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Spectral properties of a ring optical resonator with an arbitrary longitudinal inhomogeneity

V.F. Soudakov

***Abstract*. A ring optical resonator with an arbitrary but continuous change in the permittivity of the filling medium along the resonator axis is considered. It is shown that in the case of a small deviation of the permittivity from its average value, the double degeneracy of eigenfrequencies inherent in a homogeneous resonator is removed and the corresponding modes acquire the properties of standing waves. Simple universal expressions are derived to calculate eigenfrequencies and distribution coefficients in modes. Conditions are found under which splitting in the frequency spectrum of an inhomogeneous resonator is absent. The general results obtained in the paper can be used in numerical experiments.**

Keywords: *ring resonator, eigenfrequency, mode, nonreciprocity.*

1. For ideal homogeneously filled stable optical cavities without intracavity, output and diffraction losses, the solution of the spectral problem is known in the high-frequency approximation. The inhomogeneous filling is one of the types of perturbation of an ideal optical resonator. In practice, inhomogeneities are caused by intracavity elements. Usually, measures are taken to eliminate backscattering from these inhomogeneities but even quality samples do not guarantee complete absence of backscattering, and consequently, the inhomogeneity as its reason. In addition, it is impossible to eliminate the possibility of using partially reflecting intracavity elements to produce the required spectral properties of a complex cavity. Although the inhomogeneity is usually of a mixed type, it is expedient to select a purely transverse (depending only on the cross section coordinates, normal to the base contour of the optical resonator) and a purely longitudinal (depending only on the coordinate z along the base contour axis) inhomogeneities. Such a division in the inhomogeneity types makes it possible to simplify considerably the optical resonator model and to obtain important qualitative results. The transverse inhomogeneity mainly determines the mode transverse structure and eigenfrequencies of higher-order modes. Zero-order (longitudinal) modes and their frequencies are changed when the longitudinal inhomogeneity is introduced.

V.F. Soudakov N.E. Bauman Moscow State Technical University, 2-ya Baumanskaya ul. 5, 107005 Moscow, Russia; e-mail: vvffss@rol.ru

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In this paper, the perturbation of an optical resonator by a longitudinal inhomogeneity is considered. In this case, it is sufficient to consider a one-dimensional resonator model and to study one of the most complicated spectral problems appearing in the theory of ring optical resonators (RORs), when an ideal resonator possesses a doubly degenerate spectrum. It was found that the inhomogeneity can eliminate this degeneracy and change the character of longitudinal modes. It is convenient to study the influence of these perturbations on the ROR spectrum by particular examples. The spectral problem for a ROR with the simplest types of the longitudinal inhomogeneity was approximately solved in [1, 2], where the splitting of the degenerate spectrum in an ideal resonator was estimated and mode peculiarities were found (the results of other papers on this subject are also presented). In the presence of splitting, the spectrum becomes simple and the modes are the standing waves (in an ideal ROR for each eigenfrequency they are usually assumed to be two counterpropagating travelling waves). The conditions under which splitting is absent were found.

It was assumed in [1, 2] that longitudinal inhomogeneities have discontinuities. This allows one to use Fresnel formulae and reflection and transmission coefficients at points of discontinuities. Thus, the spectral problem for differential operators can be reduced to algebraic. Although the method of papers [1, 2] remains suitable in the case of any piecewise-uniform filling medium, the disadvantage of this model is a cumbersome computation algorithm. Therefore, it is reasonable to abandon the model of 'sharp' inhomogeneities (significantly changing at the wavelength) and consider 'soft' inhomogeneities. First, inhomogeneities of this type can exist in the resonator. Second, it is impossible to predict beforehand how adequate 'sharp' inhomogeneities can be described within the framework of a 'soft' model.

