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Peculiarities of periodic and aperiodic energy-exchange regimes in the cascade quasi-synchronous parametric frequency conversion

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***Abstract*. The domains of existence and peculiarities of exact analytic solutions of the problem of quasi-synchronous interaction of four plane collinear monochromatic waves – modes in a quadratically nonlinear medium during cascade frequency conversion are analysed. It is shown that the unusual types of multicomponent cnoidal and solitary soliton-like waves (of periodic and aperiodic energy-exchange regimes) are realised. Two of the four components of the latter are proportional to the real and imaginary parts of the well-known Lorentzian dependence, which is commonly used to describe the dispersion of contributions from resonance transitions to the complex permittivity in the case of homogeneous line broadening.**

Keywords: *quadratic nonlinearity, cascade quasi-synchronous frequency conversion, periodic and aperiodic energy exchange, multicomponent cnoidal and solitary waves.*

1. Introduction

The exact solution of the problem of quasi-synchronous interaction of four plane collinear monochromatic waves (the subscript $i = 1-4$) – modes with frequencies ω_1, $\omega_2 = \omega_1$, $\omega_3 = \omega_1 + \omega_2 = 2\omega_1$ and $\omega_4 = \omega_1 + \omega_3 = 3\omega_1$, wave vectors $\boldsymbol{k}_{1-4}$, and complex amplitudes A_{1-4} in a medium with the quadratic nonlinearity during cascade frequency conversion (simultaneous processes $\omega_1 + \omega_{2,3} \rightarrow \omega_{3,4}$) was presented in [1]. It was shown that the initial system of truncated equations

$$\frac{\mathrm{d}A_1}{\mathrm{d}z} = -\mathrm{i}\gamma_1 A_2^* A_3 - \mathrm{i}\gamma_2 A_3^* A_4, \tag{1a}$$

$$\frac{\mathrm{d}A_2}{\mathrm{d}z} = -\mathrm{i}\gamma_1 A_1^* A_3, \tag{1b}$$

$$\frac{\mathrm{d}A_3}{\mathrm{d}z} = -\mathrm{i}2\gamma_1^* A_1 A_2 - \mathrm{i}2\gamma_2 A_1^* A_4, \tag{1c}$$

$$\frac{\mathrm{d}A_4}{\mathrm{d}z} = -\mathrm{i}3\gamma_2^* A_1 A_3 \tag{1d}$$

with the boundary conditions $A_i\big|_{z=0} = A_{i0}$ describing this problem can be reduced to the closed system of two coupled stationary nonlinear Schrödinger equations (NSEs)

$$\frac{\mathrm{d}^2 A_1}{\mathrm{d}z^2} = -G_+|A_1|^2 A_1 + \tfrac{3}{2}G_-|A_3|^2 A_1 + \left(|\gamma_1|^2 J_1 + 3|\gamma_2|^2 J_3\right)A_1, \tag{2a}$$

$$\frac{\mathrm{d}^2 A_3}{\mathrm{d}z^2} = -3G_+|A_1|^2 A_3 + \tfrac{1}{2}G_-|A_3|^2 A_3 + \left(|\gamma_1|^2 J_1 + 3|\gamma_2|^2 J_3\right)A_3 \tag{2b}$$

for the amplitudes $A_{1,3}$ of two waves involved in both nonlinear processes with the boundary conditions $A_1\big|_{z=0} = A_{10}$, $(\mathrm{d}A_1/\mathrm{d}z)\big|_{z=0} = -\mathrm{i}\gamma_1 A_{20}^* A_{30} - \mathrm{i}\gamma_2 A_{30}^* A_{40}$, $A_3\big|_{z=0} = A_{30}$, $(\mathrm{d}A_3/\mathrm{d}z)\big|_{z=0} = -\mathrm{i}2\gamma_1^* A_{10}A_{20} - \mathrm{i}2\gamma_2 A_{10}^* A_{40}$. The use of Eqns (2a)–(2b) is equivalent to the introduction of the efficient cubic nonlinearity describing the competition of processes of merging and decomposition of photons. Here, the axis z is directed along vectors $\boldsymbol{k}_i$; $\gamma_{1,2}$ are nonlinear coupling constants for processes $\omega_1 + \omega_{2,3} \rightarrow \omega_{3,4}$ averaged over the domain structure period (see [1]); $G_\pm = |\gamma_1|^2 \pm 3|\gamma_2|^2$; and $J_1 = I_1 - 2I_2 - I_3/2$ and $J_3 = I_1 + I_3/2 + 2I_4/3$ are integrals of system (1); $I_i = A_i A_i^*$ are variables proportional to the intensities. It was found that system (2) can be transformed to two identical independent NSEs, which determined its self-consistent solutions (cnoidal waves) in a rather unusual form as the sum and difference of two identical solutions of the same NSE with shifted arguments. In this case, although equations for the amplitudes $A_{2,4}$ of two other waves cannot be reduced to a system similar to (2), their intensities $I_{2,4}$ can be easily found from the relations

$$I_2 - I_{20} = \tfrac{1}{2}(I_1 - I_{10}) - \tfrac{1}{4}(I_3 - I_{30}),$$

$$I_4 - I_{40} = -\tfrac{3}{2}(I_1 - I_{10}) - \tfrac{3}{4}(I_3 - I_{30}), \tag{3}$$

which follow from the conservation laws. Here, $I_{i0} = I_i\big|_{z=0}$. The analytic solutions obtained in this way in [1] completely overlap the range of variations of boundary conditions, allowing the optimisation of energy-exchange regimes in any particular situation.

