Quantum Electronics **39** (12) 1125 – 1130 (2009)

PACS numbers: 42.55.Vc; 23.20.Lv
DOI: 10.1070/QE2009v039n12ABEH014041

Pump and amplification dynamics of gamma rays in a nuclear medium with the hidden population inversion

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***Abstract*. The features of the pump dynamics of isomeric nuclei excited by X-rays of a repetitively pulsed relativistic electron beam followed by the production of a medium with the negative absorption of gamma quanta are analysed. In the extended nuclear medium, the pump excites a travelling hidden-population-inversion wave with the anisotropic gamma amplification, which becomes positive in the case of the excess over the critical pump parameter equal to the product of the peak spectral power density of the X-ray source and the relative duration of an ultrashort relativistic electron bunch. In the alternative scheme with orthogonal directions of pumping X-rays and a flux of amplified gamma quanta, the absence of the amplification anisotropy opens up the possibility for constructing a standard two-mirror resonator with Bragg single-crystal reflectors. The critical peak value of the spectral pump power density is compared with the known characteristics of relativistic-electron X-ray sources by examples of some nuclides.**

Keywords: *quantum nucleonics, stimulated emission of gamma quanta by isomeric nuclei, amplification anisotropy, hidden population inversion.*

1. Introduction

The observation of stimulated emission of gamma rays and the construction of a nuclear gamma laser (NGL) is one of the urgent problems of modern physics [1], which has been in the view of researchers already for more than half a century without any noticeable experimental success [2].

The main obstacle for solving this problem by the methods well developed for the construction of lasers is a fundamental conflict which follows in fact from the (energy $\times$ time) uncertainty relation [2]. The concept of gamma amplification in an ensemble of free nuclei with the hidden inversion and the emission line narrowed down by laser cooling methods is possibly most close to experimental implementation [2]. The aim of the present paper is to analyse the features of the dynamics of accumulation of excited nuclei and amplification in this scheme caused by the complex temporal structure of pumping X-rays produced by relativistic electrons.

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Received 12 January 2009; revision received 20 May 2009
Kvantovaya Elektronika **39** (12) 1125 – 1130 (2009)
Translated by M.N. Sapozhnikov

The hidden inversion [3 – 5] appears because the energy of absorbed and emitted gamma quanta $\hbar\omega_{\rm a,e} = E \pm E_{\rm rec}$ differs from the excited-state energy E by the recoil energy

$$E_{\rm rec} \approx E^2/2Mc^2 \approx 0.5E^2A^{-1}, \tag{1}$$

where M is the mass of an atom; c is the speed of light; A is the mass number of a nucleus; E is measured in the numerical expression in keV; and $E_{\rm rec}$ – in meV. This circumstance, pointed out already by Einstein in 1916 [6], opens up the possibility [7] of observing stimulated emission without the presence of the real population inversion, i.e. without the excess of the number of excited nuclei over unexcited ones. Because of this, the standard expression for the gain of a photon flux

$$g = \sigma\beta n_2 - \chi n \tag{2}$$

in a medium with nucleus concentrations n_2 and n_1 on the upper and lower levels of the laser transition does not contain n_1. Here, $\sigma = \lambda^2/2\pi$; λ is the wavelength;

$$\beta = \frac{\Delta\omega_\gamma}{\Delta\omega_{\rm tot}} = \left[(1+\alpha)(1+\tau\Delta\omega_{\rm br}/2\pi)\right]^{-1} \tag{3}$$

is the ratio of the natural linewidth $\Delta\omega_\gamma$ of the radiative transition to the total linewidth $\Delta\omega_{\rm tot}$, which takes into account all the types of the excess homogeneous and inhomogeneous broadening $\Delta\omega_{\rm br}$; $\tau = 2\pi/\Delta\omega_\gamma(1+\alpha)$ is the excited-state lifetime; α is the internal electron conversion coefficient; χ is the averaged cross section for photonic losses of all types; and n is the total concentration of atoms of different types.

The main source of the undesirable decrease of β (3) in nuclei of free atoms is the thermal Doppler broadening $\Delta\omega_{\rm br} \approx \Delta\omega_{\rm D}$. Therefore, β can be reduced to its minimal value $\beta \to (1+\alpha)^{-1}$ by decreasing temperature down to

$$T \ll (\pi\hbar/\tau E)^2(Mc^2/2k_{\rm B}\ln 2) \approx 40A/(\tau E)^2, \tag{4}$$

where T is measured in μK and τ – in ns. The estimate by (4) (for example, $T \ll 40$ μK for $E = 10$ keV, $A = 100$ and $\tau = 1$ ns) shows that for not too exotically low temperatures, which can be produced by the known laser cooling methods, the excited-state lifetime τ should not exceed a few nanoseconds.

