\right)^{1/2},$$

we find that when $\cos\phi = 0$, H_0 is a purely imaginary matrix:

$$H_0(\cos\phi = 0) = \mathrm{i}L^{-1}(1-\gamma^2+\delta^2)^{1/2}$$

$$\times\operatorname{diag}\left[\left(\frac{\gamma+\delta}{\gamma-\delta}\right)^{1/2}, \left(\frac{\gamma-\delta}{\gamma+\delta}\right)^{1/2}\right], \qquad (19)$$

and $H_\pm = H_0 \mp \Psi/2$ are diagonal matrices.

It follows from (19) that on boundary straights $\delta = \pm\gamma$ one of the eigenvalues of the matrix H_0 vanishes, the other turns into imaginary infinity, while H_0 becomes a zero matrix on the hyperbola $\gamma^2 - \delta^2 = 1$. This means that at large $|\gamma|$ the matrix H_0 changes very quickly in the intervals between the hyperbola and its asymptotes.

9. The case when $\delta = \gamma = \cos\phi = 0$ requires a special consideration. In this case, matrix elements in (9), (10) are undefined and the solution $H_0 = \mathrm{i}L^{-1}E$ obtained from (17) is not the only one. At this point matrices $T_\pm$ have two eigenvalues ($+1$ and -1) with corresponding intrinsic subspaces having no fixed sign [22]. For this reason the sought-for set of solutions is defined by two complex parameters $\zeta_{1,2}$ rather than one and can be written, for example, in the form

$$H_0(\delta = \gamma = \cos\phi = 0) = \frac{2\mathrm{i}\operatorname{diag}[\zeta_1,\zeta_2] + (1-\zeta_1\zeta_2)\sigma}{(1+\zeta_1\zeta_2)L}, \qquad (20)$$

where $\operatorname{Re}\zeta_{1,2} > 0$. As a matter of fact, this set includes all symmetric matrices with a positive definite imaginary part and a determinant equal to $-L^{-2}$. We do not present the expressions for $\operatorname{Re} H_0$ and $\operatorname{Im} H_0$ because they are too cumbersome when complex parameters $\zeta_{1,2}$ have a non-zero imaginary part.

It is interesting to analyse the changes in the behaviour of the matrix H_0 in the vicinity of the point under study. Let $\gamma = \varepsilon\gamma'$, $\delta = \varepsilon\delta'$, $\cos\phi = \varepsilon c$, where $|\delta'| < ||\gamma'| - |c||$. If we let ε approach zero, we obtain in the limit the matrix

$$H_0(\varepsilon\to 0) =$$

$$= \frac{\mathrm{i}t'\operatorname{diag}[(\gamma'+\delta')^2 - c^2 + d', -(\gamma'-\delta')^2 + c^2 - d']}{4L\gamma'\delta'|c|} \pm$$

$$\pm\frac{\gamma'^2 - \delta'^2 - c^2 + d'}{2Lc\delta'}\sigma, \qquad (21)$$

where the sign of $\operatorname{Re} H_0$ coincides with that of $\sin\phi$;

$$d' = \left\{\left[(\gamma'+\delta')^2 - c^2\right]\left[(\gamma'-\delta')^2 - c^2\right]\right\}^{1/2}\operatorname{sign}(|c| - |\gamma'|);$$

$$t' = \left[2(\gamma'^2 + c^2 - \delta'^2 + d')\right]^{1/2}.$$

When γ', δ' or c vanish, it is necessary to additionally multiply the numerators and denominators in (21) by conjugate expressions or transform somehow the expression for t' by deriving factor c from it. In this case, if $\delta' = 0$, $|\gamma'| = |c|$, the matrix H_0 is not uniquely defined again. All these transformations are similar to those we described above for the general case and repeating them here does not make point.

Matrices belonging to set (21) can be written in the form (20), where

$$\zeta_1 = \frac{t'\left[\delta'^2 - (\gamma' \pm c)^2 + d'\right]}{4\gamma'|c|(\delta' - \gamma' \mp c)},$$

$$\zeta_2 = \frac{t'\left[\delta'^2 - (\gamma' \mp c)^2 + d'\right]}{4\gamma'|c|(\delta' + \gamma' \mp c)}.$$

It is obvious that set (20) is knowingly wider than this set in which $\zeta_{1,2}$ can be only real.

