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Cascade of torus doubling bifurcations in a detuned laser

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***Abstract*.** **By using a simplified system of Maxwell – Bloch equations (with the adiabatically excluded polarisation of the medium), we studied the processes proceeding in the cross section of a light wave propagating in a wide-aperture laser emitting at the frequency detuned from the transition-line centre. It is shown that in the model under study the passage to the chaotic regime during a change in the wave propagation velocity across the aperture occurs via the doubling bifurcations of an ergodic two-dimensional torus. The spectrum of Lyapunov exponents is found and it is established that at bifurcation points a structurally unstable three-dimensional torus is produced, which gives rise to a stable doubled ergodic torus.**

Keywords: *wide-aperture lasers, torus doubling bifurcation, ergodic torus, Lyapunov characteristic exponent, chaos.*

1. Introduction

At present processes proceeding in the cross section of a light wave propagating in wide-aperture lasers and passive optical systems are being extensively studied. It has been shown in [1] that as the Fresnel number in a wide-aperture Nd : YAG laser is increased, the stationary transverse intensity patterns change to periodic, quasi-periodic and chaotic patterns. Similar results have been obtained in paper [2] for an electric-discharge single-longitudinal mode CO_2 laser. In a wide-aperture laser (for Fresnel numbers above 30), the intensity that is almost constant over the beam cross section becomes periodically modulated in space and weakly modulated in time with the frequency ~ 150 kHz depending on the mode-frequency detuning from the transition-line centre. As the Fresnel number, frequency detuning or the pump power are increased, the dependences of the intensity on the transverse coordinate and time become strongly irregular and the spatial and temporal correlations are completely lost. It was shown for the first time in [3] that these effects can be qualitatively explained by using a simple system of Maxwell – Bloch equations with a detuning from the frequency of a longitudinal mode describing the production of travelling periodic waves due to the Andronov – Hopf bifurcation.

The authors of paper [4] found the conditions under which it is possible to exclude adiabatically the medium polarisation from Maxwel – Bloch equations. The study of such simplified equations also showed that in the case of negative frequency detuning, the periodic intensity modulations of waves travelling across the aperture may arise; the growth increments, the frequency and the velocity of these waves were found analytically [5 – 7].

In this paper, we used a self-similar system of equations [5] to study the change in the optical field structure in the beam cross section from the periodic to chaotic during the variation of the wave propagation velocity across the aperture and showed that the passage to chaos proceeds via a cascade of torus doubling bifurcations.

2. Basic equations. Linear stability analysis

Consider the initial system of equations, as in [5]:

$$\frac{\partial E}{\partial t} - \mathrm{i}\frac{\partial^2 E}{\partial x^2} = \frac{\nu}{2}E\left(\frac{N}{1+\Delta_0^2} - 1\right) + \frac{\mathrm{i}\nu E}{2}\left(2\Delta_{\mathrm{cav}} - \frac{\Delta_0}{1+\Delta_0^2}N\right), \quad (1)$$

$$\frac{\partial N}{\partial t} = N_{\mathrm{un}} - N\left(1 + \frac{J}{1+\Delta_0^2}\right).$$

The system of equations (1) describes the spatiotemporal dynamics of laser radiation assuming that the medium polarisation instantly follows changes in the optical field. This model is often called in the literature the standard adiabatic exclusion of polarisation. It is also assumed that lasing occurs at one longitudinal mode of the Fabry – Perot cavity. Here, E is a slowly varying field amplitude in a wide-aperture laser in a one-dimensional (planar) approximation, which is normalised to the saturation field amplitude E_s in the active medium; $N = g/g_t$; g, g_t are the gains in the active medium at the central frequency of the laser transition and losses averaged over the cavity length, respectively; the dimensionless time t and coordinate x are related with