The aim of this paper is to solve the spectral problem for the ROR with a rather smooth and weak change in a filling inhomogeneity of an arbitrary type and to determine the spectral properties and the mode structure depending on the integral parameters of the variable refractive index. The general solution of this problem can be rather simple and easily reproducible when the inhomogeneity type is specified. Limitations on the degree of the inhomogeneity smoothness should be also specified for which this solution can be obtained.

2. Let us formulate the spectral problem for a longitudinally inhomogeneous ROR and transform it to the form convenient for deriving the approximate solution. Because the optical properties of a filling medium change only along the ROR axis, the derivatives over the transverse coordinates can be neglected and the field $E(z,t)$ can be assumed one-dimensional ($0 \leqslant z \leqslant L$, where L is the ROR perimeter). For simplicity, its polarisation is assumed linear. By considering the stationary oscillations with the frequency ω, we obtain the equation

$$\frac{d^2E}{dz^2} + k^2Q^2(z)E = 0 \tag{1}$$

for the amplitude of these oscillations. The wave number k and the refractive index $Q(z)$ are introduced by the relations $k^2 = \omega^2\mu\varepsilon_0$ and $\varepsilon(z)/\varepsilon_0 = Q^2(z)$, where ε_0 is the permittivity of an empty resonator; the magnetic permeability μ is constant and the permittivity $\varepsilon(z)$ arbitrarily depends on the coordinate. We assume that $Q^2(z) = 1 + \delta(z)$, where $0 \leqslant \delta(z) \ll 1$. In the optical frequency range, k is large and solutions (1) can be sought for in the asymptotic approximation. However, expression (1) is hardly suitable for this type of approximations. Therefore, we will use the known transformation to write (1) in the normal Liouville form:

$$-\frac{d^2u}{dx^2} + q(x)u = \chi^2u. \tag{2}$$

Here, we introduce a new variable

$$x(z) = \frac{1}{\Lambda}\int_0^z Q(z)dz, \tag{3}$$

a new function

$$u(x) = Q^{1/2}(z(x))E(z(x)) \tag{4}$$

and a constant

$$\Lambda = \frac{1}{\pi}\int_0^L Q(z)dz. \tag{5}$$

The variation interval in z is transformed into the variation interval $(0,\pi)$ in x. The parameter k of Eqn (1) changes to a large parameter $\chi = \Lambda k$. The variable coefficient $q(x)$ is independent of k, which allows one to seek for the asymptotic approximation of the solution of Eqn (2) and by using expressions (3) and (4) to pass from it to the required solution of expression (1).

The ROR modes should satisfy the boundary conditions

$$E(0) = E(L), \quad \frac{dE(0)}{dz} = \frac{dE(L)}{dz}.$$

In this connection, it is necessary to seek for the eigenfunctions and their corresponding eigenvalues of equation (2) with boundary conditions of a periodic type:

$$u(0) = u(\pi), \quad \frac{du(0)}{dx} = \frac{du(\pi)}{dx}. \tag{6}$$

3. Consider an auxiliary spectral problem generated by relations (2), (6), where the spectral parameter χ is related to the frequency and can be assumed large. The coefficient $q(x)$ is uniformly smaller than χ, which imposes certain limitations on the smoothness degree of the function $\delta(z)$ (see below). We will solve this spectral problem (determination of eigenvalues and eigenfunctions) in an explicit form but approximately for $\chi \to \infty$.

As the fundamental system of solutions (FSS) of expression (2) we will select two its solutions, i.e. $\phi(x,\chi)$ and $\theta(x,\chi)$.

$$\phi(0,\chi) = 1, \ \frac{d\phi(0,\chi)}{dx} = 0, \ \theta(0,\chi) = 0, \ \frac{d\theta(0,\chi)}{dx} = 1.$$

It is obvious that these solutions also satisfy the integral equations

$$\phi(x,\chi) = \cos(\chi x) + \frac{1}{\chi}\int_0^x \sin[\chi(x-x')]q(x')\phi(x',\chi)dx', \tag{7}$$

$$\theta(x,\chi) = \frac{1}{\chi}\sin(\chi x) + \frac{1}{\chi}\int_0^x \sin[\chi(x-x')]q(x')\phi(x',\chi)dx'. \tag{8}$$