10. Consider another special case when $\phi = n\pi$, $\sin\phi = 0$, $\cos\phi = \pm 1$. In this case, $d = h_1h_2$, $t = h_1 + h_2$, $H_0 = \mathrm{i}L^{-1}\operatorname{diag}[h_1, h_2]$, where $h_{1,2} = [1-(\gamma\pm\delta)^2]^{1/2} = [1-(L\psi_{1,2}/2)^2]^{1/2}$; as with $\cos\phi = 0$, matrices $H_\pm = H_0 \mp \Psi/2$ are diagonal ones. It is the case of simple astigmatism (if $\delta \neq 0$), which means that the blocks of the monodromy matrix are diagonal matrices and the beam can be described by two independent 2×2 *ABCD* matrices. When $\delta = 0$, the beam is axially symmetric and, despite of multiple eignevalues, the solution remains the only one (unlike the case with an odd number of mirrors [22]).

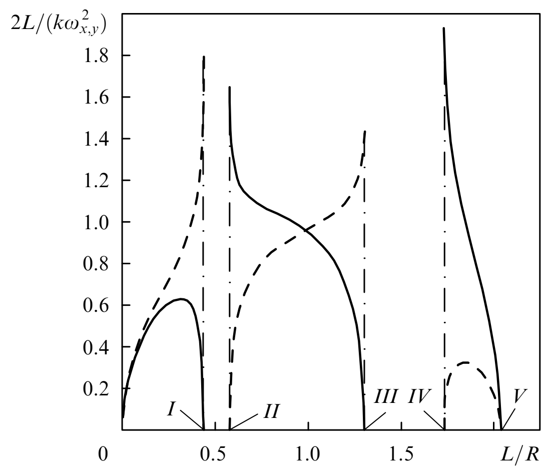

Figure 9. Dependences of eigenvalues of the matrix $L\operatorname{Im} H_0$ on L/R ($\alpha = \pi/3$, $\phi = \pi/3$). The solid curves are the computation along the x axis, the dashed lines – along the y axis. Numbers $I - V$ point to the values of L/R corresponding to the expressions given in the caption of Fig. 4.

11. Let us return to the case when the mirror is spherical. Figure 9 presents the diagonal elements of the dimensionless matrix $L\,\mathrm{Im}\,H_0$ as a function of L/R for constant ϕ and α. It corresponds to the above-considered case when straight $\alpha = \mathrm{const}$ (in Fig. 4) intersects all three subzones of the stability region. Figure 10 presents the dependence of KL on L/R for the same conditions, where K is the proportionality coefficient at the matrix σ in expression (10). Figure 11 shows the dependence of the same characteristics of the matrix H_0 (dimensional) on $|\cos\phi|$ for a resonator considered in [9]. According to our calculations, when we approach the upper and lower bounds of the instability region, different eigenvalues of the matrix $\mathrm{Im}\,H_0$ vanish, which results in the beam size growing infinitely along the x axis in one case and along the y axis in the other, the size in the direction of the other axis decreasing drastically in this case. (Note that at this point our results contradict the dependences presented in Fig. 3 [9] according to which the beam size along the x axis grows in both cases.) When the parameters $\gamma\delta$ approach the instability region boundaries, the modulus of the coefficient K grows quickly, remaining finite.

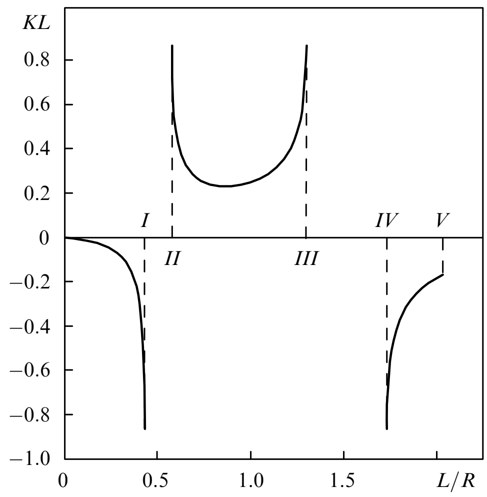

Figure 10. Dependence of KL on L/R for $\alpha = \pi/3$, $\phi = \pi/3$. Numbers $I - V$ point to the values of L/R corresponding to the expressions given in the caption of Fig. 4.