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dimensional quantities t_d and x_d as $t = t_d/T_i$; $x = x_d \times (2k/T_i c)^{1/2}$; k is the wave number; c is the speed of light; T_i is the population relaxation time of the active-medium levels; $v = cT_i g_t$ is the coefficient determining the ratio of the relaxation time of the active-medium population to the photon lifetime in the cavity; $\Delta_{cav} = (\omega - \omega_{cav})/cg_t$ is the laser frequency detuning ω from the mode frequency of an empty cavity ω_{cav} normalised to the linewidth in the cavity; $\Delta_0 = (\omega_0 - \omega)T_p$ is the laser frequency detuning from the central frequency ω_0 of the gain line of the active medium normalised to the half-width of the gain line; T_p is the relaxation time of polarisation; $J = |E|^2$; $N_{un} = g_{un}/g_t$; and g_{un} is the unsaturated gain at the frequency ω_0.

Applicability conditions (1) were discussed in papers [8–10] and are as follows. First, the field amplitude weakly changes during the round-trip transit in the cavity; second, a rather large number of transverse modes or one longitudinal cavity mode or several longitudinal modes whose transverse structures differ insignificantly fall into the homogeneous gain line. This is fulfilled under the condition $\Delta\omega/\omega \ll N_F^{-1/2}$, where $\Delta\omega/\omega$ is the relative width of the laser emission spectrum and N_F is the Fresnel number.

The system of equations (1) has two homogeneous equilibrium states. The first of them corresponds to the absence of lasing ($E = 0$, $N = N_{un}$) and the second ($E = E_{st}$, $N = N_{st} = 1 + \Delta_0^2$, $\Delta_{cav} = \Delta_0/2$) – to stationary lasing with the intensity $J_{st} \equiv |E_{st}|^2 = N_{un} - 1 - \Delta_0^2$.

We will seek for the self-similar solution of the system of equations (1) in the form of intensity waves travelling with a constant velocity. For this purpose, we will use the self-similar change of variables $\xi = t - \beta x$, where β^{-1} is the velocity of the wave propagating across the cavity axis. The model neglects the cavity aperture finiteness.

The linear analysis performed in [5] showed that if the frequency detuning is $\Delta_0 < 0$, then at

$$\beta > \beta_{cr} = \left[\frac{-(1 + I_{st})\Delta_0}{(1 + I_{st})^2 + \Delta_0^2 v I_{st}}\right]^{1/2}, \quad I_{st} = \frac{J_{st}}{1 + \Delta_0^2}$$

the regime of uniform stationary lasing proves to be unstable. At $\beta = \beta_{cr}$, the Andronov–Hopf bifurcation takes place, which produces a family of periodic small-amplitude intensity waves parametrised with the help of β. The intensity modulation frequency at the bifurcation point is

$$\omega_{bif} = \frac{(N_{un}^2 + \Delta_0^2 v I_{st})^{1/2}}{|\Delta_0|}.$$

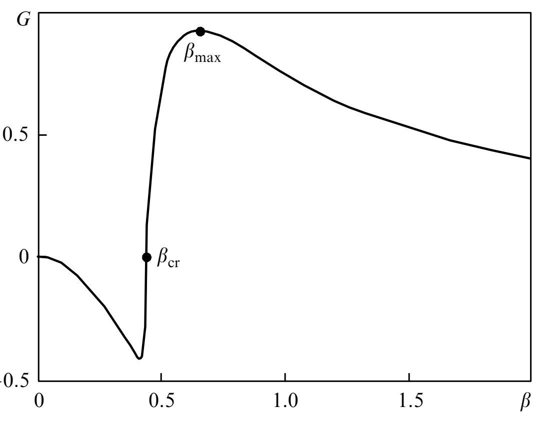

Figure 1. Intensity growth increment G as a function of β.