Any solutions of problem (2), i.e. the eigenfunction and its derivative can be represented in the form of linear combinations of functions from the FSS and their derivatives, respectively:

$$u(x,\chi) = c_1\phi(x,\chi) + c_2\theta(x,\chi),$$

$$\frac{du(x,\chi)}{dx} = c_1\frac{d\phi(x,\chi)}{dx} + c_2\frac{d\theta(x,\chi)}{dx}.$$

The constants c_1, c_2 are such that the reduced expansions satisfy boundary conditions (6):

$$c_1 = c_1\phi(\pi,\chi) + c_2\theta(\pi,\chi), \quad c_2 = c_1\frac{d\phi(\pi,\chi)}{dx} + c_2\frac{d\theta(\pi,\chi)}{dx}. \tag{9}$$

The nonzero solution of uniform system (9) exists only at

$$\det\begin{pmatrix} \phi(\pi,\chi) - 1 & \theta(\pi,\chi) \\ \dfrac{d\phi(\pi,\chi)}{dx} & \dfrac{d\theta(\pi,\chi)}{dx} - 1 \end{pmatrix} = 0.$$

It follows that the eigenvalues should be the roots of the equation

$$\frac{d\theta(\pi,\chi)}{dx} + \phi(\pi,\chi) - 2 = 0. \tag{10}$$

To derive asymptotic expressions for the roots of Eqn (10), we will first find the asymptotic expressions for the FSS. For this purpose, we will use integral expressions (7), (8). Because the integral term is of the order $O(1/\chi)$, we can pass to the first iterations of their solution:

$$\phi(x,\chi) \approx \cos(\chi x) + \frac{1}{\chi}\int_0^x \sin[\chi(x-x')]q(x')\cos(\chi x')dx',$$

$$\frac{d\theta(x,\chi)}{dx} \approx \cos(\chi\pi) + \int_0^x \cos[\chi(x-x')]q(x')\cos(\chi\pi)dx'.$$

By using the second iteration, we can obtain approximations of the order $O(1/\chi^2)$ for $\phi(\pi,\chi)$ and $d\theta(\pi,\chi)/dx$:

$$\phi(\pi,\chi) \approx \cos(\chi\pi) + \frac{1}{\chi}\int_0^\pi dx'\sin[\chi(\pi-x')]q(x')\cos(\chi x') +$$

$$+\frac{1}{\chi^2}\int_0^{\pi}\mathrm{d}x'\sin[\chi(\pi-x')]q(x')$$

$$\times\int_0^{x'}\mathrm{d}x''\sin[\chi(x'-x'')]q(x'')\cos(\chi x''), \qquad (11)$$

$$\frac{\mathrm{d}\theta(\pi,\chi)}{\mathrm{d}x}\approx\cos(\chi x)+\frac{1}{\chi}\int_0^{\pi}\mathrm{d}x'\cos[\chi(\pi-x')]q(x')\sin(\chi x')$$

$$+\frac{1}{\chi^2}\int_0^{\pi}\mathrm{d}x'\cos[\chi(\pi-x')]q(x')$$

$$\times\int_0^{x'}\mathrm{d}x''\sin[\chi(x'-x'')]q(x'')\sin(\chi x''). \qquad (12)$$