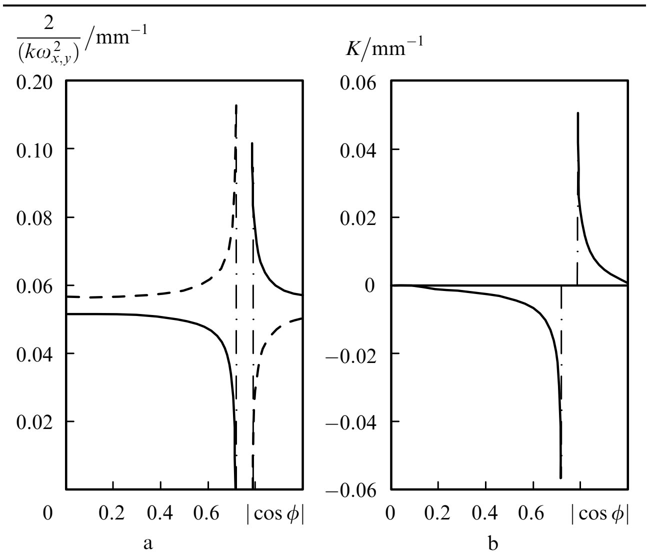

Figure 11. Dependences of eigenvalues of the matrix $\mathrm{Im}\,H_0$ (a) and coefficient K (b) on $|\cos\phi|$ for $R = 50$ mm, $L \approx 12.22$ mm, $\alpha = \pi/3$ ($\gamma \approx 0.7531$, $\delta \approx -0.0353$). When $\tan\phi < 0$, Fig. 11b should be mirrored with respect to the x axis.

The eigenvectors of matrices $\mathrm{Re}\,H_{\mp}$ (8), which define the wavefront shapes of the incident and reflected beams, make the angles

$$\theta_{\mp} = \pm\frac{1}{2}\arctan\frac{\left(\cos^2\phi - \gamma^2 - d + \delta^2\right)\tan\phi}{2\delta^2} \tag{22}$$

with the coordinate axes or the angles differing from the above ones by an integer which is multiple of $\pi/2$ [the expresions correspond to the case when $\gamma\delta\cos\phi \neq 0$ and expressions (9), (10) hold true for H_0]. The dependence of θ_+ on ϕ for a resonator whose parameters are borrowed from [9] ($\gamma \approx 0.7531$, $\delta \approx -0.0353$ and diagonal elements of the matrix $\mathrm{Re}\,H_+$ independent of $|\cos\phi|$ are roughly equal to -0.0231 and $-0.0173\ \mathrm{mm}^{-1}$) is given in Fig. 12a. Figure 12b shows the eigenvalues of the matrix $\mathrm{Re}\,H_+$ (curvatures of the wavefront in the direction of the eigenvectors) as a function of $|\cos\phi|$ (here we do not give the expressions for computing them because they are too cumbersome). In order to obtain similar dependences for the matrix $\mathrm{Re}\,H_-$, all curves in Fig. 12 should be mirrored with respect to the x axis.

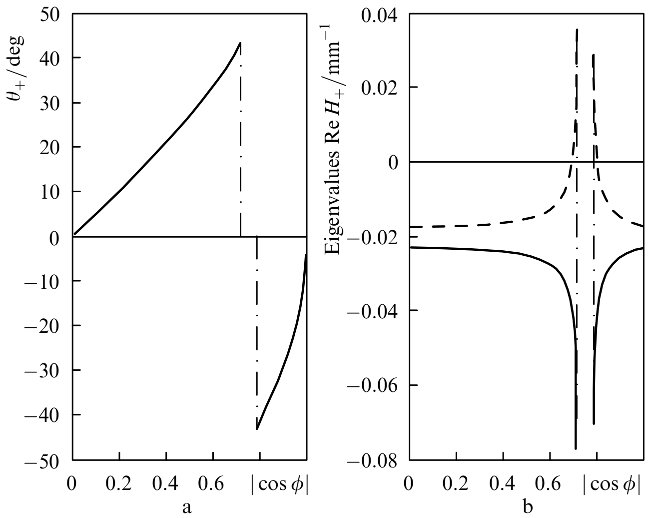

Figure 12. Dependences of the rotation angle θ_+ (a) and eigenvalues of the matrix $\mathrm{Re}H_+$ (b) on $|\cos\phi|$ for $\gamma \approx 0.7531$, $\delta \approx -0.0353$. The eigenvalues corresponding to the solid and dashed curves in Fig. 12b make the angle θ_+ with the x and y axis, respectively. When $\tan\phi < 0$, Fig. 12a should be mirrored with respect to the x axis.