However, even for $\beta \to 1$ and under hidden inversion conditions, the gain g remains extremely small. Thus, for $\lambda = 10^{-8}$ cm, $\beta = 1$, $n_2 = 10^{13}$ cm^{-3}, and $\chi n \ll \sigma n_2$, the gain is only $\sim 10^{-4}$ cm^{-1}. Therefore, it is commonly supposed that the total exponential gain $G = \exp(gL) \gg 1$ can be achieved only when the length L of the amplifying medium is large (exceeding tens of metres). This estimate is based on the assumption that the concentration n_2 is uniformly distributed over the medium length L. The geometry in the form of a cylinder with length L greatly exceeding its transverse size requires that the flux of pumping X-rays should be coaxial with the expected stimulated gamma quantum flux. This leads to the gain anisotropy in a medium with the hidden inversion because after the absorption of a pump photon, the velocity of a nucleus increases by $\Delta v/c = E/Mc^2 \approx 2E_{\rm rec}/E$ and the emission line experiences the corresponding frequency shift $\hbar\omega_{\rm e} \approx E(1 - E_{\rm rec}/E \pm 2E_{\rm rec}/E)$, where the signs $\pm$ correspond to the emission of a gamma quantum parallel or antiparallel to the pump direction, i.e.

$$\hbar\omega_{\rm e} \approx E\left(1 + \frac{E_{\rm rec}}{E}\right) = \hbar\omega_{\rm a} \text{ or } \omega_{\rm e} \approx E\left(1 - \frac{3E_{\rm rec}}{E}\right), \quad (5)$$

respectively. Therefore, when the emission direction coincides with the pump direction, the hidden inversion is absent ($\hbar\omega_{\rm e} = \hbar\omega_{\rm a}$), and stimulated emission can occur only toward the pump flux (backward stimulated scattering of pump photons with the Stokes frequency shift [5]). The gain anisotropy in the case of longitudinal pumping makes impossible the use of simple two-mirror open resonators.

A negative aspect of longitudinal pumping is the decrease in the brightness (spectral density) Ψ of the photon flux density over length L caused by scattering processes of different types and the finite divergence of the X-ray beam. The brightness decreases by a factor of

$$\frac{\Psi(L)}{\Psi_0} \approx \left[1 + \left(\frac{\zeta L^2}{S}\right)^{1/2}\right]^{-2}, \quad (6)$$

where ζ is the solid angle, Ψ_0 is the spectral photon density, and S is the pump-beam cross section at the nuclear channel input.

The features of the temporal structure of high-spectral brightness X-ray sources available at present require an additional analysis of the pumping process to reconsider the simple estimate of the total gain G in an extended medium.

2. Pumping of nuclei in a longitudinal medium with hidden inversion

The most suitable pumping can be performed by X-rays from relativistic electrons, which provide the highest spectral brightness among other radiation sources. Electrons are usually accelerated up to relativistic energies by a microwave electromagnetic field with frequency $\Delta t_{\rm m}^{-1}$ and wavelength $\Lambda = c\Delta t_{\rm m}$, the electron bunches of duration $\Delta t_{\rm e}$ filling only a small part of the microwave period. Correspondingly, an X-ray quantum flux represents a train of pulses of approximately the same duration $\Delta t_{\rm e}$ separated by time intervals $\Delta t_{\rm m} - \Delta t_{\rm e}$. The highest peak brightness of pulsed X-ray radiation ($\sim 5 \times 10^{23}$ phot$\times$ s^{-1} mm^{-2} mrad^{-2} in the relative 0.1 % band) are achieved in free-electron lasers [8, 9].

Such a repetitively pulsed structure of the X-ray flux determines the features of the pump dynamics manifested in the form of cycles repeated at the frequency $\Delta t_{\rm m}^{-1}$. Nuclear populations at a point of a homogeneous medium with a fixed coordinate z follow the rhythm of the train of X-ray pulses: the nuclei are excited during a short interval $\Delta t_{\rm e}$ and then decay spontaneously for a longer interval $\Delta t_{\rm m} - \Delta t_{\rm e}$. The general time dependence of the concentration of excited nuclei has the saw-tooth shape with asymmetric teeth with a sharp leading edge and a gently sloping trailing edge.

The total concentration of cooled atoms is the sum $n = n_1 + n_2 + n_3$ of the concentration n_1 of unperturbed nuclei in the ground state suitable for the production of excited nuclei at the concentration n_2, and the concentration n_3 of ballast nuclei at the lower level. The latter are formed from excited nuclei during spontaneous decay or upon simulating the action of pumping. Because of recoil and also due to the violation of the requirement $\beta \to 1$, they are unsuitable for the further production of the amplifying medium. It is important to emphasise that the depletion of atoms at concentration n_1 and accumulation of ballast atoms at concentration n_3 leads to a constant decrease in the number of nuclei involved in pumping.

