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Structure of modes of a smoothly irregular integrated-optical four-layer three-dimensional waveguide

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Abstract. **The asymptotic method and the method of coupled waves used to study an integrated-optical multilayer three-dimensional waveguide satisfying the conditions of a continuously variable effective refractive index are considered. Three-dimensional fields of smoothly deforming modes of a four-layer integrated-optical waveguide are described analytically. Explicit dependences of the contributions of the first order of smallness to the electric and magnetic field amplitudes of quasi-waveguide modes are obtained. The canonical type of quasi-wave equations describing the structure of quasi-TE and quasi-TM modes in a smoothly irregular four-layer integrated-optical three-dimensional waveguide is presented for the asymptotic method. By using the perturbation theory, shifts of complex propagation constants are obtained in an explicit form for these modes. The elaborated theory can be used to analyse structures from dielectric, magnetic and metamaterials in a rather broad wavelength range of electromagnetic waves.**

Keywords: *integrated optics, waveguide mode, smooth three-dimensional irregularities, asymptotic method, quasi-wave equations, generalised Luneburg lens, coupled waves.*

1. Introduction

Guided modes propagating along the regular segment of an integrated-optical waveguide are independent: they do not exchange energy with each other and the medium surrounding the waveguide [1, 2]. In the waveguide section with smooth refractive index irregularities of layers or their thicknesses, the guided mode experiences perturbation. This weakly-perturbed mode can be considered as a quasi-waveguide mode, which is characterised by the fact that in the transverse cross section of the waveguide, the wave is standing and the number of nodes (zeros) of the electric field strength during the waveguide propagation of the mode remains invariable. Quasi-waveguide modes can exchange energy with each other and the surrounding medium [2 – 18]. Because this energy is a small part of the energy transferred by separate modes, approximate methods can be used to study smoothly irregular waveguides (see, for example, [3 – 10]).

For the efficient energy transfer through different coupling elements (lenses, couplers, prisms, multiplexors), it is necessary to take into account the vector character of the fields at all the stages in solving the electrodynamic problem of propagation of a plane monochromatic light wave in a planar multilayer integrated-optical structure. The coupling efficiency, as is known, strongly depends on the field matching in front of and behind the coupling element [2, 7 – 13, 16 – 18].

The analysis of the processes [2 – 17, 19 – 21] shows that the modes of a smoothly irregular waveguide segment are weakly hybrid quasi-TE and quasi-TM modes [9, 13, 16, 17, 21]. The retention of terms proportional to the dielectric constant gradient in boundary conditions and in the solution of quasi-wave equations allows one to take into account the vector character of the monochromatic electromagnetic field propagation along the smoothly irregular segments of the multilayer multimode integrated-optical waveguide [2, 8 – 10, 13, 16, 17, 21]. Note that the vector scattering of the waveguide mode in a statistically irregular waveguide was considered in detail in papers [8 – 10, 13, 21] including in the presence of noise [15].

In Section 2 of the present paper, we discuss the peculiarities of propagation of monochromatic electromagnetic waves in smoothly irregular four-layer integrated-optical three-dimensional waveguide. We found that the boundary conditions 'couple' two modes of this waveguide with different polarisations into one hybrid mode. The physical reason for this discrepancy between TE and TM modes in a regular planar waveguide is the polarisation dependence of the plane wave reflectivity [2, 9]. In the smoothly irregular section of the three-dimensional waveguide at the oblique incidence of the waves (at the interface of the media forming the waveguide), polarisations are mixed and the linearly polarised mode becomes a hybrid one. Thus, our consideration is based on the method of short-wave asymptotics [4, 6], in which the solution of U is represented in the form of some asymptotic series: $U \sim \sum_m u_m/k_0^m$, where the terms of the series are proportional to $k_0^{-m} = (2\pi/\lambda_0)^{-m}$; λ_0 is the wavelength of monochromatic light in vacuum; k_0 is the modulus of the wave vector $\boldsymbol{k}_0$. In the visible wavelength range, $\lambda_0 \to 0$ $(k_0 \to \infty)$, which allows one to use the solution in the form of a finite asymptotic series known as the adiabatic approximation [4, 6, 16, 17, 22].

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We show in Section 3 that the allowance for the adiabaticity of the quasi-waveguide mode in the smoothly irregular four-layer three-dimensional waveguide leads to quasi-wave equations, which are solved asymptotically in the zero and first approximations. In both cases, we present the structure of the mode field components. The canonical type of quasi-wave equations is presented for the asymptotic method.

The quasi-TM and quasi-TE modes in the smoothly irregular four layer integrated-optical three-dimensional waveguide are analysed in Section 4. By using the perturbation theory we obtained in the explicit form the shifts of complex propagation constants (spectral numbers) for weakly coupled quasi-TE and quasi-TM modes. We found that these shifts are imaginary and different for different modes. Note that the development of the vector three-dimensional theory of waveguide propagation and scattering of light is one of the urgent topics in optoelectronics [8–10, 12–21]. Indeed, the use of the scalar two-dimensional wave equation [1–5, 9, 16] upon passage to the submicrone and, all the more, to the nanometer range of linear dimensions restricts the possibilities in solving the problem of analysis and synthesis of elements in integrated-optical devices [17].

The aim of this study is solve the electrodynamic three-dimensional problem formulated in this paper and to analyse the quasi-wave equations and analytic expressions obtained for the fields of deforming modes of a four-layer smoothly irregular integrated-optical three-dimensional waveguide both in the zero and first approximations of the asymptotic method.

2. Formulation of the problem. General analysis of the problem

We studied smoothly irregular integrated-optical three-dimensional structures containing regular (waveguide) and irregular (of the type of the generalised waveguide Luneburg lens) segments (Fig. 1). Coupling devices widely used for connecting different elements of the integrated-optical processor as well as, for example, such elements of integrated-optical schemes as prisms and film-based lenses refer to smooth 'irregularities' [5, 11, 17, 18]. The latter include, in particular, a thin-film waveguide generalised Luneburg lens [5, 17]. The requirement to the calculation accuracy of the parameters of a similar waveguide lens on passing to the nanometer range strongly increases due to the presence of restrictions caused by diffraction effects [2, 5, 17], which strongly affect the spectrum analyser resolution.

Similar problems exist in coupling different elements of one integrated-optical scheme. In particular, in an integrated-optical high-frequency spectrum analyser there exists the problem of coupling of a laser beam with a waveguide and in optical data transmission systems it is necessary to couple optical fibres with radiation sources and with signal detectors.

To transmit efficiently energy through three-dimensional smoothly irregular elements of a multilayer integrated-optical scheme, the vector character of the fields at all the stages in solving the electrodynamic problem of the light wave propagation should be taken into account [2, 8–10, 16, 17].