Figure 1 shows the dependence of the instability increment G in the stationary state on β for the laser system parameters $N_{un} = 5$, $v = 4.4$, $\Delta_0 = -1$ taken from [5]. One can see from the figure that the increment is maximal for $\beta = \beta_{max}$. It can be expected that without the external action (for example, the tilt of the mirror), the waves with the maximum growth increment will dominate on the aperture. The value of β_{max} is determined by the laser system parameters and can significantly exceed the bifurcation value β_{cr}. Below, we study in detail the change in the nonlinear dynamics of laser radiation with increasing the parameter $\varepsilon = (\beta - \beta_{cr})/\beta_{cr}$.

3. Dynamics of a laser system at $\beta > \beta_{cr}$

We study the nonlinear stage in the development of perturbations of the uniform stationary lasing regime at $\Delta_0 < 0$ by the numerical simulation of the system of equations (1) written in the self-similar form (by using the change of variables $\xi = t - \beta x$). For this purpose, the complex field is represented in the form $E = E' + iE''$. Thus, we obtain the dynamic system with the dimensionality $n = 5$ and initial conditions corresponding to small deviations E and N from their stationary values.

The parameters of the laser medium corresponded to those indicated in the previous section. As a result, we obtained that at $\varepsilon < 0$, the uniform stationary lasing regime is stable and the established periodic solutions appear only at $\varepsilon > 0$. A further increase in the parameter ε leads to doubling bifurcations of radiation intensity oscillation period, which was first pointed out in [5].

Figure 2 presents the phase portraits in the plane $(J, dJ/dt)$, the shape of intensity oscillations, and the intensity spectrum. Apart from the fundamental frequency, the spectrum exhibits its harmonics – 2ω, 3ω, etc. Subharmonics of the frequency ω and its linear combinations appear in the sequence of period doubling bifurcations. We found that upon passing through the critical value of the parameter $\varepsilon \approx 0.078$, the spectrum becomes continuous.

Figure 3 demonstrates the phase parametric diagram illustrating the passage to chaos via the period doubling cascade. In this figure, the governing parameter is plotted on the abscissa and the possible values of the intensity maxima – on the ordinate. One can see from this diagram and from the structure of the phase portrait (see Fig. 2a) that the 2^l periods ($l > 1$) are not produced simultaneously for different 'intensity branches'.

So far we discussed the dynamics of the radiation intensity defined as $J = E'^2 + E''^2$. Another scenario of a passage to chaos is observed when the dynamics of the field components (E', E'') is considered.

Figure 4 shows the projections of the trajectories in the space (E', E'', N), the Poincare cross section by the plane $N = N_{st}$ (we took into account only the points at which the difference $N - N_{st}$ changes its sign from plus to minus), and the oscillation spectrum E'. At small values of the governing parameter ε, any small deviations from the equilibrium position are drawn to the attractor, which is a two-dimensional ergodic torus. One can see from Fig. 4 that when the governing parameter ε increases, torus doubling bifurcations leading finally to a chaotic oscillation regime are observed.