If we substitute (11), (12) into Eqn (10), which is used to derive eigenvalues, after some transformations we obtain

$$\frac{\mathrm{d}\theta(\pi,\chi)}{\mathrm{d}x}+\phi(\pi,\chi)-2=2[\cos(\chi\pi)-1]+\frac{A_{-1}}{\chi}\sin(\chi\pi)$$

$$+\frac{A_{-2}(\chi)}{\chi^2}, \qquad (13)$$

where

$$A_{-1}=\int_0^{\pi}q(x')\mathrm{d}x';$$

$$A_{-2}(\chi)=\int_0^{\pi}\mathrm{d}x'q(x')\int_0^{x'}\mathrm{d}x''\sin[\chi(x'-x'')] \qquad (14)$$

$$\times\sin[\chi(\pi-x'+x'')]q(x'').$$

The order of the quantity $A_{-2}(\chi)$ is equal to $O(1/\chi)$, which follows from the properties of the Fourier transform of a rather smooth function $q(x)$. Roots of Eqn (10) taking (13) into account can be found in the same approximation: $\chi_n\pi=2\pi n+\delta_n$ $(n=0,1,2,\ldots)$. Substituting this expression into (1) makes it possible to derive the approximate equation with respect to δ_n:

$$0=2[\cos(\chi_n\pi)-1]+\frac{A_{-1}}{\chi_n}\sin(\chi_n\pi)+\frac{A_{-2}(\chi_n)}{\chi_n^2}$$

$$\approx-\delta_n^2+\frac{A_{-1}}{2\pi n}\delta_n+\frac{A_{-2}(2\pi n)}{(2\pi n)^2}.$$

From here we obtain two small corrections $\delta_n^{\pm}$:

$$\delta_n^{\pm}=\frac{1}{2\pi n}\left\{\frac{A_{-1}}{2}\pm\left[\left(\frac{A_{-1}}{2}\right)^2+A_{-2}(2\pi n)\right]^{1/2}\right\}. \qquad (15)$$

The order of these corrections is equal to $O(1/\chi)$ but they also take into account the components of the order $O(1/\chi^2)$. It follows from (15) that the eigenvalues produce two series of discrete real values:

$$\chi_n^{\pm}=2\pi n+\frac{1}{2\pi n}\frac{A_{-1}}{2}\pm\delta_{n0},$$

$$\delta_{n0}=\frac{1}{2\pi n}\left[\left(\frac{A_{-1}}{2}\right)^2+A_{-2}(2\pi n)\right]^{1/2}. \qquad (16)$$

The decomposition of the spectral set into two series of eigenvalues fully agrees with the known theorem of the general spectral theory (see, for example [3]). At the same time, the asymptotic accuracy found by us is higher that in known papers of the general type and is sufficient to obtain physically significant results. It follows from (16) that eigenvalues produced doublets with centres at points

$$\chi_{n0}=2\pi n+\frac{1}{2n}\frac{A_{-1}}{2\pi} \qquad (17)$$

upon splitting in the doublet

$$\Delta\chi_n=\frac{2}{\pi}\delta_n=\frac{1}{\pi n}\left[\left(\frac{A_{-1}}{2}\right)^2+A_{-2}(2\pi n)\right]^{1/2}. \qquad (18)$$

Expressions (17), (18) can be specified if we take into account explicit expression (14) for $A_{-2}(\chi)$. In the approximation $\chi_n^{\pm}=2\pi n$, we obtain

$$A_{-2}(2\pi n)=-\int_0^{\pi}\mathrm{d}x'q(x')\int_0^{x'}\mathrm{d}x''\sin^2\left[2\pi n(x'-x'')\right]q(x'')$$

$$\approx-\frac{1}{4}A_{-1}^2+\frac{1}{4}\left(\frac{\pi}{2}\right)^2(q_{2c}^2+q_{2s}^2),$$

where

$$q_{2c}=\frac{2}{\pi}\int_0^{\pi}q(x'')\cos(4\pi nx'')\mathrm{d}x'';$$

$$q_{2s}=\frac{2}{\pi}\int_0^{\pi}q(x'')\sin(4\pi nx'')\mathrm{d}x''. \qquad (19)$$

Thus, splitting in the doublet of eigenvalues significantly depends on the amplitudes of the sine (odd) and cosine (even) harmonics with the spatial frequency $4\pi n$ in the expansion of the periodic function $q(x)$ in the Fourier series. It follows that using (18) it is easy to obtain that

$$2\Delta\chi_n=\frac{1}{4\pi n}\left(q_{2c}^2+q_{2s}^2\right)^{1/2}. \qquad (20)$$

In a homogeneous resonator, $q(x)=\text{const}$, the amplitudes of the harmonics vanish and splittings in the doublets (i.e. the proper doublets) disappear as is the case in the absence of nonreciprocity.