The accumulation of the concentration of atoms in an individual Nth pumping cycle at a point of the medium with a fixed coordinate z (in the weak gamma-signal regime) for the X-ray pulse duration $\Delta t_{\rm e}$ is governed by the system of rate equations

$$\frac{{\rm d}n_1}{{\rm d}\theta} = -b_1 n_1, \quad \frac{{\rm d}n_2}{{\rm d}\theta} = b_1 n_1 - (\mu + b_2)n_2, \quad (7)$$

where $\mu \equiv \Delta t_{\rm m}/\tau$; $b_1 \equiv a/(2J_1 + 1)$; $b_2 \equiv a/(2J_2 + 1)$; $a \equiv 2\sigma\Psi\mu(1 + \alpha)^{-1}$; J_1 and J_2 are the spins of the lower and upper states of the laser transition. It is also assumed that the spectral band of the X-ray pump pulse covers both lines at frequencies $\omega_{\rm a}$ and $\omega_{\rm e}$. Due to the limited value of the spectral brightness Ψ obtained at present the parameter a is small ($a \ll 1$), and, therefore, $b_1 \ll 1$, $b_2 \ll 1$, $b_1\theta \ll 1$, and $b_2\theta \ll 1$, which is used below in approximate expressions. As for the parameter μ, for the given value of $\Delta t_{\rm m}$ it cannot be extremely small due to temperature restrictions (4).

The integration of system of equations (7) with the initial conditions $n_2 = n_2(0)$ and $n_1 = n_1(0)$ gives the time dependence of concentration during the X-ray pulse ($0 \leqslant \theta \leqslant \theta_{\rm e} \equiv \Delta t_{\rm e}/\Delta t_{\rm m} \ll 1$):

$$n_1(\theta) = n_1(0)\exp(-b_1\theta) \approx n_1(0)(1 - b_1\theta), \quad (8)$$

$$n_2(\theta) = [n_2(0) + n_1(0)D(\theta)]\exp[-(\mu + b_2)\theta]$$

$$\approx [n_2(0) + n_1(0)b_1\theta](1 - \mu\theta), \quad (9)$$

where

$$D(\theta) \equiv b_1\frac{\exp[(\mu - \Delta b)\theta] - 1}{\mu - \Delta b} \approx b_1\theta; \ \Delta b \equiv b_1 - b_2.$$

The concentrations $n_1(\theta)$ and $n_2(\theta)$ achieve the minimum and maximum values, respectively, by the X-ray pulse end, when $\theta = \theta_{\rm e} \ll 1$ and $D(\theta) = D(\theta_{\rm e}) \equiv D_{\rm e} \approx b_1\theta_{\rm e}$.

Then the excited states decay spontaneously during the interval of duration $\Delta t_{\rm m}(1 - \theta_{\rm e})$ according to the equation ${\rm d}n_2/{\rm d}\theta = -\mu n_2$ and ballast nuclei are accumulated up to the concentration n_3. Upon integration with the initial condition $n_2 = n_2(\theta_e)$, the concentration of the excited nuclei during this interval is

$$n_2(\theta) = n_2(\theta_e)\exp(-\mu\theta) = [n_2(0) + n_1(0)D_e] \times \exp[-(\mu + b_2)\theta_e - \mu\theta] \approx [n_2(0) + n_1(0)b_1\theta_e] \times (1 - \mu\theta_e)\exp(-\mu\theta) \quad (10)$$

and achieves the value

$$n_2(\theta_m) = [n_2(0) + n_1(0)D_e]\exp(-\mu - b_2\theta_e) \approx [n_2(0) + n_1(0)b_1\theta_e]\exp(-\mu) \quad (11)$$

by the end of the cycle ($\theta_m = 1$); in this case, $n_1(\theta_m) = n_1(\theta_e)$ and $n_3(\theta_m) = n - n_2(\theta_m) - n_1(\theta_e)$.

It follows from (11) that excited nuclei can be accumulated by the end of a cycle up to the concentration $n_2(\theta_m) > n_2(0)$ (for example, already in the first cycle $n_2(\theta_m) = nD_e\exp(-\mu - b_2\theta_e) > n_2(0) = 0$).

The final concentrations at the end of a current cycle N are at the same time the initial concentrations for the next cycle $N+1$: $n_1(\theta_m, N) = n_1(\theta_e, N) \to n_1(0, N+1)$, $n_2(\theta_m, N) \to n_2(0, N+1)$, $n_3(\theta_m, N) \to n_3(0, N+1)$.