Maxwell's equations for an unabsorbed inhomogeneous medium in the SI system in the absence of sources can be written in the form:

$$\mathrm{rot}\tilde{\boldsymbol{H}} = \varepsilon\frac{\partial\tilde{\boldsymbol{E}}}{\partial t}, \quad \mathrm{rot}\tilde{\boldsymbol{E}} = -\mu\frac{\partial\tilde{\boldsymbol{H}}}{\partial t}, \tag{1}$$

where $\varepsilon = \varepsilon_r\varepsilon_0$ is the dielectric constant of the medium; $\mu = \mu_r\mu_0$ is the magnetic permeability of the medium; ε_r, μ_r are the relative dielectric constant and magnetic permeability, respectively; ε_0 and μ_0 are the electric and magnetic constants, respectively; $\omega\sqrt{\mu\varepsilon} = nk_0$; n is the refractive index of the medium (layer); $k_0 = 2\pi/\lambda_0 = \omega/c$ is the modulus of the wave vector $\boldsymbol{k}_0$; λ_0 is the wavelength of monochromatic light in vacuum; c is the speed of light in vacuum; $\omega = 2\pi f$; f is the electromagnetic field frequency; $\boldsymbol{E}$, $\boldsymbol{H}$ are the vectors of electric and magnetic field strengths; the tilde above the field vectors refers to their complex character.

In a regular four-layer waveguide, the thickness of the second waveguide layer $h(y,z) = \mathrm{const}$, and the fields of eigenmodes propagating along the z axis have the form [1, 16]:

$$\begin{aligned} \tilde{\boldsymbol{E}}(x,y,z;t) &= \boldsymbol{A}\exp(\mathrm{i}\omega t)\exp(-\mathrm{i}k_x x)\exp(-\mathrm{i}k_z z), \\ \tilde{\boldsymbol{H}}(x,y,z;t) &= \boldsymbol{B}\exp(\mathrm{i}\omega t)\exp(-\mathrm{i}k_x x)\exp(-\mathrm{i}k_z z). \end{aligned} \tag{2}$$

In this case, for TE modes $\boldsymbol{A} = (0, A_y, 0)^{\mathrm{t}}$ and $\boldsymbol{B} = (B_x, 0, B_z)^{\mathrm{t}}$, and for TM modes $\boldsymbol{A} = (A_x, 0, A_z)^{\mathrm{t}}$ and $\boldsymbol{B} = (0, B_y, 0)^{\mathrm{t}}$, where $(*,*,*)^{\mathrm{t}}$ is the column transposed to the line $(*,*,*)$. Between the mode components, the following relations are fulfilled [1, 16]: for TE modes $B_x = (-\beta/\mu)A_y$, $B_z = (\mathrm{i}/k_0\mu)\,\mathrm{d}A_y/\mathrm{d}x$; for TM modes $A_x = (\beta/\varepsilon)B_y$, $A_z = (-\mathrm{i}/k_0\varepsilon)\mathrm{d}B_y/\mathrm{d}x$, where $\beta = k_z/k_0$ is the coefficient of the phase deceleration of the mode (effective refractive index of the waveguide).

In a smoothly irregular four-layer three-dimensional waveguide (see Fig. 1), the thickness of the second waveguide layer is not constant, $h(y,z) \neq \mathrm{const}$, so that $\partial h/\partial y \neq 0$, $\partial h/\partial z \neq 0$. It is assumed in this case that $|\partial h/\partial y|$, $|\partial h/\partial z| \ll 1$.

In the regular segment of the four-dimensional waveguide, a TE or TM mode (2) propagates along the z axis.

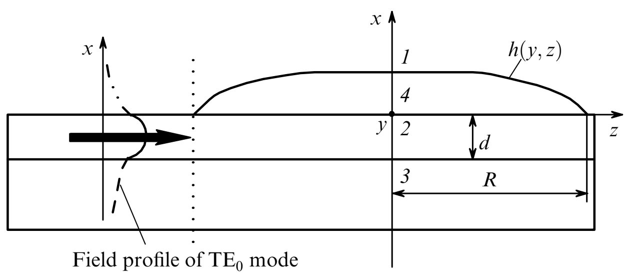

Figure 1. Cross section of the considered three-dimensional integrated-optics structure produced by the surrounding medium (cover layer is air) with the refractive index n_c (*1*), the first waveguide layer (the regular segment of the structure) with the refractive index n_f (*2*), the substrate with the refractive index n_s (*3*) and by the thin-film waveguide Luneburg lens (the irregular segment of the structure) with the refractive index n_L (second waveguide layer) (*4*); $h(y,z)$ is the thickness of the layer forming the Luneburg lens; R is the radius of the lens aperture; d is the thickness of the regular segment of the waveguide integrated-optics structure. The propagation direction of the TE_0 mode is shown by the thick arrow.

The waveguide layer – air interface $x - h(y, z) = \text{const}$ in the regular segment is horizontal and at any point $(h(y, z), y, z)^{\mathrm{t}}$ the plane tangent to it coincides with the yz plane. The tangent boundary conditions $H_z|_{h-0} = H_z|_{h+0}$, $E_y|_{h-0} = E_y|_{h+0}$ and $E_z|_{h-0} = E_z|_{h+0}$, $H_y|_{h-0} = H_y|_{h+0}$ are fulfilled separately for both modes.

At the surface interface $x = h(y, z)$ of the three-dimensional irregular segment of the waveguide, the tangent plane at the point $(h(y, z), y, z)^{\mathrm{t}}$ in the general case does not coincide with the horizontal plane yz. In this case, the tangent boundary conditions $\boldsymbol{E}_\tau|_{h-0} = \boldsymbol{E}_\tau|_{h+0}$, $\boldsymbol{H}_\tau|_{h-0} = \boldsymbol{H}_\tau|_{h+0}$ in the general case are not fulfilled separately for TE and TM modes. Thus, it is the boundary conditions that combine two initially independent waveguide modes into one weakly coupled hybrid mode. The coupling is weak due to the estimate $|\partial h/\partial y|, |\partial h/\partial z| \ll 1$.

In paper [5] for a thin-film waveguide Luneburg lens, which is an example of smoothly irregular integrated waveguide structures under study, Southwell obtained dispersion relations in the approximation when 'inclined' tangent boundary conditions were replaced by their projections to the horizontal plane, i.e. the solution obtained in [5] was two-dimensional. The account for the conditions $|\partial h/\partial y| \neq 0, |\partial h/\partial z| \neq 0$ introduces into Southwell's relations a small correction with respect to the parameter δ, where $\delta = \max|\nabla_{y,z}\beta|(k_0\beta^2)^{-1}$ (this is a two-dimensional analogue of the quantity $|\nabla\varepsilon/\varepsilon|$).

This approach explains the appearance of hybrid modes with six field components in the waveguide with smooth irregularities (see Sections 3, 4) and not with three as is the case of TE and TM modes [1 – 13]. For hybrid modes, the conditions $\partial\boldsymbol{E}/\partial y \equiv 0$, $\partial\boldsymbol{H}/\partial y \equiv 0$ [1, 2, 9] are not fulfilled, i.e. there exist variations of fields in the direction of the y axis.