12. Let us return to the evolution of the matrix H when the beam propagates along the resonator contour, which we briefly discussed earlier while considering a four-mirror resonator. Let us transform expression (5) for $H(z)$ taking into account the fact that any matrix satisfies its characteristic equation:

$$H(z) = \frac{H_+ + (z\det H_+)E}{1 + z\,\mathrm{tr}\,H_+ + z^2\det H_+}, \tag{23}$$

where

$$\mathrm{tr}\,H_+ = L^{-1}[\mathrm{i}t|\cos\phi|^{-1} + 2(\gamma - 1)];$$

$$\det H_+ = L^{-2}\left(\gamma^2 - 2\gamma + \cos^2\phi - \delta^2 - d\right)$$
$$\times[1 + \mathrm{i}t(2\gamma|\cos\phi|)^{-1}].$$

The fulfillment of similarity relations (6) between $H(L)$ and H_- is checked immediately.

By multiplying denominator (23) by a complex conjugate expression, we can derive the real and imaginary parts of the matrix $H(z)$. We do not present these expressions because they are too cumbersome. We present here only the expressions for the angles of inclination of the semi-axes of the ellipses of intensity and phase θ_{Im} and θ_{Re} [i.e. eigenvectors of matrices $\mathrm{Im}\,H(z)$ and $\mathrm{Re}\,H(z)$]:

$$\theta_{\mathrm{Im}}(\kappa) = \frac{1}{2}\Big\{-\phi - \arctan\frac{(2\kappa-1)\sin\phi\cos\phi w_-}{[1-2\kappa(1-\kappa)]\cos^2\phi w_- + 2\kappa(1-\kappa)\gamma w_+} + n\pi\Big\}, \tag{24}$$

$$\theta_{\mathrm{Re}}(\kappa) = \frac{1}{2}\Big\{-\phi - \operatorname{arccot}\frac{(2\kappa-1)\cot\phi(\cos^2\phi w_- - w_+)}{[1-2\kappa(1-\kappa)]\cos^2\phi w_- + 2\kappa(1-\kappa)\gamma w_+} + n\pi\Big\}, \tag{25}$$

where

$$\kappa = z/L; \quad w_\pm = \cos^2\phi - \gamma^2 - d \pm \delta^2$$

(the expressions correspond to the case when $\gamma\delta\cos\phi \neq 0$).

Angles (24), (25) are defined up to term $n\pi/2$ (the semi-axes of the ellipses are perpendicular) and counted off from the initial direction of the x axis or from the direction that it assumes after one or several reflections from flat mirrors; the direction of counting is defined by the orientation of the coordinate axes, i.e. by the number of reflections. Passing to another coordinate system involves adding a constant and (when the orientation changes) changing the sign in (24), (25). Note that when deriving expressions (24), (25) we passed to another coordinate system by rotating it by the angle $\phi/2$ with respect to the original one (for the four-mirror resonator presented in Fig. 6, this is a coordinate system linked to the BCD plane). Incidentally, in this system the square matrix calculated at $z = L/2$, $\kappa = 1/2$ (point C in Fig. 6) looks fairly simple:

$$U_{\phi/2}H(L/2)U_{-\phi/2} = \frac{\mathrm{i}t(2\gamma|\cos\phi|)^{-1}\mathrm{diag}\,[u + v\cos\phi, u - v\cos\phi] + (v\sin\phi)\sigma}{L\delta\left[(\gamma+1)(\gamma+\cos^2\phi) - \delta^2\right]}, \tag{26}$$

where

$$u = \delta[w_- + 2\gamma(\gamma + \cos^2\phi)]; \quad v = -\gamma w_+ - w_-,$$

which agrees with the results of paper [20] given different notation.

It is easy to make sure that at some n, the angle $\theta_{\mathrm{Im}}(0) = 0$ (in particular, $n = 0$ when $|\phi| < \pi/2$): the semi-axes of the intensity ellipse are located at the coordinate axes. For this particular n, the angle $\theta_{\mathrm{Im}}(1) = -\phi + n\pi$ (it is measured in the coordinate system rotated by the angle $+\phi$ as a result of the resonator round trip). It is more difficult to verify that the equality $\theta_{\mathrm{Re}}(0) = \theta_+$ holds true for some n; in this case it is enough, in particular, to make sure that $\tan[2\theta_{\mathrm{Re}}(0)] = \tan(2\theta_+)$. The equality $\theta_{\mathrm{Re}}(1) = -\phi + (n - 1/2)\pi + \theta_-$ is also true for such n. In the middle point of the resonator axial contour $\theta_{\mathrm{Im}}(1/2) = (-\phi + n\pi)/2$, $\theta_{\mathrm{Re}}(1/2) = [-\phi + (n-1/2)\pi]/2$ (for arbitrary n), the semi-axes of the phase ellipse are located along the bisectors between the semi-axes of the intensity ellipse. The latter fact is consistent with the form of matrix (26) whose imaginary part is diagonal and real part is antidiagonal. The dependences of θ_{Im} and θ_{Re} on κ for a resonator with the selected parameters are presented in Fig. 13.