By considering the pump dynamics over numerous repeating cycles, it is convenient to construct new differential equations by treating the finite differences of variables of the current cycle as the differentials of a 'long' pump process, namely, to treat the duration Δt_m as the differential of a 'long' time $\vartheta \equiv t/\Delta t_m$ $(0 \leqslant \vartheta \leqslant \infty)$ and the differences of the final and start concentrations in one cycle – as the differentials of concentrations: $\Delta n_1 = n_1(\theta_m) - n_1(0) = -n_1(0)D_1$ (here $D_1 \equiv 1 - \exp(-b_1\theta_e) \approx b_1\theta_e$). The passage from finite differences to differentials gives the differential equation $\mathrm{d}n_1/\mathrm{d}\vartheta = \Delta n_1$ governing the depletion of the concentration $n_1(\vartheta)$ during many repeating pump cycles, with the integral at the initial condition $n_1(\vartheta = 0) = n$

$$n_1(\vartheta) = n\exp(-D_1\vartheta) \approx n\exp(-b_1\theta_e\vartheta), \quad (12)$$

demonstrating the monotonic decrease of $n_1(\theta)$.

Similarly, taking dependence $n_1(\vartheta)$ (12) into account, we have

$$\Delta n_2 = n_2(\theta_m) - n_2(0) = -n_2(0)D_2 + nD_e(1 - D_2)\exp(-D_1\vartheta),$$

By using the same procedure of passing from finite differences to differentials, we obtain from this relation the equation $\mathrm{d}n_2/\mathrm{d}\vartheta = \Delta n_2$ for the accumulation of concentration $n_2(\vartheta)$ (here, $D_2 \equiv 1 - \exp(-\mu - b_2\theta_e)$. The integration of this equation with the initial condition $n_2(\theta = 0) = 0$ gives the change in the concentration of excited nuclei during many repeating cycles:

$$n_2(\vartheta) = \frac{nD_e(1 - D_2)}{D_2 - D_1}[\exp(-D_1\vartheta) - \exp(-D_2\vartheta)] \approx \frac{nb_1\theta_e}{e^{\mu} - 1}\{\exp(-b_1\theta_e\vartheta) - \exp[-(1 - e^{-\mu})\vartheta]\}, \quad (13)$$

which for $\vartheta \to \infty$ approaches asymptotically the zero value $[n_2(\vartheta) \to 0]$, passing through the maximum

$$n_2(\vartheta_{max}) = nD_e(D_2^{-1} - 1)\left(\frac{D_1}{D_2}\right)^{\frac{D_1}{D_2 - D_1}} \approx \frac{nb_1\theta_e}{e^{\mu} - 1} \quad (14a)$$

at

$$\vartheta_{max} = \frac{\ln(D_2/D_1)}{D_2 - D_1} \approx (1 - e^{-\mu})^{-1}\ln\left(\frac{1 - e^{-\mu}}{b_1\theta_e}\right). \quad (14b)$$

The long accumulation of excited nuclei (because $\vartheta_{max} \gg 1$) during many pump cycles produces the concentration $n_2(\vartheta_{max})$, which approximately $(1 - e^{-\mu})^{-1}$ times exceeds concentration (11) produced in the first cycle.

A change in the concentration n_2^{peak} at the peak of a tooth during many cycles follows from (9) with the argument $\theta = \theta_e$, where the initial concentrations $n_1(0)$ and $n_2(0)$ are replaced by current dependences $n_1(\vartheta)$ (12) and $n_2(\vartheta)$ (13):

$$n_2^{peak}(\vartheta) = \frac{nD_e}{D_2 - D_1}\exp[-(\mu + b_2)\theta_e]\{(D_2 - D_1)\exp(-D_1\vartheta) + (1 - D_2)[\exp(-D_1\vartheta) - \exp(-D_2\vartheta)]\} \approx nb_1\theta_e\exp(-b_1\theta_e\vartheta)\frac{1 - \exp[-\mu - (1 - e^{-\mu})\vartheta]}{1 - e^{-\mu}}. \quad (15)$$

For $\vartheta \to 0$, peak concentration (15) asymptotically tends to zero, $n_2^{peak}(\vartheta) \to 0$, passing through a maximum

$$n_2^{peak}(\vartheta_{max}^{peak}) = nD_e\exp[-(\mu + b_2\theta_e)] \times \left[\left(\frac{D_2}{1 - D_1}\right)^{D_2}\left(\frac{1 - D_2}{D_1}\right)^{D_1}\right]^{\frac{1}{D_2 - D_1}} \approx \frac{nb_1\theta_e}{1 - e^{-\mu}} \quad (16a)$$

for

$$\vartheta_{max}^{peak} = \frac{1}{D_2 - D_1}\ln\left(\frac{D_2}{D_1}\frac{D_2 - 1}{D_1 - 1}\right) \approx \frac{1}{1 - e^{-\mu}}\ln\frac{1 - e^{-\mu}}{b_1\theta_e e^{\mu}}. \quad (16b)$$

The peak maximum is only slightly ahead of the concentration maximum approximately by μ, i.e. both these maxima belong to the same cycle, the peak maximum exceeding the concentration maximum approximately by e^{μ} times. Both maxima are rather flat.