4. Let us find asymptotic expressions for the eigenfunctions of spectral problem (2), (6). In the mathematical literature (see, for example, [3]), instead of problem with boundary conditions (6) two other problems with other zero boundary conditions of a regular type [4] are proposed to be solved. This methyod is rather complicated, and therefore, we offer here another purely formal (without rigorous mathematical substantiation) approach, which is quite suitable for physical applications. It is intuitively understandable, easily reproducible and can be used in other similar situations. For further action, it is sufficient to seek for the eigenfunctions in the approximation $O(1/\chi)$. It is obvious that the eigenfunction corresponding to $\chi_n^{\pm}$ is defined with an accuracy to the arbitrary constant c_2:

$$u_n^{\pm}(x)=c_2\left[\frac{c_1}{c_2}\phi(x,\chi_n^{\pm})+\chi_n^{\pm}\theta(x,\chi_n^{\pm})\right]. \qquad (21)$$

The ratio c_1/c_2, which we will call the distribution coefficient, is determined unambiguously from the boundary conditions, i.e. the system of equations (9) for the zero determinant of its matrix coefficients:

$$\left(\frac{c_1}{c_2}\right)_n^{\pm} = \frac{\chi_n^{\pm}\theta(\pi,\chi_n^{\pm})}{1-\phi(\pi,\chi_n^{\pm})}.$$

The Wronskian of the FSS at the point $x=0$ is 1, which means that it is identically equal to 1. This allows one to rewrite the last equality in the form:

$$\left(\frac{c_1}{c_2}\right)_n^{\pm} = -\frac{\chi_n^{\pm}\theta(\pi,\chi_n^{\pm})}{1-\mathrm{d}\theta(\pi,\chi_n^{\pm})/\mathrm{d}x}. \tag{22}$$

Let us present the asymptotic estimates necessary for the calculation. It follows from (8) that

$$\chi_n^{\pm}\theta(\pi,\chi_n^{\pm}) \approx \sin(\chi_n^{\pm}\pi) + \frac{1}{\chi_n^{\pm}}\int_0^{\pi}\sin\left[\chi_n^{\pm}(\pi-x')\right]q(x')\sin(\chi_n^{\pm}x')\mathrm{d}x'.$$

By using estimate (16), we obtain the asymptotics

$$\chi_n^{\pm}\theta(\pi,\chi_n^{\pm}) \approx \pm\sin\delta_n + \frac{1}{2\pi n}\int_0^{\pi}q(x')\sin(\pm\delta_n-2\pi nx')\sin(2\pi nx')\mathrm{d}x'.$$

Further approximations and simple trigonometric transformations allow one to reduce this expression to the final form:

$$\chi_n^{\pm}\theta(\pi,\chi_n^{\pm}) \approx \frac{1}{4\pi n}\frac{\pi}{2}\left[\pm\left(q_{2c}^2+q_{2s}^2\right)^{1/2}+q_{2c}\right]. \tag{23}$$

It follows from (12) that

$$\frac{\mathrm{d}\theta(\pi,\chi_n^{\pm})}{\mathrm{d}x} = \cos(\chi_n^{\pm}\pi) + \frac{1}{\chi_n^{\pm}}\int_0^{\pi}\cos[\chi_n^{\pm}(\pi-x')]q(x')\sin(\chi_n^{\pm}x')\mathrm{d}x'.$$