3. Quasi-wave equations for adiabatic modes of a smoothly irregular four-layer three-dimensional waveguide

3.1 Asymptotic method for solving quasi-wave equations

In the analysis of the propagation of polarised monochromatic electromagnetic field in the multilayer integrated-optical three-dimensional waveguide with smoothly varying layer thicknesses, we used the combination of the short-wave asymptotic method [22] and the modified averaging method [23] (see Appendix 1). As a result, we obtain three separate problems:

(i) the nonlinear equation coupling the phase $\varphi(y, z)$ and its partial derivatives $\partial\varphi/\partial y$ and $\partial\varphi/\partial z$ with the profile of the waveguide layer thickness $h(y, z)$ and its partial derivatives $\partial h/\partial y$ and $\partial h/\partial z$ generalising the dispersion relation for the regular waveguide;

(ii) ordinary differential equations of the second order for the amplitudes $E(x)$ and $H(x)$ with small ($\sim\delta$) right-hand sides specifying the interaction between them;

(iii) integral equations of type A1.4, where integration in expressions for the electromagnetic field $\tilde{\boldsymbol{E}}(x; y, z)$, $\tilde{\boldsymbol{H}}(x; y, z)$ is performed along the rays, which are the solution of the system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\beta\frac{\mathrm{d}y}{\mathrm{d}s}\right) = \frac{\partial\beta}{\partial y},\ \ \frac{\mathrm{d}}{\mathrm{d}s}\left(\beta\frac{\mathrm{d}z}{\mathrm{d}s}\right) = \frac{\partial\beta}{\partial z}.$$

Our method for solving the formulated electrodynamic problem has five advantages:

(i) the solution obtained satisfies Maxwell's equations;

(ii) there is no need to solve the search problem of the orthogonal basis;

(iii) at each point of the horizontal plane (y, z) the vertical distribution of the electromagnetic field is calculated separately independently of the variables (y, z); however, the solutions at two neighbouring points are related parametrically through $\beta(y, z)$;

(iv) the dispersion relations are solved separately from other factors of the required solution of Maxwell's equations;

(v) the obtained solution $\beta(y, z)$ of the dispersion relations makes it possible to calculate separately the rays and wave fronts in the horizontal plane, after which the phase delay $\varphi(y, z)$ can be determined by integration along the rays.

Note that in this case both the Fourier method of separation of variables used for regular waveguides [1, 2] and the method of expansion in the total system of guided and radiation modes of a regular two-dimensional waveguide [2, 7 – 13] are inapplicable because the propagation constant is here complex and there exists the problem of orthogonality of the corresponding modes [9, 13, 15, 19].

Thus, we will seek for the solutions of Maxwell's equations (1) in the form:

$$\begin{aligned}\tilde{\boldsymbol{E}}(x, y, z; t) &= \boldsymbol{E}(x; y, z)\exp(\mathrm{i}\omega t)\\ &\times\exp\left[-\mathrm{i}k_0\int_{y_0,z_0}^{y,z}\beta(y', z')\mathrm{d}s(y', z')\right][\beta(y, z)]^{-1/2},\\ \tilde{\boldsymbol{H}}(x, y, z; t) &= \boldsymbol{H}(x; y, z)\exp(\mathrm{i}\omega t)\\ &\times\exp\left[-\mathrm{i}k_0\int_{y_0,z_0}^{y,z}\beta(y', z')\mathrm{d}s(y', z')\right][\beta(y, z)]^{-1/2},\end{aligned} \tag{3}$$

where $\mathrm{d}s = (\mathrm{d}y^2 + \mathrm{d}z^2)^{1/2}$ is the element of the ray length.

The method for solving three independent problems is iterative with respect to the smallness parameter δ and consists of the following stages*:

(i) The solution of the wave equation in the zero approximation for $E_y(x; y, z)$ and $H_y(x; y, z)$ for TE and TM modes, respectively, in the homogeneous subregions [see below (7)];

(ii) the substitution of the solutions into the boundary conditions (of the zero order of smallness in δ) to obtain dispersion relations for the zero-order TE and TM modes – algebraic transcendental equations for β;

(iii) the solution of the ray equation in the zero approximation;

(iv) the integration of phases along the rays and obtaining of full solutions (in the zero approximation with respect to δ).

After this, it is necessary to repeat all the stages for the corresponding equations in the first approximation with respect to δ. Equations of dispersion relations for TE and TM modes in this cycle are nonlinear first-order differential

*A detailed analysis of these problems is beyond the scope of this paper and will be presented elsewhere.

equations in partial derivatives. Quasi-wave equations for $E_y(x;y,z)$ and $H_y(x;y,z)$ in the first approximation prove to be coupled. All this considerably hampers the calculations of electromagnetic fields in the first approximation.

In this paper we pay the main attention to the analysis of the dependence of the electromagnetic field components on the argument x. Let us write vector Maxwell equations with respect to the components and construct certain linear combinations from differential expressions from them (see Appendix 2). Relations (A2.8)–(A2.11) obtained in Appendix 2 we reduce by the common multiplier, which is not identically equal to zero. As a result we obtain expressions for the field components $E_x(x;y,z), E_y(x;y,z)$ and $H_y(x;y,z), H_x(x;y,z)$:

$$\begin{aligned} H_y &= \chi_z^{-2}\left\{\left[\frac{\partial}{\partial z}\left(-\mathrm{i}\beta\frac{\partial s}{\partial y}\right)-\beta^2\frac{\partial s}{\partial z}\frac{\partial s}{\partial y}\right]H_z-\mathrm{i}\varepsilon\omega\frac{\mathrm{d}E_z}{\mathrm{d}x}\right\},\\ H_x &= \chi_z^{-2}\left[\left(-\mathrm{i}\beta\frac{\partial s}{\partial z}\right)\frac{\mathrm{d}H_z}{\mathrm{d}x}+\mathrm{i}\varepsilon\omega\beta\frac{\partial s}{\partial y}E_z\right],\\ E_y &= \chi_z^{-2}\left\{\left[\frac{\partial}{\partial z}\left(-\mathrm{i}\beta\frac{\partial s}{\partial y}\right)-\beta^2\frac{\partial s}{\partial z}\frac{\partial s}{\partial y}\right]E_z+\mathrm{i}\mu\omega\frac{\mathrm{d}H_z}{\mathrm{d}x}\right\},\\ E_x &= \chi_z^{-2}\left[\left(-\mathrm{i}\beta\frac{\partial s}{\partial z}\right)\frac{\mathrm{d}E_z}{\mathrm{d}x}-\mathrm{i}\mu\omega\beta\frac{\partial s}{\partial y}H_z\right]. \end{aligned} \tag{4}$$

For the components $E_z(x;y,z), H_z(x;y,z)$ we obtain the quasi-wave equations:

$$\begin{aligned} \frac{\mathrm{d}^2E_z}{\mathrm{d}x^2}+\chi_z^2E_z &= -\left(p_yp_y+\frac{\partial p_y}{\partial y}-p_y\chi_z^2\frac{\partial\chi_z^{-2}}{\partial y}\right)E_z\\ &\quad -\frac{\mathrm{i}}{\omega\varepsilon}\left(\frac{\partial p_y}{\partial z}-\frac{\partial p_z}{\partial y}+p_z\chi_z^{-2}\frac{\partial\chi_z^2}{\partial y}\right)\frac{\mathrm{d}H_z}{\mathrm{d}x}, \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{\mathrm{d}^2H_z}{\mathrm{d}x^2}+\chi_z^2H_z &= -\left(p_yp_y+\frac{\partial p_y}{\partial y}-p_y\chi_z^2\frac{\partial\chi_z^{-2}}{\partial y}\right)H_z\\ &\quad +\frac{\mathrm{i}}{\omega\mu}\left(\frac{\partial p_y}{\partial z}-\frac{\partial p_z}{\partial y}+p_z\chi_z^{-2}\frac{\partial\chi_z^2}{\partial y}\right)\frac{\mathrm{d}E_z}{\mathrm{d}x}. \end{aligned} \tag{6}$$