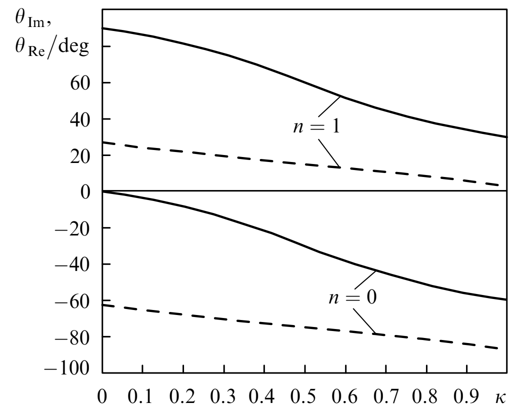

Figure 13. Dependence of the inclination angles of the semi-axes of the intensity ellipse θ_{Im} (solid lines) and phase ellipse θ_{Re} (dashed lines) on $\kappa = z/L$ for $\gamma \approx 0.7531$, $\delta \approx -0.0353$, $\phi = \pi/3$.

13. Consider now the behaviour of the eigenvalues of the matrix $\mathrm{Im}\,H(z)$ equal to $2/(k\omega_{1,2}^2)$, where $\omega_{1,2}$ are the semi-axes of the intensity ellipse, and the eigenvalues of the matrix $\mathrm{Re}\,H(z)$ equal to $R_{1,2}^{-1}$, where $R_{1,2}$ are the major radii of curvature of the wavefront. The analytic expressions for these eigenvalues are too cumbersome to write them directly. However, it did not prevent us from performing numerical calculations whose results for a resonator with the selected parameters are presented in Fig. 14. One can see that in the middle of the beam path the eigenvalues of the matrix $\mathrm{Im}\,H(z)$ achieve the greatest values; hence, $\omega_{1,2}$ prove to be smallest. The eigenvalues of the matrix Re $H(z)$ have equal moduli and opposite signs at this point, and the wavefront resembles a saddle. At the points located symmetrically with respect to the middle of the path where one of the eigenvalues vanishes, the wavefront has a cylindrical shape.

14. It makes sense to compare our results with the alternative representation for a symmetric beam [10]. With the accuracy to notations, we have

$$U_{\phi/2}H(z)U_{-\phi/2} = \frac{(z - L/2)E + \mathrm{i}\,\mathrm{diag}\left[b_1\sinh^2\Phi - b_2\cosh^2\Phi, -b_1\cosh^2\Phi + b_2\sinh^2\Phi\right] - [(b_1 - b_2)\sinh\Phi\cosh\Phi]\sigma}{(z - L/2 - \mathrm{i}b_1)(z - L/2 - \mathrm{i}b_2)}. \tag{27}$$

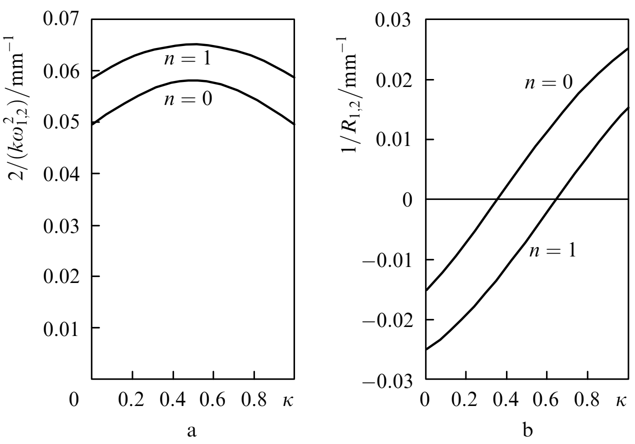

Figure 14. Dependences of eigenvalues of matrices $\mathrm{Im}\,H$ (a) and $\mathrm{Re}\,H$ (b) on $\kappa = z/L$ for $\gamma \approx 0.7531$, $\delta \approx -0.0353$, $\phi = \pi/3$.