The substitution of asymptotic representation (16) allows one to simplify this expression:

$$\frac{\mathrm{d}\theta(\pi,\chi_n^{\pm})}{\mathrm{d}x} = \cos\delta_n^{\pm} + \frac{1}{4\pi n}A_{-1}\left(\delta_n^{\pm}+\frac{\pi}{2}q_{2s}-\frac{\pi}{2}q_{2c}\delta_n^{\pm}\right) \approx 1+\frac{1}{4\pi n}A_{-1}\frac{\pi}{2}q_{2s}. \tag{24}$$

By substituting (23) and (24) into (22), we obtain

$$\left(\frac{c_1}{c_2}\right)_n^{\pm} = -\frac{1}{4\pi n}\left[\pm\frac{\pi}{2}\left(q_{2c}^2+q_{2s}^2\right)^{1/2}+\frac{\pi}{2}q_{2c}\right]\left(\frac{1}{4\pi n}\frac{\pi}{2}q_{2s}\right)^{-1} = -\frac{\pm\left(q_{2c}^2+q_{2s}^2\right)^{1/2}+q_{2c}}{q_{2s}}. \tag{25}$$

Further study of distribution coefficient (25) requires taking into account the specific character of the longitudinal inhomogeneity. If a reference cross section $x=0$ can be chosen so that the auxiliary function $q(x)$ should be either even with respect to it or should differ insignificantly from even, then $|q_{2s}/q_{2c}| \ll 1$. In this case, expression (25) can be further simplified:

$$\left(\frac{c_1}{c_2}\right)_n^{\pm} \approx -\frac{\pm q_{2c}\pm 2q_{2s}^2/q_{2c}+q_{2c}}{q_{2s}}.$$

It follows from here that the distribution coefficients are markedly different for the eigenvalues of this doublet and the corresponding eigenfunctions are also different:

$$\left(\frac{c_1}{c_2}\right)_n^{-} = 2\frac{q_{2s}}{q_{2c}}, \quad u_n^{-}(x) = c_2\chi\theta(x,\chi) \equiv \sin(\chi_n^{-}x), \tag{26}$$

$$\left(\frac{c_1}{c_2}\right)_n^{+} = -2\left(\frac{q_{2c}}{q_{2s}}+\frac{q_{2s}}{q_{2c}}\right), \quad u_n^{+}(x) = c_2\phi(x,\chi_n^{+}) \equiv \cos(\chi_n^{+}x). \tag{27}$$

In (26) the distribution coefficient in modulus is much smaller than unity (the eigenfunction is close to the odd function), while in (27) – much larger than unity (the eigenfunction is close to the even function). One can write out more exact expressions for the eigenfunction, which is necessary for their following application:

$$u_n^{-}(x) \equiv 2q_{2s}(\cos\chi_n^{-}x)+q_{2c}\sin(\chi_n^{-}x),$$

$$u_n^{+}(x) \equiv -2q_{2c}\cos(\chi_n^{+}x)+q_{2s}\sin(\chi_n^{-}x). \tag{28}$$

Here, eigenfunctions are determined accurate to the constants, which can be unambiguously selected from the ordinary normalisation conditions (to unity) of the eigenfunction. It is easy to show than in both cases these constants are equal to $(2/\pi)^{1/2}$. The orthogonality of real eigenfunctions is considered in the general sense. The orthogonality, as the reality of the eigenvalue, follows from the general theorems of the spectral theory [3].