With the help of notations $\chi_z^2=k_0^2\varepsilon\mu+p_zp_z+\partial p_z/\partial z$, $p_y=-\mathrm{i}k_0\beta\,\partial s/\partial y-(2\beta)^{-1}\partial\beta/\partial y$, $p_z=-\mathrm{i}k_0\beta\,\partial s/\partial z-(2\beta)^{-1}\times\partial\beta/\partial z$, used in (4)–(6), the derivatives with respect to field strengths E_i are expressed in the form: $\partial E_i/\partial y=p_yE_i$, $\partial^2E_i/\partial y^2=(p_yp_y+\partial p_y/\partial y)E_i$, $\partial E_i/\partial z=p_zE_i$, $\partial^2E_i/\partial z^2=(p_zp_z+\partial p_z/\partial z)E_i$. The derivatives with respect to field strengths H_i have similar forms.

In each of the regions I_j (j = 1, 2, 3, 4) with the constants ε and μ,

$$\begin{aligned} I_1 &= \{(x,y,z): x\in[h(y,z),+\infty);\ y,z\in(-\infty,+\infty)\},\\ I_2 &= \{(x,y,z): x\in[-d,0];\ y,z\in(-\infty,+\infty)\},\\ I_3 &= \{(x,y,z): x\in(-\infty,-d];\ y,z\in(-\infty,+\infty)\},\\ I_4 &= \{(x,y,z): x\in[0,h(y,z)];\ y,z\in(-\infty,+\infty)\}, \end{aligned} \tag{7}$$

equations for the x dependence of longitudinal components $E_z(x;y,z)$, $H_z(x;y,z)$ have the form:

$$\frac{\mathrm{d}^2E_z}{\mathrm{d}x^2}+\chi_j^2E_z=\frac{\mathrm{i}}{\omega\varepsilon_j}\left(\frac{\partial p_z}{\partial y}-\frac{\partial p_y}{\partial z}\right)\frac{\mathrm{d}H_z}{\mathrm{d}x}, \tag{8}$$

$$\frac{\mathrm{d}^2H_z}{\mathrm{d}x^2}+\chi_j^2H_z=\frac{\mathrm{i}}{\omega\mu_j}\left(\frac{\partial p_y}{\partial z}-\frac{\partial p_z}{\partial y}\right)\frac{\mathrm{d}E_z}{\mathrm{d}x}. \tag{9}$$

Here, $\chi_j^2=\chi^2|_{I_j}$ are the values of χ^2 in the region I_j [see (7)]; $\chi^2=\chi_z^2+p_yp_y+\partial p_y/\partial y$. Let us substitute χ^2 in the form of the sum of terms of the zero, first and second order of smallness with respect to δ:

$$\chi^2=(\chi^2)^{(0)}+(\chi^2)^{(1)}+(\chi^2)^{(2)},$$

where

$$\begin{aligned} (\chi^2)^{(0)} &= k_0^2(\varepsilon\mu-\beta^2);\\ (\chi^2)^{(1)} &= \mathrm{i}k_0\left(\frac{\partial s}{\partial y}\frac{\partial\beta}{\partial z}+\frac{\partial s}{\partial z}\frac{\partial\beta}{\partial y}\right)\\ &\quad -\mathrm{i}k_0\left[\frac{\partial}{\partial y}\left(\beta\frac{\partial s}{\partial y}\right)+\frac{\partial}{\partial z}\left(\beta\frac{\partial s}{\partial z}\right)\right];\\ (\chi^2)^{(2)} &= \left[\left(\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\beta}{\partial z}\right)^2\right](4\beta)^{-1}\\ &\quad -\left[\frac{\partial}{\partial y}\left(\frac{\partial\beta}{\partial y}(2\beta)^{-1}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\beta}{\partial z}(2\beta)^{-1}\right)\right]. \end{aligned} \tag{10}$$

Expressions in braces in the right-hand side of Eqns (8), (9) can be also presented as a sum of terms of the first and second order of smallness with respect to δ:

$$\frac{\partial p_y}{\partial z}-\frac{\partial p_z}{\partial y}=p^{(1)}+p^{(2)},$$

where

$$\begin{aligned} p^{(1)} &= \mathrm{i}k_0\left[\frac{\partial}{\partial y}\left(\beta\frac{\partial s}{\partial z}\right)-\frac{\partial}{\partial z}\left(\beta\frac{\partial s}{\partial y}\right)\right];\\ p^{(2)} &= \frac{\partial}{\partial y}\left(\frac{\partial\beta}{\partial z}\Big/2\beta\right)+\frac{\partial}{\partial z}\left(\frac{\partial\beta}{\partial y}\Big/2\beta\right). \end{aligned} \tag{11}$$

We will solve equations (8), (9) asymptotically [24] by representing the fields in the form of asymptotic (with respect to δ) series:

$$E_z(x)=\sum_{m=0}^{\infty}E_z^m(x)\,\delta^m,\quad H_z(x)=\sum_{m=0}^{\infty}H_z^m(x)\,\delta^m, \tag{12}$$

where m, as a rule, is bounded above: $m\ll\infty$.

3.2 Zero approximation of quasi-wave equations

In each of the homogeneous regions I_1-I_4 in (7), expressions (8), (9) take in the zero approximation (with respect to the order of smallness δ) a simpler form of wave equations

$$\frac{\mathrm{d}^2 E_z^{(0)}}{\mathrm{d}x^2} + k_0^2(\varepsilon_j\mu_j - \beta^2)E_z^{(0)} = 0, \quad (13)$$

$$\frac{\mathrm{d}^2 H_z^{(0)}}{\mathrm{d}x^2} + k_0^2(\varepsilon_j\mu_j - \beta^2)H_z^{(0)} = 0, \quad (14)$$

where $\varepsilon_1\mu_1 = n_c^2$; $\varepsilon_2\mu_2 = n_f^2$; $\varepsilon_3\mu_3 = n_s^2$; $\varepsilon_4\mu_4 = n_L^2$. Thus,

$$\chi_s^2 = k_0^2(n_s^2 - \beta^2) = -\gamma_s^2 < 0, \quad \chi_f^2 = k_0^2(n_f^2 - \beta^2) > 0,$$

$$\chi_L^2 = k_0^2(n_L^2 - \beta^2) > 0, \quad \chi_c^2 = k_0^2(n_c^2 - \beta^2) = -\gamma_c^2 < 0.$$