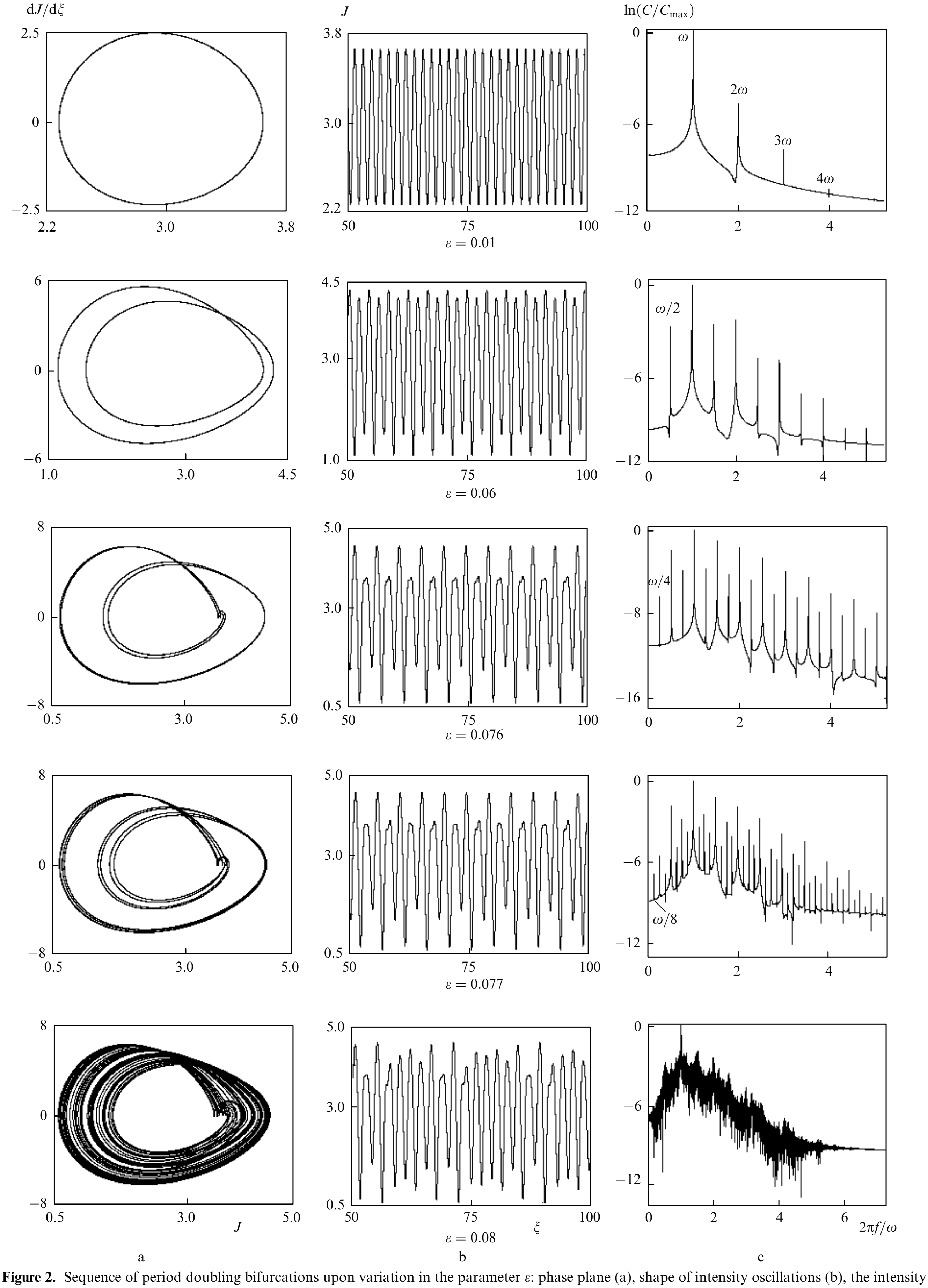

Figure 2. Sequence of period doubling bifurcations upon variation in the parameter ε: phase plane (a), shape of intensity oscillations (b), the intensity spectrum (c); C is the spectrum power, $C_{\max}$ is the maximum power, f is the linear frequency.