5. The obtained results allow one to make a conclusion about the spectrum of ROR eigenfrequencies and modes. In this case, it is sufficient to use coupling relations (3)–(5). It is obvious that the ROR eigenfrequencies produce a discrete sequence of doublets

$$\omega_n^{\pm} = \frac{1}{\mu\varepsilon_0}\frac{1}{\Lambda}\chi_n^{\pm},$$

where the eigenvalues $\chi_n^{\pm} = \chi_{n0} \pm \frac{1}{2}\Delta\chi_n$ are defined by expressions (17), (18) and the resonator optical length Λ is calculated by using expression (5). The ROR modes represent inhomogeneous standing (real) waves. The spectrum is simple, i.e. only one mode corresponds to the eigenfrequency. To derive an explicit expression for the mode, we will use substitutions (3), (4) in expressions (28) for the eigenfunctions:

$$E_n^{\pm}(z) = Q^{-1/2}(z)u_n^{\pm}\left(\frac{1}{\Lambda}\int_0^z Q(z)\mathrm{d}z\right). \tag{29}$$

The orthonormalisation of modes in an inhomogeneous ROR considered here differs from the orthonormalisation of eigenfunctions. Let $E_{1,2}(z)$ be two modes and $u_{1,2}(x)$ be the eigenfunctions corresponding to them. By using expressions (3), (4), we obtain

$$\frac{1}{\Lambda}\int_0^L Q^2(z)E_1(z)E_2(z)\mathrm{d}z = \int_0^L u_1(x)u_2(x)\mathrm{d}x.$$

Therefore, the mode normalisations has the form

$$\frac{1}{\Lambda}\int_0^L q^2(z)E_{1,2}^2(z)\mathrm{d}z = 1,$$

and the condition for their orthogonality is

$$\int_0^L Q^2(z)E_1(z)E_2(z)\mathrm{d}z = 0.$$

6. The frequency spectrum of a one-dimensional lossless ROR with an arbitrary nonuniform filling represents a discrete doublet sequence of real numbers, the doublet centres with a higher degree of accuracy corresponding to the frequencies of a homogeneous ROR with the same perimeter (corrected to the average refractive index). We have obtained asymptotic estimates of eigenfrequencies:

$$\omega_n^{\pm} = \frac{1}{\mu\varepsilon_0}\frac{1}{\Lambda}\chi_n^{\pm},$$

where $\chi_n^{\pm}$ are defined by expressions (16)–(18). The frequency interval in each doublet is determined by expression (18) and associated with the harmonic amplitude of the refractive index at the dimensionless spatial frequency approximately equal to $4\pi n$, where n is the doublet number. If the fundamental frequency (i.e. the number $n = n_0$) corresponds to the wavelength λ_0, the mentioned complex amplitude corresponds to the spatial harmonic with the wavelength $\lambda_0/2$. If the inhomogeneous ROR tends to an ideal one (resonator 'bleaching'), it is desirable to suppress maximally this harmonic.

In the same approximation we have found the distribution coefficient in the inhomogeneous ROR mode [see expressions (28), (29)]. In some particular cases, the distribution coefficient depends only on the ration of the real and imaginary parts of the above complex amplitude, i.e. on its phase. In the general case, the modes of each doublet represent standing waves orthogonal in the generalised sense and nearly orthogonal in the general sense (both sine and cosine). If the conditions for which splitting disappears are fulfilled, the spectrum approximately can be assumed degenerate and the corresponding modes – travelling waves (although in the exactly degenerate spectrum, they can be treated as orthogonal standing waves or any their linear combination).

The general results obtained in this paper are rather universal and simple. They can be easily used in numerical calculations. We have established that the derived expressions are valid if the largest change in the permittivity δ_{pl} occurs in the intermediate layer, whose size is

$$\Delta z > L\left(\frac{\delta_{\mathrm{pl}}}{\chi_{n_0}}\right)^{1/2} = \left(L\lambda_0\frac{\delta_{\mathrm{pl}}}{2\pi}\right)^{1/2}.$$

Under typical conditions, $\Delta z > 0.1$ mm, although this restriction, in our opinion, can be less stringent (the obtained expressions will be valid at smaller Δz).

Usually, a quantitative analysis should precede a numerical experiment. In the case under study, general expressions derived in this paper can be used for this analysis (in this case, it is possible to do without the numerical experiment). A significant result is the analytic determination of conditions for which spectrum splitting disappears.

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