For the quantities E_x, E_y, H_y, H_x expressions

$$E_x^{(0)} = \frac{1}{(\chi_z^2)^{(0)}}\left(-\mathrm{i}\beta\frac{\partial s}{\partial z}\frac{\mathrm{d}E_z^{(0)}}{\mathrm{d}x} - \mathrm{i}\omega\mu\beta\frac{\partial s}{\partial y}H_z^{(0)}\right),$$

$$H_y^{(0)} = \frac{1}{(\chi_z^2)^{(0)}}\left(-\beta^2\frac{\partial s}{\partial y}\frac{\partial s}{\partial z}H_z^{(0)} - \mathrm{i}\omega\varepsilon\frac{\mathrm{d}E_z^{(0)}}{\mathrm{d}x}\right), \quad (15)$$

$$H_x^{(0)} = \frac{1}{(\chi_z^2)^{(0)}}\left(-\mathrm{i}\beta\frac{\partial s}{\partial z}\frac{\mathrm{d}H_z^{(0)}}{\mathrm{d}x} + \mathrm{i}\omega\varepsilon\beta\frac{\partial s}{\partial y}E_z^{(0)}\right),$$

$$E_y^{(0)} = \frac{1}{(\chi_z^2)^{(0)}}\left(-\beta^2\frac{\partial s}{\partial y}\frac{\partial s}{\partial z}E_z^{(0)} + \mathrm{i}\omega\mu\frac{\mathrm{d}H_z^{(0)}}{\mathrm{d}x}\right) \quad (16)$$

are valid in the zero approximation.

The solutions of Eqns (13), (14) for the components E_z, H_z (taking into account the boundary conditions $|H_z|, |E_z| < \infty$ in the region I_3 at $x \to -\infty$ and the boundary conditions $|H_z|, |E_z| < \infty$ in the region I_1 at $x \to \infty$) take the form:

$$E_s^{(0)}(x) = A_s \exp(\gamma_s x), \quad x \in (-\infty, -d],$$

$$E_f^{(0)}(x) = A_f^+ \exp(\mathrm{i}\chi_f x) + A_f^- \exp(-\mathrm{i}\chi_f x), \quad x \in [-d, 0],$$

$$E_L^{(0)}(x) = A_L^+ \exp(\mathrm{i}\chi_L x) + A_L^- \exp(-\mathrm{i}\chi_L x), \quad x \in [0, h], \quad (17)$$

$$E_c^{(0)}(x) = A_c \exp(-\gamma_c x), \quad x \in [h, \infty),$$

$$H_s^{(0)}(x) = B_s \exp(\gamma_s x), \quad x \in (-\infty, -d],$$

$$H_f^{(0)}(x) = B_f^+ \exp(\mathrm{i}\chi_f x) + B_f^- \exp(-\mathrm{i}\chi_f x), \quad x \in [-d, 0],$$

$$H_L^{(0)}(x) = B_L^+ \exp(\mathrm{i}\chi_L x) + B_L^- \exp(-\mathrm{i}\chi_L x), \quad x \in [0, h], \quad (18)$$

$$H_c^{(0)}(x) = B_c \exp(-\gamma_c x), \quad x \in [h, \infty).$$

Thus, according to (2), the TE mode has the following structure of the electromagnetic field components: $E_z^{(0)} = 0$, $H_y^{(0)} = 0$, $E_x^{(0)} = 0$, $H_z^{(0)} \neq 0$, $H_x^{(0)} \neq 0$, $E_y^{(0)} \neq 0$; the explicit expressions for them are given by relations (16), (18). For the TM mode, $H_z^{(0)} = 0$, $H_x^{(0)} = 0$ $E_y^{(0)} = 0$, $E_z^{(0)} \neq 0$, $H_y^{(0)} \neq 0$, $E_x^{(0)} \neq 0$ [according to (2)]; explicit expressions for these components are given by relations (15), (17).

3.3 First approximation of quasi-wave equations

For the contributions of the first order of smallness with respect to δ to amplitudes (12), we obtain, according to the rules of the asymptotic method, a system of equations with a small parameter in the right-hand side:

$$\frac{\mathrm{d}^2 E_z^{(1)}}{\mathrm{d}x^2} + k_0^2(\varepsilon\mu - \beta^2)E_z^{(1)} = f_1 E_z^{(0)} + \frac{f_2}{\varepsilon}\frac{\mathrm{d}H_z^{(0)}}{\mathrm{d}x}, \quad (19)$$

$$\frac{\mathrm{d}^2 H_z^{(1)}}{\mathrm{d}x^2} + k_0^2(\varepsilon\mu - \beta^2)H_z^{(1)} = -f_1 H_z^{(0)} - \frac{f_2}{\mu}\frac{\mathrm{d}E_z^{(0)}}{\mathrm{d}x}, \quad (20)$$

where

$$f_1 = \mathrm{i}k_0\beta\left(\frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}\right); \quad f_2 = \frac{k_0}{\omega}\left(\frac{\partial\beta}{\partial y}\frac{\partial s}{\partial z} + \frac{\partial\beta}{\partial z}\frac{\partial s}{\partial y}\right).$$

By substituting expressions for the field amplitudes from (17), (18) into expressions (19), (20), we obtain

$$\frac{\mathrm{d}^2 E_z^{(1)}}{\mathrm{d}x^2} + \chi_j^2 E_z^{(1)} = f_1[A_j^+ \exp(\mathrm{i}\chi_j x) + A_j^- \exp(-\mathrm{i}\chi_j x)]$$
$$+ \mathrm{i}\chi_j\frac{f_2}{\varepsilon}[B_j^+ \exp(\mathrm{i}\chi_j x) - B_j^- \exp(-\mathrm{i}\chi_j x)], \quad (21)$$

$$\frac{\mathrm{d}^2 H_z^{(1)}}{\mathrm{d}x^2} + \chi_j^2 H_z^{(1)} = -f_1[B_j^+ \exp(\mathrm{i}\chi_j x) + B_j^- \exp(-\mathrm{i}\chi_j x)]$$
$$- \mathrm{i}\,\chi_j\frac{f_2}{\mu}[A_j^+ \exp(\mathrm{i}\chi_j x) - A_j^- \exp(-\mathrm{i}\chi_j x)]. \quad (22)$$

After grouping similar terms, the system takes the form canonical for the asymptotic method:

$$\frac{\mathrm{d}^2 E_z^{(1)}}{\mathrm{d}x^2} + \chi_j^2 E_z^{(1)} = C_{j1}^+ \exp(\mathrm{i}\chi_j x) + C_{j1}^- \exp(-\mathrm{i}\chi_j x), \quad (23)$$

$$\frac{\mathrm{d}^2 H_z^{(1)}}{\mathrm{d}x^2} + \chi_j^2 H_z^{(1)} = D_{j1}^+ \exp(\mathrm{i}\chi_j x) + D_{j1}^- \exp(-\mathrm{i}\chi_j x). \quad (24)$$

Because the left-hand and right-hand sides of equations (23), (24) are periodic with one 'frequency' χ_j, the solutions of these equations reflect the appearance of resonance phenomena in the system under study. Because of the first order of smallness f_1 and f_2 following from (10), (11), both (19), (20) and equations from them are weakly coupled.

The solution of equations (23), (24) has the form

$$E_z^{(1)}(x) = x[C_{j1}^+ \exp(\mathrm{i}\chi_j x) - C_{j1}^- \exp(-\mathrm{i}\chi_j x)](2i\chi_j)^{-1}, \quad (25)$$

$$H_z^{(1)}(x) = x[D_{j1}^+ \exp(\mathrm{i}\chi_j x) - D_{j1}^- \exp(-\mathrm{i}\chi_j x)](2i\chi_j)^{-1}. \quad (26)$$

The multiplier x in the resonance terms does not violate the solution regularity because its linear growth is suppressed at $x \to \pm\infty$ by the exponentially decreasing multiplier (see details in [24]).