Thus, as mentioned above, the accumulation of the concentration of excited nuclei in time at a fixed point z in an extended nuclear medium can be described by a saw-tooth sequence of cycles. The travelling wave of the population 'saw' moves together with the X-ray pump flux in the positive direction along the z axis at the speed of light. [In this case, the postulates of the special theory of relativity are not violated because only the state ('phase') of the medium (the population of nuclear states) moves at the speed of light, but not the nuclei themselves or other material bodies.] As a result, the concentration of excited nuclei produced upon longitudinal pumping by X-rays from relativistic electrons is strongly spatially and temporally inhomogeneous, and, therefore, the assumption about the uniform distribution of n_2 over the length is no loner valid and the amplification of gamma quanta cannot be described by the simple expression $G = \exp(gL) \geqslant 1$ used earlier.

3. Local amplification in a medium with a travelling nuclear population wave

The gamma quantum flux propagates oppositely to the travelling population wave in the negative direction along the z axis and, therefore, passes through each saw tooth for the time $\Delta t_m/2$, while the local amplification within one tooth occurs over the length $\Lambda/2$, the total number of nuclei in the tooth remaining invariable. Taking these factors into account, the calculation of amplification can be more conveniently performed at invariable concentrations (9) and (11) in the coefficient $g = \sigma\beta n_2 - \chi n$ (1), but then integration should be performed overt the invariable total length Λ.

The maximum value of $g(\theta)$

$$g_{\max} \approx n[\sigma\beta b_1\theta_e(1-e^{-\mu})^{-1} - \chi] \quad (17)$$

is achieved at the tooth peak at the moment of 'long' time $\vartheta = \vartheta_{\max}^{\text{peak}}$ (16b).

In this top case, it is useful to estimate for $g = g_{\max} = 0$ the critical value of the pump parameter $P \equiv \Psi\theta_e = \Psi\Delta t_e/\Delta t_m$ characterising minimal requirements to the X-ray source:

$$P_{\text{crit}} \equiv (\Psi\theta_e)_{\text{crit}} = \chi\frac{1-\exp(-\mu)}{2\sigma^2\beta\mu}(2J_1+1)(1+\alpha)$$

$$= 2\pi^2\chi\frac{1-\exp(-\mu)}{\mu}(2J_1+1)\left(\frac{1+\alpha}{\lambda^2}\right)^2. \quad (18)$$

The value P_{crit} very strongly depends on the wavelength and energy ($\sim\lambda^{-4}$ and $\sim E^4$, respectively) and weaker depends on χ and α, the latter dependences weakening with increasing E. As for the dependence of P_{crit} on $\mu\sim\tau^{-1}$, a decrease in τ, which is desirable to reduce the value of P_{crit}, results in too rapid exponential decrease $g\sim\exp(-\mu\theta)$ behind the tooth peak. This variety of the dependences makes the choice of the optimal nuclide a complicated multifactor problem.

The achievement of the critical pump value P_{crit} (18) does not mean that the real amplification can be observed because $g < 0$ everywhere on the tooth except the peak point only. The gain for $P > P_{\text{crit}}$ within the entire tooth can be calculated by neglecting the contribution from a site located in the interval $c\Delta t_e$ because of the strong inequality $\theta_e \ll 1$. Then, the gain $g(N^*) \approx n\chi[(P/P_{\text{crit}})\exp(-\mu\theta) - 1]$ decreases exponentially both with time θ and coordinate $z = \Lambda\theta$, and the total local gain over the entire length of the Nth cycle ($N = N^* = \vartheta_{\max}^{\text{peak}}$) is

$$G(N^*, \Lambda) = \exp\int_\Lambda g\mathrm{d}z \approx \exp\left\{cn\chi\tau\left[\frac{P}{P_{\text{crit}}}(1-e^{-\mu}) - \mu\right]\right\} \quad (19)$$

and $G(N^*, \Lambda) \geqslant 1$, if

$$P/P_{\text{crit}} \geqslant \mu(1-e^{-\mu})^{-1}. \quad (20)$$

Even when condition (20) is fulfilled, local gain (19) only slightly exceeds unity because the product $n\chi$ is small [for example, $G(N^*, \Lambda) \geqslant 1$ for $\mu = 1$, $P/P_{\text{crit}} \geqslant 1.6$, while $G(N^*, \Lambda) \approx 1 + (2\times10^{-7})$ for $P/P_{\text{crit}} = 5$ and $n\chi = 10^{-8}\ \text{cm}^{-1}$].