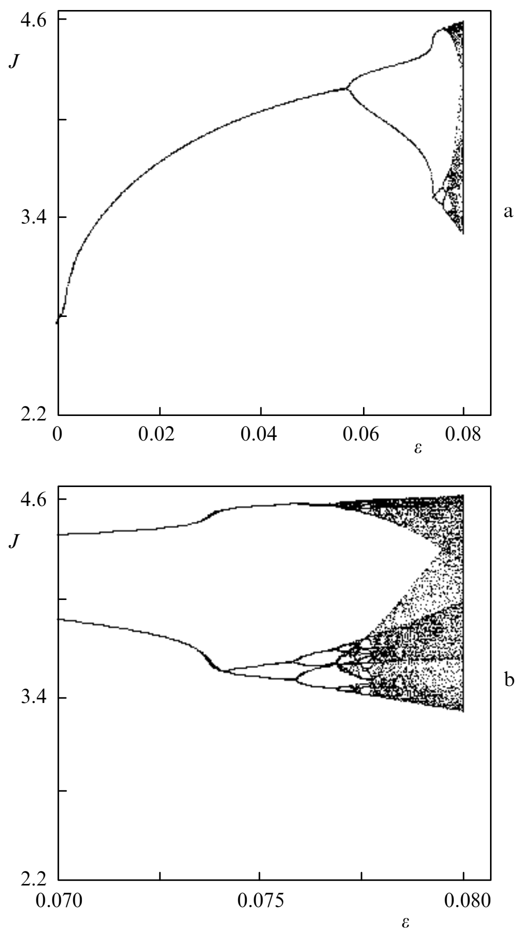

Figure 3. Phase parametric diagram (a) and its fragment on an enlarged scale (b).

After the first bifurcation at $\varepsilon \approx 0.056$ and the appearance of the second (internal) torus, next bifurcations proceed, first of all, with internal tori. Apart from the process at the frequency ω, which experiences period doubling bifurcations, the spectrum exhibits a signal at the second frequency Ω, which does not experience doubling bifurcations, and a harmonic $\omega + 2\Omega$ associated with the cubic nonlinearity of system (1) (Fig. 3c).

The form of oscillations of the field component $E'(\xi)$ and field phase $\varphi(\xi) = \arctan[E''(\xi)/E'(\xi)]$ at $\varepsilon = 0.05$ is presented in Fig. 5. One can see that the frequency ratio changes approximately as $\omega/\Omega \sim 1/\varepsilon$. In turn, it means that the torus is ergodic except those values of the governing parameter at which the frequency ratio ω/Ω becomes a rational number.

From the point of view of nonlinear dynamics of the system under study, of special interest is the spectrum of Lyapunov characteristic exponents (LCEs) especially at the points of period doubling bifurcations because it allows one to understand the bifurcation mechanism. Figure 6 shows the dependences of five LCEs ($\lambda_1 - \lambda_5$) on the parameter ε. Two exponents ($\lambda_1 - \lambda_5$) are independent of ε and equal to zero, which corresponds to a two-dimensional torus. At bifurcation points (points A, B, C, D, E), three exponents turn out to be zero. The signature of the LCE spectrum at bifurcation points B, C, D changes as

$$0,0,-,-,- \to 0,0,0,-,- \to 0,0,-,-,-,$$

which corresponds to the production of a structurally unstable three-dimensional torus at the bifurcation point, which then produces a stable doubled ergodic torus [11].

The signature of the LCE spectrum in the case of bifurcation at the point E changes as

$$0,0,-,-,- \to 0,0,0,-,- \to +,0,0,-,-,$$

i.e. the principle Lyapunov exponent becomes positive, which indicates the passage to chaotic oscillations.

It is easy to calculate the field divergence of phase velocities of our system averaged over the trajectory: $\langle \mathrm{div} F \rangle = -1 - J_{\mathrm{st}}/(1 + \Delta_0^2) = -2.5$ (the system parameters are mentioned above). On the other hand, $\langle \mathrm{div} F \rangle = \sum \lambda_i \approx -2.5$, where λ_i is the corresponding Lyapunov exponent. This identity is confirmed by the results of numerical calculation of the LCE spectrum with an error no worse than 1.5 %.

4. Conclusions

It is known that the torus doubling bifurcation is possible only in systems with the dimensionality $n \geqslant 4$, which are less studied than the systems with a lower dimensionality. As mentioned in [11], despite the fact that doubling of a two-dimensional torus was discovered many years ago [12–14], the details of this bifurcation have not been elucidated so far and the search for simple autonomous models, which will make it possible to realise the regimes of a stable two-dimensional torus, torus doubling bifurcation, and passage to chaos, is urgent. The authors of paper [11] found the conditions for torus doubling bifurcations in the modified model of the inertial Anishchenko–Astakhov generator. The authors of paper [15] proposed for the first time an autonomous point model describing the torus doubling bifurcation in a two-mode laser with a saturable filter.