The comparison of expressions (23)–(26) yields the equalities

$$C_{j1}^+ = A_j^+ f_1 + B_j^+(\mathrm{i}\chi_j f_2/\varepsilon), \quad C_{j1}^- = A_j^- f_1 - B_j^-(\mathrm{i}\chi_j f_2/\varepsilon)$$
$$D_{j1}^+ = -B_j^+ f_1 - A_j^+(\mathrm{i}\chi_j f_2/\mu), \; D_{j1}^- = -B_j^- f_1 + A_j^-(\mathrm{i}\chi_j f_2/\mu). \quad (27)$$

From (25)–(27) we obtain according to (12) the explicit dependence of contributions of the first order of smallness in δ to the amplitudes of electric and magnetic fields:

$$E_z^{(1)}(x) = -\frac{xf_1}{2\chi_j^2}\frac{\mathrm{d}E_z^{(0)}(x)}{\mathrm{d}x} + \frac{xf_2}{2\varepsilon}H_z^{(0)}(x), \tag{28}$$

$$H_z^{(1)}(x) = \frac{xf_1}{2\chi_j^2}\frac{\mathrm{d}H_z^{(0)}(x)}{\mathrm{d}x} - \frac{xf_2}{2\mu}E_z^{(0)}(x). \tag{29}$$

Note also that the coefficients $C_{j1}^{\pm}$ and $D_{j1}^{\pm}$ at the oscillating multipliers $\exp(\pm\mathrm{i}\chi_j x)$ in the first (with respect to the smallness parameter δ) approximation of solutions $E_z^{(1)}(x), H_z^{(1)}(x)$ depend on the coefficients $A_j^{\pm}$, $B_j^{\pm}$ and β in the zero approximation.

The solutions for the electric and magnetic field amplitudes in the first approximation have the asymptotics

$$E_z(x) = E_z^{(0)}(x) + E_z^{(1)}(x) = E_z^{(0)}(x) + \frac{x}{2}\left[\frac{f_2}{\varepsilon}H_z^{(0)}(x) - \frac{f_1}{\chi_j^2}\frac{\mathrm{d}E_z^{(0)}(x)}{\mathrm{d}x}\right] + O(\delta^2), \tag{30}$$

$$H_z(x) = H_z^{(0)}(x) + H_z^{(1)}(x) = H_z^{(0)}(x) - \frac{x}{2}\left[\frac{f_2}{\mu}E_z^{(0)}(x) - \frac{f_1}{\chi_j^2}\frac{\mathrm{d}H_z^{(0)}(x)}{\mathrm{d}x}\right] + O(\delta^2) \tag{31}$$

uniformly along the entire x axis at arbitrary $(y,z) \in yz$ (see details in [24]).

4. Quasi-TE and quasi-TM modes of a smoothly irregular four-layer integrated-optical three-dimensional waveguide

We analysed above the solutions in the zero and first approximations of Eqns (5), (6) for an arbitrary polarised mode propagating from the three-layer regular two-dimensional waveguide into the four-layer irregular three-dimensional waveguide.

In this section we will consider successively two particular cases: propagation of the TE or TM mode from the regular three-layer integrated-optical two-dimensional waveguide (the left part in Fig. 1 up to the dashed vertical line) into the four-layer smoothly irregular integrated-optical three-dimensional waveguide (the right part in the figure). We will pay the main attention to the transformation of the vertical field distribution of these modes in the four-layer waveguide.

The boundary conditions at the three boundaries of the integrated-optical waveguide structure under study (see Fig. 1) produce a homogeneous system of twelve linear algebraic equations for twelve undetermined amplitude coefficients $A_\mathrm{s}, B_\mathrm{s}, A_\mathrm{f}^{\pm}, B_\mathrm{f}^{\pm}, A_\mathrm{L}^{\pm}, B_\mathrm{L}^{\pm}, A_\mathrm{c}, B_\mathrm{c}$ [17]. This system has a nontrivial solution, if it is degenerate, i.e. if the determinant of the matrix corresponding to it is zero.

We consider first the TE mode [see the first expression in (2)]. The coefficient of the phase deceleration β satisfies the dispersion relations for six amplitudes B_j of the longitudinal component of the field strength H_z. The admissible values β are given by the roots $\beta_1^H, \beta_2^H, \ldots$ of the dispersion relations (they are also the roots of the dispersion relations in the generally accepted trigonometric notation [2, 16]).

In the irregular segment of the three-dimensional waveguide, the cofactors $H_z(x;y,z)$ and $E_z(x;y,z)$ entering the longitudinal components of the fields $\boldsymbol{H}$ and $\tilde{\boldsymbol{E}}$ of type (3) satisfy equations (8), (9). These equations include functions f_1 and f_2 depending in the case of the TE mode on β_m^H $(m = 1, 2, \ldots)$ beyond the boundary of the irregularity region. Upon passage of the TE mode in the irregularity regions it deforms and begins to satisfy the dispersion relations, which include all twelve amplitudes A_j, B_j [17]. We will denote the solution of the zero approximation of these dispersion relations coinciding with one of the roots β_m^H at the boundary of the irregularity region by $\beta_m^H(y,z)$. This solution can be found numerically.

By denoting $f_1(\beta_m^H(y,z))$ by f_1^H and $f_2(\beta_m^H(y,z))$ by f_2^H, we will write equations (8), (9) in the form:

$$\frac{\mathrm{d}^2E_z}{\mathrm{d}x^2} + \chi_j^2(\beta_m^H)E_z = f_1^H E_z + \frac{f_2^H}{\varepsilon}\frac{\mathrm{d}H_z}{\mathrm{d}x},$$

$$\frac{\mathrm{d}^2H_z}{\mathrm{d}x^2} + \chi_j^2(\beta_m^H)H_z = -f_1^H H_z - \frac{f_2^H}{\mu}\frac{\mathrm{d}E_z}{\mathrm{d}x}.$$

The solutions of these equations in the first approximation are given by expressions (30), (31) in which the dependences of the functions f_1, f_2 and χ_j on the coefficient $\beta_m^H(y,z)$ are specified.

Thus, upon the passage of the TE mode to the irregularity region and upon the further propagation of the deforming mode in the irregular three-dimensional waveguide, the amplitudes $H_z(x;y,z)$, $E_y(x;y,z)$ and $H_x(x;y,z)$ parametrically depend via $\beta_m^H(y,z)$ on the arguments (y,z). Simultaneously, the amplitudes $E_z(x;y,z)$, $H_y(x;y,z)$ and $E_x(x;y,z)$ stop being equal to zero and also parametrically depend via $\beta_m^H(y,z)$ on the variables (y,z). In the zero approximation, the right-hand side of quasi-wave equation (8) is equal to zero for the TM mode in the regions with the constant dielectric constant and magnetic permeability and the solution of equation (8) for the component $E_z^{(0)} \neq 0$ has the form of (17). Due to (15), the components $E_x^{(0)}$ and $H_y^{(0)}$ are also nonzero for the TM mode. Similarly, the right-hand side of quasi-wave equation (9) is equal to zero for the TE mode in the zero approximation, and the solution of equation (9) for the component $H_z^{(0)} \neq 0$ has the form of (18). Due to (16), the components $H_x^{(0)}$ and $E_y^{(0)}$ are also nonzero for the TE mode. Thus, in the zero approximation propagating modes retain the initial linear polarisation and in the first approximation the propagating waveguide mode is depolarised, i.e. it changes the initial polarisation.