The gain $g(N^*) > 0$ remains positive in the entire interval from $\theta = \theta_e$ to $\theta = 1$ only if the inequality

$$P/P_{\text{crit}} \geqslant \exp\mu \quad (21)$$

is fulfilled, which is stronger than (20). Therefore, although the local gain can exceed unity $[G(N^*, \Lambda) \geqslant 1]$ under condition (20), the positive gain ($g > 0$) in the pump brightness interval $\mu[1-\exp(-\mu)]^{-1} \leqslant P/P_{\text{crit}} \leqslant \exp\mu$ is achieved only in the restricted ($\Delta l < \Lambda$) region of the tooth in the peak vicinity, while the gain is negative ($g < 0$) in the rest of the absorbing region of length $\Lambda - \Delta l$. The presence of this absorbing region reduces the potentially realised local gain $G(N^*) \geqslant 1$.

The negative action of absorbing regions with $g < 0$ can be obviously eliminated by excluding their interaction with the gamma-quantum flux being amplified by retaining regions with $g > 0$. In this case, despite the decrease in the length of the nuclear medium, the total local amplification $G(N^*)$ in one tooth increases. The relative length of the amplifying part of the tooth is restricted by the inequality

$$\Delta l/\Lambda \approx \mu^{-1}\ln(P/P_{\text{crit}}) \geqslant \frac{1}{\mu}\ln\left[\frac{\mu}{1-\exp(-\mu)}\right],$$

which depends only on the parameter μ. In this case, integration in (19) over Δl rather than over the total length Λ of the tooth gives

$$G(N^*, \Delta l) = \exp\int_{\Delta l} g\mathrm{d}z$$

$$\approx \exp\left[n\chi\Lambda\left(\frac{P}{P_{\text{crit}}}\frac{1-\exp(-\mu\Delta l/\Lambda)}{\mu} - \frac{\Delta l}{\Lambda}\right)\right]$$

$$= \exp\left[cn\chi\tau\left(\frac{P}{P_{\text{crit}}} - \ln\frac{P}{P_{\text{crit}}} - 1\right)\right], \quad (22)$$

which for the same value of pump parameter P/P_{crit} (20) yields

$$\frac{G(N^*, \Delta l)}{G(N^*, \Lambda)} = \exp\left\{cn\chi\tau\left[\mu + \frac{P}{P_{\text{crit}}}\right.\right.$$

$$\left.\left.\times\exp(-\mu) - \ln\frac{P}{P_{\text{crit}}} - 1\right]\right\} > 1 \quad (23)$$

despite the decreased integration length ($\Delta l < \Lambda$). Thus, for $P/P_{\text{crit}} = 5$ and $\mu = 1$, the exponent in (22) exceeds this exponent in (19) by a factor of 1.15, but the ratio of local gains (23) itself remains very small because the coefficient $cn\chi\tau \ll 1$. The influence of the shortening of the medium part within one tooth is more noticeable when condition (20) is not fulfilled and $G(N^*, \Lambda) < 1$. Then, $G(N^*, \Lambda) > 1$ and amplification becomes possible if

$$\frac{P}{P_{\text{crit}}} \geqslant \frac{\Delta l}{\Lambda}\frac{\mu}{1-\exp(-\mu\Delta l/\Lambda)} \approx 1 + \frac{\mu}{2}\frac{\Delta l}{\Lambda}. \quad (24)$$

For example, $G(N^*, \Lambda) > 1$ for $\Delta l/\Lambda = 0.1$, $\mu = 1$ and $P/P_{\text{crit}} = 1.05$.

Thus, the relative shortening of the nuclear medium region down to $\Delta l/\Lambda^* < 1$ can be used to produce amplification when the pump parameter P is restricted.

4. Total amplification in an extended medium with a travelling population wave

A nuclear medium with shortened amplification regions consists of a series of quantum traps (the operating length is Δl) with cooled atoms, which are located one over another and are separated by passive gaps that contain no substance (all the traps can be enclosed within a common vacuum case). The total gain $G(L)$ after the passage of the gamma quantum flux through a series of N teeth (traps) is equal to the product of N local gains $G(N, \Lambda) > 1$:

$$G(L) = \prod_N G(N, \Lambda) \approx [G(N^*, \Delta l)]^N$$
$$\approx \exp\left[Ncn\chi\tau\left(\frac{P}{P_{\text{crit}}} - \ln\frac{P}{P_{\text{crit}}} - 1\right)\right]$$
$$\approx 1 + 2Ncn\chi\tau(P/P_{\text{crit}} - 1). \qquad (25)$$

The estimate of $G(L)$ is equal to the product of the maximum value $G(N^*, \Lambda)$ (19) and the operating number N of teeth, which is admissible (with some overstating) because of flat maximum (16) [the second approximate equality (25) is related to the case $P/P_{\text{crit}} - 1 \ll 1$].