Comparing this expression with (26), we find that $\tanh(2\Phi) = 2\gamma \sin\phi |\cos\phi|/t$, the eigenvalues of the matrix $U_{\phi/2} H(L/2) U_{-\phi/2}$

$$\frac{\mathrm{i}}{b_{1,2}} = \frac{\mathrm{i}\left[tu \pm \cos\phi \sqrt{t^2 - (2\gamma \sin\phi)^2}\,\right]}{2\gamma |\cos\phi| L\delta \left[(\gamma + 1)(\gamma + \cos^2\phi) - \delta^2\right]},$$

where the signs in front of the radical are determined, for example, from the condition that the signs of the coefficients for σ in (26) and (27) coincide. The use of the notation of expression (27) allows a fairly compact form of analytical expressions for $R_{1,2}$ and $\omega_{1,2}$, as well as expressions for the rotation angles of the axes of the ellipses similar to (24), (25) (see, for example, [10]).

15. Let us say a few words about possible generalisations of our results. So far we have talked about a perfect lossless resonator which has a real monodromy matrix. However, the expressions describing the transverse field distribution can be easily transformed for the case when γ and δ have a nonzero imaginary part, i.e. the matrix T_Ψ already describes an amplitude-phase quadratic corrector rather than a phase corrector; in other words, for a focusing element the dependence of the coefficient of reflection (transmission) on transverse coordinates obeys the Gaussian law. Of course, expressions (9), (10) (or rather, their analogues) will no longer have the meaning of the real and imaginary parts of the matrix H_0: only the expression for the matrix itself as a linear combination of matrices (9) and (10) with coefficients 1 and i remains valid. The same remark is valid for other similar expressions.

The use of this approach may bring about problems related to the choice of the branch of the root in expressions similar to (11), (12) and to the transformation of expressions containing moduli of some quantities: complex expressions will hold these quantities with signs '+' or '−' instead. The right choice should provide concentration of the solution in the vicinity of the optical axis, i.e. positive definiteness of the imaginary part of the matrix responsible for the transverse field distribution.

It is clear that if the imaginary parts of γ and δ are small and the real parts lie deep inside region (4), there should not be particular difficulties; however, a special analysis can be required in the vicinity of boundary (4). Note that in the case of a complex monodromy matrix the conditions ensuring the existence of solutions of equation (1) with a positive definite imaginary part (i.e. stability conditions) differ significantly from those which we got used to in the real-value case [15, 16].

Another generalisation involves the above-mentioned problem of a resonator with an odd number of mirrors. In this case, the block B of the monodromy matrix proves to be symmetric, which facilitates solution of equation (1) allowing the use of the simplified method [23]. In all other respects this problem is almost as difficult as that under study.

A more complicated problem is considered in [11]. It concerns a nonplanar ring resonator with a symmetrically positioned nonplanar mirror and Gaussian aperture (or two nonplanar mirrors). At the same time, a particular case of $\phi = \pi/2$ investigated in this paper permits a fairly simple analytic solution.

In our paper we have studied only the forms of the transverse distribution of the fundamental mode using a scalar approach and ignoring such important issues as polarisation effects resulted from the rotation of the coordinate system, consideration of higher modes, the spectrum of characteristic frequencies, evaluation of zero-approximation errors, construction of complete asymptotic expansion, etc. All these problems deserve a special study.

16. Let us state the main results of the research. For a ring resonator with a nonplanar axial contour performing image rotation and incorporating an even number of mirrors (including one nonplanar mirror), we have studied in detail the geometry of the stability region in the space of dimensionless parameters inherent in this resonator. We have presented the explicit expressions for the square matrices describing the transverse field distribution of the fundamental mode for all admissible values of these parameters. We have investigated singular points and critical surfaces of the stability region for which the square matrix is not uniquely defined, as well as the behaviour of the matrix in the vicinity of these singular points and critical surfaces. We have studied the dependence of the transverse field distribution on the longitudinal coordinate and presented, in particular, explicit expressions for the angles of inclination of the semi-axes of the intensity and phase ellipses as functions of this coordinate.

The results are detailed for the case when the nonplanar mirror has a spherical shape. The beam parameters as functions of the radius of curvature and rotation angle of the image have been investigated for this resonator. In the particular case of a four-mirror resonator, the relation between the image rotation angle and the ring path geometry is characterised.

In our calculations we have used a resonator whose parameters are borrowed from paper [9]. Comparison of the results of calculations and the dependences based on them with the results presented in this paper and obtained by using conventional methods has revealed both similarities and differences in the qualitative and quantitative results.

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