In the case of the TM mode [see the second expression in (2)], the consideration, similar to that for the TE mode, allows us to write equations (8), (9) in the form:

$$\frac{\mathrm{d}^2E_z}{\mathrm{d}x^2} + \chi_j^2(\beta_m^E)E_z = f_1^E E_z + \frac{f_2^E}{\varepsilon}\frac{\mathrm{d}H_z}{\mathrm{d}x},$$

$$\frac{\mathrm{d}^2H_z}{\mathrm{d}x^2} + \chi_j^2(\beta_m^E)H_z = -f_1^E H_z - \frac{f_2^E}{\mu}\frac{\mathrm{d}E_z}{\mathrm{d}x}.$$

Here, as in the zero approximation (8), (9), propagating modes retain the initial polarisation and in the first approximation they are depolarised.

The author of paper [21] studied 'the eigenwaves of the mean field' in a statistically irregular waveguide. The solutions obtained in [21] have the structure similar to that of quasi-TE and quasi-TM waves. The expressions for the shifts of spectral (eigenwave) numbers were obtained in [21] by the method of the perturbation theory in the form of the formal expression via the multiple integral. We obtained similar results in the explicit form. The distributions of complex propagation constants, distortions in their spectrum, the structure of the corresponding modes should be analysed on the complex plane [2, 7–10, 13, 15, 19].

When the structure of the modes under study is taken into account, equations (8), (9) for the quasi-TM mode take the form:

$$\frac{\mathrm{d}^2 E_z}{\mathrm{d}x^2} + \chi_j^2(\beta_m^E(y,z))E_z = f_1(\beta_m^E(y,z))E_z,$$
$$\frac{\mathrm{d}^2 H_z}{\mathrm{d}x^2} + \chi_j^2(\beta_m^E(y,z))H_z = \frac{f_2}{\mu}(\beta_m^E(y,z))\frac{\mathrm{d}E_z}{\mathrm{d}x}. \tag{32}$$

One can see from first equation (32) that for the quasi-TM mode, the eigenvalue, equal to χ_j^2 in the zero approximation, is

$$(\chi_j^2)^{(1)} = \chi_j^2(\beta_m^E(y,z)) - f_1(\beta_m^E(y,z)) \tag{33}$$

in the first approximation.

Similarly, equations (8), (9) for the quasi-TE mode assume the from

$$\frac{\mathrm{d}^2 H_z}{\mathrm{d}x^2} + \chi_j^2(\beta_m^H(y,z))H_z = -f_1(\beta_m^H(y,z))H_z,$$
$$\frac{\mathrm{d}^2 E_z}{\mathrm{d}x^2} + \chi_j^2(\beta_m^H(y,z))E_z = -\frac{f_2}{\varepsilon}(\beta_m^H(y,z))\frac{\mathrm{d}H_z}{\mathrm{d}x}. \tag{34}$$

Thus, for the quasi-TE mode the eigenvalue in the first approximation is

$$(\chi_j^2)^{(1)} = \chi_j^2(\beta_m^H(y,z)) + f_1(\beta_m^H(y,z)). \tag{35}$$

Note that due to (33) and (35) additions of the first order to eigenvalues $(\chi_j^2)^{(0)}(\beta_m^{E,H})$ are purely imaginary and differ for different quasi-waveguide modes.

When the adiabatic waveguide mode propagates along the irregular segment of the multilayer integrated-optical three-dimensional waveguide, the following characteristic features take place.

The phase distribution of the electromagnetic field is described by the fast-oscillating multiplier. The polarisation of the electromagnetic field is described by the evolution of multipliers $\boldsymbol{E}(x;y,z), \boldsymbol{H}(x;y,z)$ during the propagation of 'centres of the transverse energy distribution' along the ray tubes. In this case, in the region with $\partial h/\partial y \neq 0$ and $\partial h/\partial z \neq 0$ the initially linearly polarised TE and TM modes experience depolarisation.

The simultaneous solution, for example, of equations (32) and (34) will make it possible to analyse the peculiarities of the energy exchange between quasi-TE and quasi-TM modes. One of the difficulties in this analysis is related to the numerical solution of the dispersion relation obtained with the help of the system consisting of twelve linear algebraic equations [17].

5. Conclusions

In this paper, we have studied a multilayer integrated-optical three-dimensional waveguide with the help of the asymptotic method and the method of coupled waves. Analytic expressions for the deforming modes of the four-layer smoothly irregular integrated-optical three-dimensional waveguide have been derived in the zero and first approximations of the perturbation theory. Quasi-wave equations describing the transformation of the vertical structure of quasi-TE and quasi-TM modes have been presented. We have obtained in the explicit form the shifts of wave number eigenvalues.

The theory developed takes into account the vector character of fields, i.e. makes it possible to describe adequately, unlike the scalar consideration, the real smoothly irregular waveguide structures. The solution obtained can be used for the analysis of similar three-dimensional integrated structures from dielectric, magnetic and metamaterials, including materials consisting of N layers, in a rather broad wavelength range of electromagnetic waves.

The results of this paper can lay the basis for developing the algorithms of the approximate numerical solution of synthesis problems of smoothly irregular integrated-optical multilayer waveguide three-dimensional structures, for example, the specified amplitude-phase transformation of electromagnetic radiation. The solution of this problem, despite its urgency and practical significance, has been restricted till recently by the absence of the analytic solution of the corresponding electrodynamic problem obtained by the authors of this paper.

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Appendix 1

When using the method of short-wavelength asymptotics, the solution of the wave equation for the electric field strength taking into account the gradient of the dielectric constant

$$\nabla^2 \boldsymbol{E} + k_0^2 n^2(\boldsymbol{r})\boldsymbol{E} = -2\nabla[\boldsymbol{E}, \nabla(\ln n(\boldsymbol{r}))], \tag{A1.1}$$

which follows from Maxwell's equations [22] is represented in the form of the finite asymptotic series called the adiabatic approximation:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\boldsymbol{E}(\boldsymbol{r}_0)\sqrt{n(\boldsymbol{r}_0)}}{\sqrt{n(\boldsymbol{r})}} \exp\left(-\mathrm{i}k_0 \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} n(\boldsymbol{r}')\mathrm{d}l\right). \tag{A1.2}$$

In the ray approximation, radiation coupled into a regular optical waveguide propagates along the z axis in the form of plane waves moving in a zigzag manner and experiencing total internal reflection at the waveguide boundaries [2, 12]. In the case of total internal reflection, as is known, the wave phase experiences a drastic jump upon each reflection from the waveguide boundaries. In addition, the waveguide as a guiding structure has different characteristic spatial scales L_j: along the x axis the scale L_x is of the order of the wavelength λ (in the visible range, $L_x \sim 1$ μm), along the y, z axes the scales $L_y, L_z \gg L_x$. As a result, the phase of the electromagnetic wave changes much faster along the x axis than along the y, z axes. This allows one to consider the variable x to be 'fast' and variables y, z – 'slow', which is the basis for the applicability of the averaging method.