5. Scheme with orthogonal beams

Another experimental configuration is possible when the local amplification $G(N^*, \Delta l)$ at one tooth (22) is not too small. In this scheme, the propagation directions of X-ray pump fluxes (along the z axis) and gamma (along the y axis) are mutually orthogonal in the yz plane, which leads to several consequences.

(i) The linear Doppler shift (5) of the radiative nuclear transition and inversion anisotropy, which are inherent in the longitudinal scheme, are absent. The quadratic Doppler shift $\Delta''_{\text{D}}(\hbar\omega_{\text{e}}) = E^3/2(Mc^2)^2 \ll \hbar\Delta\omega_\gamma$, which is retained, does not cause anisotropy and does not prevent the appearance of hidden inversion and gamma amplification in both directions along the y axis. This allows one to construct a standard open resonator with efficient Bragg mirrors (see, for example, [10]).

(ii) A small length of the nuclear medium along the pump beam, equal to the transverse size of the gamma-quantum flux in the resonator, excludes the longitudinal decrease in the spectral pump density Ψ (6). On the other hand, the transverse size of the pump beam, which is usually smaller than Λ, restricts the length of the gain region.

As a result, a nuclear medium of length no more than Δl in the transverse scheme is located along the gamma-quantum flux midway between Bragg mirrors separated by the distance $L_{\text{res}} = \Lambda$ or close to one of the mirrors (in this case, $L_{\text{res}} = \Lambda/2$). Then, the gamma-quantum flux in the resonator propagates in the material medium with $g > 0$ only when the positive gain exists in it and, as in Section 3, is not absorbed during the rest of the time within the cycle in the region of the resonator that does not contain substance, which solves the problem of exclusion of excess losses of gamma quanta in regions with $g < 0$. If the resonator Q factor is high enough, we can expect that usual lasing threshold conditions can be fulfilled within a single tooth and a train of gamma pulses synchronised with population oscillations will be generated with a repetition rate of Δt_{m}^{-1}.
In this scheme, it is convenient to enrich continuously the amplifying medium with nuclei in the ground state, whose concentration monotonically decreases during pumping ($\text{d}n_1/\text{d}\theta < 0$), by directing the flux of deeply cooled nucleus-containing atoms along the third Cartesian coordinate x. Due to such a three-dimensional orthogonality of the directions of three main fluxes (of nuclei, X-ray pump quanta and gamma quanta), this experimental configuration can be called Cartesian.

Taking into account all these circumstances and the total local amplification $G(N^*, \Delta l)$ (22), the threshold pump condition in a resonator with the reflectance R is $P_{\text{thr}}/P_{\text{crit}} \approx 1 + [2(1-R)/cn\chi\tau R]^{1/2}$.

6. Numerical example

The choice of a nuclide suitable for the demonstrating calculation and so much for experiments is complicated because various and sometimes mutually contradictive requirements to nuclear characteristics and atomic properties should be satisfied (the latter should be suitable for laser cooling and long enough confinement in traps). In particular, by considering a complicated relation of critical pump parameter P_{crit} (18) with the transition energy E and the internal conversion coefficient α, we should take into account the empirical dependence of $\log B$ for $B = [(1+\alpha)E^2]^2$ in (18) on the atomic number Z of isomers with $E \leqslant 100$ keV and the lifetime τ providing the fulfilment of inequality (4). The representing points (for example, for isomers $^{40\text{m}}_{19}$K, $^{54\text{m}}_{25}$Mn, $^{119\text{m}}_{50}$Sn, $^{57\text{m}}_{26}$Fe, $^{83\text{m}}_{37}$Rb, $^{134\text{m}}_{55}$Cs, $^{137\text{m}}_{57}$La, $^{151\text{m}}_{63}$Eu, $^{155\text{m}}_{64}$Gd, $^{161\text{m}}_{66}$Dy, and $^{185\text{m}}_{76}$Os) are located within a narrow band with the general tendency satisfactorily described by the linear expression $\log B = 5.4 + Z/20$ (E in keV). This expression shows that to decrease the values of B and P_{crit}, nuclides from the beginning of the periodic table should be used. Because the choice of nuclides is complicated, the nuclear isomers $^{54\text{m}}_{25}$Mn and $^{40\text{m}}_{19}$K considered below play only an illustrative role.