We have shown in this paper that realisation of the scenario of passage to chaos via torus doubling bifurcation is also possible in a simple self-similar system of equations describing the dynamics of a single-mode detuned laser in the case of adiabatic exclusion of the medium polarisation. The governing parameter here is actually the inverse velocity $\beta_{\max}$ of the wave, which implicitly depends on the laser system parameter (first of all on $\nu, N_{\mathrm{un}}, \Delta_0$). The numerical study of distribution model (1) (not in the self-similar form) performed in our previous paper [6] also demonstrated the passage from regular optical patterns with waves travelling in the transverse direction (with the velocity depending on $\nu, N_{\mathrm{un}}, \Delta_0$) to chaotic patterns when the mentioned parameters of the laser system were varied.

The scenario of the passage to chaos was not studied in [6]. In this connection, of interest for further investigations is the verification of the existence of the scenario of the passage to chaos via the torus doubling bifurcation in more complex models including both distribution model (1) and a complete system of Maxwell–Bloch equations (taking into account the finiteness of the relaxation time of the medium polarisation).

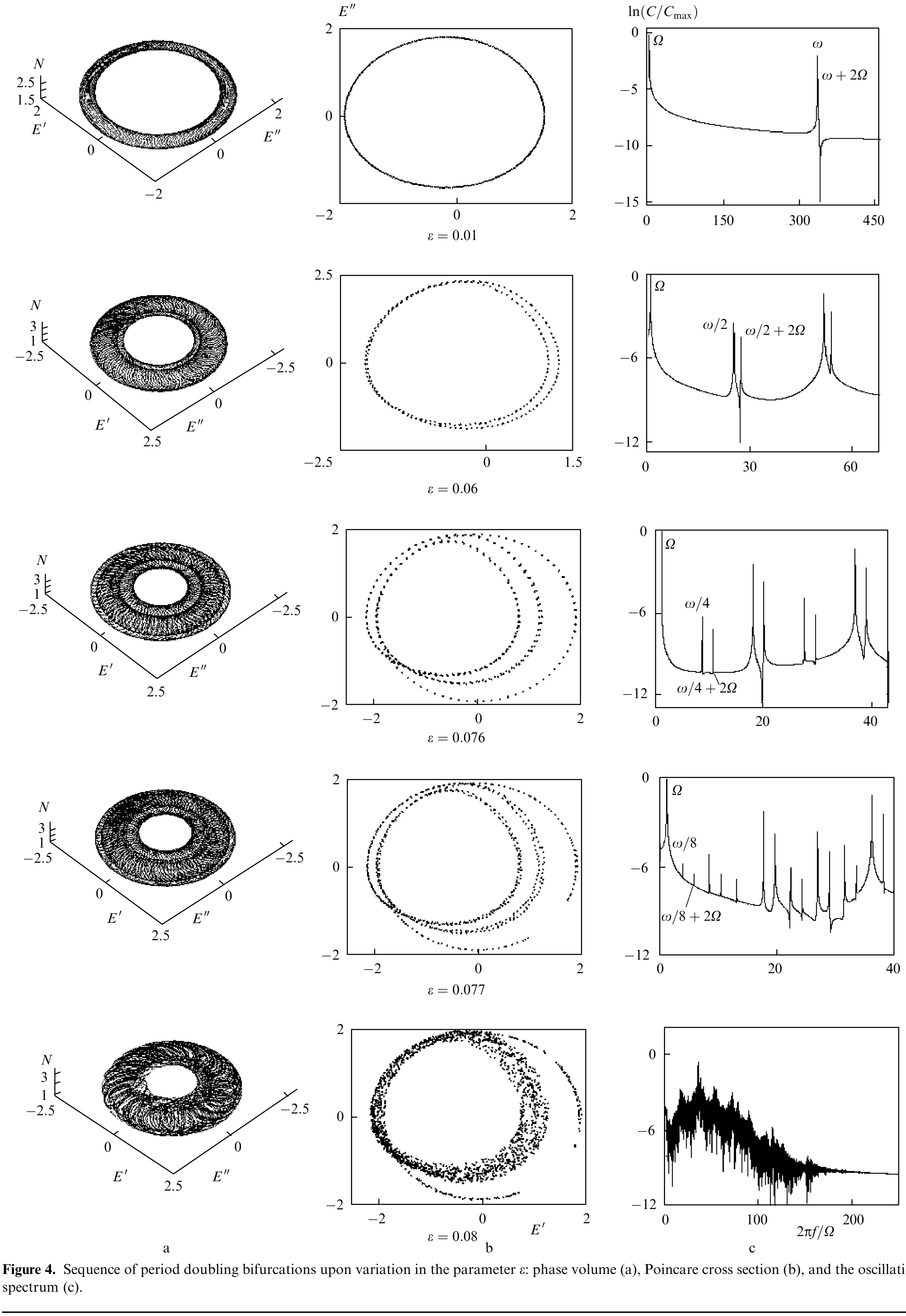

Figure 4. Sequence of period doubling bifurcations upon variation in the parameter ε: phase volume (a), Poincare cross section (b), and the oscillation spectrum (c).

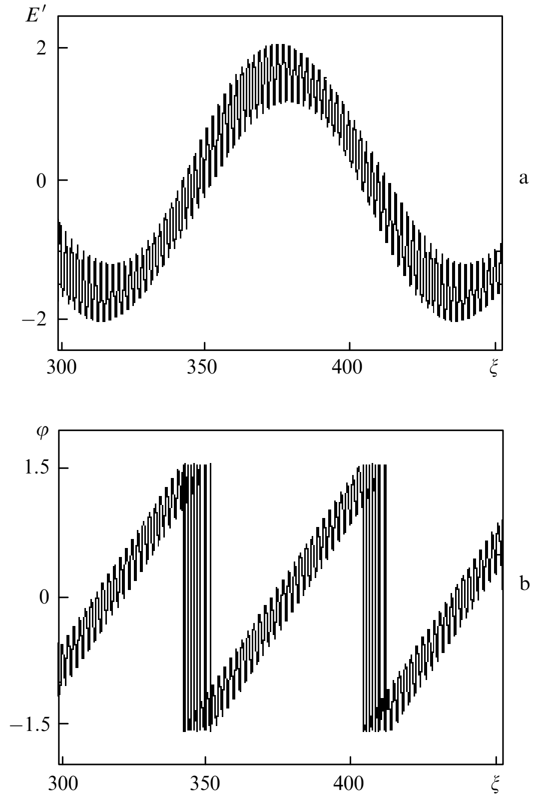

Figure 5. Oscillations of the field component $E'(\xi)$ (a) and the field phase $\varphi(\xi)$ (b).

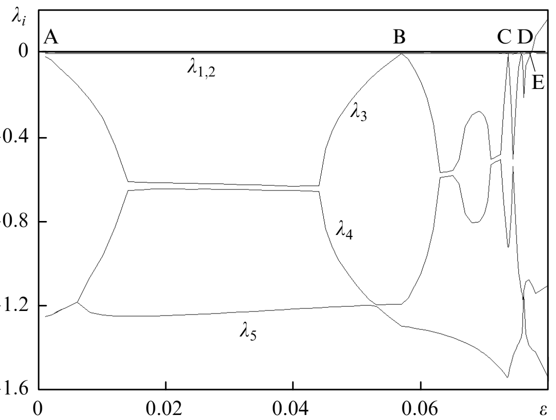

Figure 6. Spectrum of Lyapunov characteristic exponents.

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