According to the averaging method [23], we first find the averaged two-dimensional solution:

$$\bar{\boldsymbol{E}}(y,z) = \frac{\bar{\boldsymbol{E}}(y_0,z_0)\beta(y_0,z_0)}{\sqrt{\beta(y,z)}} \times \exp\left[-\mathrm{i}k_0\int_{y_0,z_0}^{y,z}\beta(y',z')\mathrm{d}s\right], \tag{A1.3}$$

where $\mathrm{d}s = (\mathrm{d}y^2 + \mathrm{d}z^2)^{1/2}$ is the element of the ray length;

$$\varphi(y,z) = k_0\int_{y_0,z_0}^{y,z}\beta(y',z')\mathrm{d}s(y',z')$$

is the wave phase of the averaged two-dimensional equation $\nabla^2_{y,z}\bar{\boldsymbol{E}}(y,z) + k_0^2\beta^2(y,z)\bar{\boldsymbol{E}}(y,z) = 0$ for the unaveraged wave equation corresponding to unaveraged wave equation (A1.1). The integral in the exponential is taken along the ray propagating through the points (y_0,z_0) and (y,z).

After this, unlike the conventional averaging method [23], we seek for the solution of the unaveraged problem in the form

$$\tilde{\boldsymbol{E}}(x,y,z) = \boldsymbol{E}(x;y,z)\frac{\exp[-\mathrm{i}k_0\int\beta(y,z)\mathrm{d}s]}{\sqrt{\beta(y,z)}},$$
$$\tilde{\boldsymbol{H}}(x,y,z) = \boldsymbol{H}(x;y,z)\frac{\exp[-\mathrm{i}k_0\int\beta(y,z)\mathrm{d}s]}{\sqrt{\beta(y,z)}} \tag{A1.4}$$

and substitute it in Maxwell's equations (1) and boundary conditions taking into account the non-horizontal tangent planes at the points of the interface between media.

Appendix 2

We will write Maxwell's equations (1) in the coordinate form:

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{\varepsilon}{c}\frac{\partial E_x}{\partial t}, \tag{A2.1}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \frac{\varepsilon}{c}\frac{\partial E_y}{\partial t}, \tag{A2.2}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{\varepsilon}{c}\frac{\partial E_z}{\partial t}, \tag{A2.3}$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\mu}{c}\frac{\partial H_x}{\partial t}, \tag{A2.4}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\mu}{c}\frac{\partial H_y}{\partial t}, \tag{A2.5}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\mu}{c}\frac{\partial H_z}{\partial t}. \tag{A2.6}$$

According to (3) each of the electromagnetic field components has the form

$$\psi(x,y,z;t) = \varphi(x;y,z)\exp(\mathrm{i}\omega t) \times \exp\left[-\mathrm{i}k_0\int_{y_0,z_0}^{y,z}\beta(y',z')\mathrm{d}s(y',z')\right][\beta(y,z)]^{-1/2},$$

and partial derivatives of ψ –

$$\frac{\partial\psi}{\partial x} = \exp\left[-\mathrm{i}k_0\int_{y_0,z_0}^{y,z}\beta(y',z')\mathrm{d}s(y',z')\right]\frac{\exp(\mathrm{i}\omega t)}{\sqrt{\beta(y,z)}}\frac{\partial\varphi}{\partial x},$$

$$\frac{\partial\psi}{\partial y} = p_y\psi,\quad \frac{\partial\psi}{\partial z} = p_z\psi,\quad \frac{\partial\psi}{\partial t} = \mathrm{i}\omega\psi.$$

Thus, relation

$$\frac{\partial^2\psi}{\partial z^2} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\psi}{\partial t^2} = \left(k_0^2\varepsilon\mu + p_z^2 + \frac{\partial p_z}{\partial z}\right)\psi \tag{A2.7}$$

is valid for each electromagnetic field component of type (3).

We obtain from Maxwell's equations (A2.1)–(A2.6) expressions of type (A2.7) for the components H_y, H_x, E_y, E_x. Namely, applying the operator $\partial/\partial z$ to relation (A2.1) and the operator $(\varepsilon/c)(\partial/\partial t)$ to relation (A2.5) and then summing these relations, we obtain a new expression:

$$\frac{\partial^2 H_y}{\partial z^2} - \frac{\varepsilon\mu}{c}\frac{\partial^2 H_y}{\partial t^2} \equiv \left(k_0^2\varepsilon\mu + p_z^2 + \frac{\partial p_z}{\partial z}\right)H_y = \frac{\partial}{\partial z}\left(\frac{\partial H_z}{\partial y}\right) - \frac{\varepsilon}{c}\frac{\partial}{\partial t}\left(\frac{\partial E_z}{\partial x}\right). \tag{A2.8}$$

Similarly, applying the operator $(\mu/c)(\partial/\partial t)$ to relation (A2.1) and the operator $\partial/\partial z$ to relation (A2.5), the operator $\partial/\partial z$ to relation (A2.2) and the operator $(\varepsilon/c)(\partial/\partial t)$ to relation (A2.4) as well as operator $(\mu/c)(\partial/\partial t)$ to relation (A2.2) and the operator $\partial/\partial z$ to relation (A2.4) and then summing the found relations in turn, we obtain, respectively,

$$\left(k_0^2\varepsilon\mu + p_z^2 + \frac{\partial p_z}{\partial z}\right)E_x = \frac{\partial}{\partial z}\left(\frac{\partial E_z}{\partial x}\right) - \frac{\mu}{c}\frac{\partial}{\partial t}\left(\frac{\partial H_z}{\partial y}\right), \tag{A2.9}$$

$$\left(k_0^2\varepsilon\mu + p_z^2 + \frac{\partial p_z}{\partial z}\right)H_x = \frac{\partial}{\partial z}\left(\frac{\partial H_z}{\partial x}\right) + \frac{\varepsilon}{c}\frac{\partial}{\partial t}\left(\frac{\partial E_z}{\partial y}\right), \tag{A2.10}$$

$$\left(k_0^2\varepsilon\mu + p_z^2 + \frac{\partial p_z}{\partial z}\right)E_y = \frac{\partial}{\partial z}\left(\frac{\partial E_z}{\partial y}\right) + \frac{\mu}{c}\frac{\partial}{\partial t}\left(\frac{\partial H_z}{\partial x}\right). \tag{A2.11}$$

Thus, we have obtained for the components H_y, H_x, E_y, E_x algebraic expressions from the components H_z, E_z and their derivatives.

Now we obtain from Maxwell's equations (A2.1)–(A2.6) second-order equations for the longitudinal components H_z, E_z of the electric field. To do this, we substitute expressions for H_y, H_x from relations (A2.8) and (A2.10) and expressions for E_y, E_x from relations (A2.9) and (A2.11) into the left-hand sides of equations (A2.3) and (A2.6), respectively.

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