	$^{54\text{m}}_{25}$Mn	$^{40\text{m}}_{19}$K
E/keV	54.4	29.56
λ/nm	0.023	0.042
$\sigma/10^{-18}$ cm^2	0.84	2.8
J_1	3^+	4^-
τ/ns	0.049	4.25
α	0.21	0.35
T/μK	< 300	< 0.1
β	0.825	0.74
$\chi/10^{-22}$ cm^2	~ 0.35	~ 0.35
μ	6	0.07
$P_{\text{crit}}/10^{13}$ cm^{-2} s^{-1} Hz^{-1}	4.3	3.6
θ_{e}	0.03	0.03
$\Psi_{\text{crit}}/10^{15}$ cm^{-2} s^{-1} Hz^{-1}	1.4	1.2
P/P_{crit} (for $\Delta l/\Lambda = 1$)	> 400	> 1.07
$G(N^*, \Lambda)$ (for $n = 10^{14}$ cm^{-3})	$1+(1.4\times10^{-5})$	$1+(1.7\times10^{-10})$

These estimates should be compared with parameters of existing X-ray sources with relativistic electrons. The most advanced among these sources is probably the XFEL HASYLSAB/DASY source (Hamburg) generating ~ 0.1-ps X-ray pulses at a wavelength of 0.1 nm with the peak brightness 5.4× 10^{33} phot s^{-1} mm^{-2} mrad^{-2} in the relative band of 0.1 % in a beam of diameter $h = 0.11$ mm with the divergence 0.8 μrad [8, 9]. To convert this peak brightness to the brightness units used above (phot s^{-1} cm^{-2} $\times mrad^{-2}$ Hz^{-1}), it should be multiplied by the coefficient $3 \times 10^{-13}\lambda$ (λ in nm), which, taking into account other parameters, gives $\sim 1.5\times10^{20}$ phot s^{-1} cm^{-2} $mrad^{-2}$ Hz^{-1} or $\Psi \approx 10^{14}$ phot $\times s^{-1}$ cm^{-2} Hz^{-1} and $P \approx 3 \times 10^{10}$ phot$\times$ $s^{-1}cm^{-2}$ Hz^{-1}.

By comparing the obtained data, we see that because of a very small relative duration ($\theta_e = \Delta t_e/\Delta t_m$) of X-ray pulses, the pump parameter of available sources considerably differs from the estimated critical value P_{crit} of this parameter, although the peak spectral density Ψ approaches the estimated critical value $\Psi_{crit} = P_{crit}/\theta_e$ in the order of magnitude.

7. Conclusions

The analysis performed in the paper has shown that the dynamics of pumping isomeric nuclei by X-rays produced by relativistic electrons and the gamma-amplification dynamics possess specific features caused by the repetitively pulsed type of the emitting electron beam.

The pump excites in the extended nuclear medium a travelling hidden-population-inversion wave with the anisotropic positive gamma amplification ($g > 0$) in the direction opposite to the X-ray flux. The coefficient $g > 0$ is achieved when the critical pump parameter P_{crit} is exceeded. This parameter is equal to the product of the peak spectral density Ψ of X-ray quanta and the relative duration $\theta_e = \Delta t_e/\Delta t_m$ of the ultrashort bunch of relativistic electrons.

The local gain in a single spatial period Λ of the travelling inversion wave can exceed unity ($G > 1$) in a part of the period Λ even for $g < 0$. The negative action of absorbing regions with $g < 0$ is eliminated by shortening the acting amplification length down to $\Delta l < \Lambda$.

The total gamma amplification $G(L)$ in an extended nuclear medium of length $L = N\Lambda$ is estimated as the product of N local gains G, the number N of acting spatial periods of the extended medium and its length L are limited by the depletion of the concentration of nuclei suitable for the production of the hidden population inversion.

In the scheme with the orthogonal X-ray pump and amplified gamma quantum beams, the hidden inversion anisotropy is absent. This opens up the possibility of constructing a standard two-mirror laser resonator with Bragg single-crystal reflectors.

Estimates of the critical peak spectral pump density Ψ_{crit} for nuclides $^{54m}_{25}Mn$ and $^{40m}_{19}K$, used as examples, are close by the order of magnitude to the known parameters of available X-ray sources of relativistic electrons; however, P_{crit} noticeably exceeds the levels of P achievable at present mainly because of a very short relative duration $\theta_e = \Delta t_e/\Delta t_m$ of X-ray pulses. Therefore, although stimulated gamma radiation in the nuclear medium with the hidden inversion appears already upon pumping by available X-ray sources with relativistic electrons, the impressive experimental results can be expected after a considerable increase in the pump parameter P which can be achieved in X-ray sources of the next generation.

Acknowledgements. This work was supported by the Federal Education Agency (Project No. 2.1.1.1.195).

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