Below, we analyse the domains of existence and peculiarities of periodic self-consistent solutions (cnoidal waves, see references in [1]) obtained for this problem in [1] and show that in this case an unusual new type of a multicomponent aperiodic (soliton-like) solution can be obtained.

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The two of its four components are proportional to the real and imaginary parts of the well-known complex Lorentzian dependence, which is commonly used to describe the dispersion of contributions of resonance transitions to the complex permittivity in the case of homogeneously broadened lines.

2. Domains of existence and peculiarities of solutions

To illustrate the type and peculiarities of the exact analytic solutions of system (1), we consider the case when $I_{10,20} \neq 0$ and $I_{30} = I_{40} = 0$, i.e. when the Hamiltonian H of system (1) is zero and the phases φ_j of complex amplitudes A_j introduced by the relation $A_j(z) = X_j(z)\exp[i\varphi_j(z)]$ are constant (see [1]). Here, the real variables X_j can be both positive and negative. In this case, two low-frequency modes $A_{1,2}$ play the role of two-component pumping used to generate two high-frequency modes $A_{3,4}$. Note that the choice of the zero position on the z axis is conditional and the argument of any of the solutions presented in [1] can be arbitrarily shifted. Therefore, to satisfy the boundary conditions chosen by us, solutions (43) and (44) from [1], unlike other solutions, will be preliminary shifted along the z axis by the quarter of a period.

Let us introduce the plane (ε, N) defined by two parameters $\varepsilon = 3|\gamma_2|^2/|\gamma_1|^2 - 1 \geqslant -1$, and $N = I_{10}/I_{20} \geqslant 0$, which describe the relation between nonlinear coupling constants and the role of boundary conditions for expressions (25), (26), (32), (33), (43)–(45) and (47) from [1]. The values of para-meters ε and N used below in calculations are indicated in Fig. 1 by points with the numbers corresponding to solutions from [1]. The domains of existence of the analytic solutions listed above are limited by separatrices in this plane (Fig. 1):

$$\varepsilon_0(N) = -1/N, \tag{4a}$$

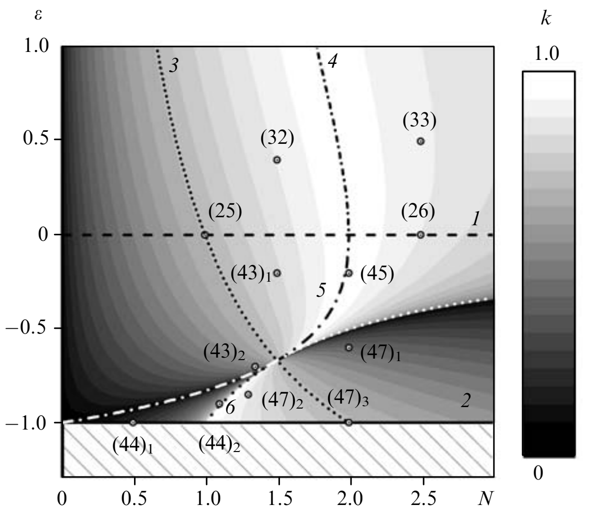

Figure 1. Gray scale map for the dependence $k(\varepsilon, N)$ for $I_{10,20} \neq 0$ and $I_{30} = I_{40} = 0$. In the plane (ε, N) are shown straight lines $\varepsilon = 0$ (*1*) and $\varepsilon = -1$ (*2*) and separatrices $\varepsilon_0(N) = -1/N$ (*6*) and $\varepsilon_\pm(N) = (2/N)[2 - N \pm \sqrt{2(2-N)}]$ (*4*, *5*). The black and white curves correspond to $k = 1$ and 0. Dotted curve (*3*) passes through the singular point $(\varepsilon = -2/3, N = 3/2)$ and corresponds to $k = 1/2$.

$$\varepsilon_\pm(N) = (2/N)\{2 - N \pm [2(2-N)]^{1/2}\}. \tag{4b}$$

In reality the separatrices $\varepsilon_\pm(N)$ represent the two branches of the two-valued solution of the equation $N(\varepsilon) = 8(\varepsilon+1)/[\varepsilon^2 + 4(\varepsilon+1)]$ and, therefore, they are analytically sewed together at the point ($\varepsilon = 0$, $N = 2$). Because tangents to separatrices $\varepsilon_0(N)$ and $\varepsilon_-(N)$ coincide at their intersection point ($\varepsilon = -2/3$, $N = 3/2$), they also continue analytically each other. All this leads to the formation of two resulting intersecting curves on which the modulus k of elliptic Jacobi functions sn(x), cn(x), and dn(x) [2], through which all the periodic solutions of interest to us are expressed, is zero and unity, respectively (Fig. 1). The intersection point of these curves is a singularity (see below), and solutions (32) and (33), (43)–(45) and (47), and (25) and (26) from [1] are responsible for domains $\varepsilon > 0$, $\varepsilon < 0$ and their interface $\varepsilon = 0$, respectively.

Expressions (25), (32) and (43) (in the latter case, after the shift of the argument by the quarter of a period) in [1] are responsible for the domain where all these three solutions can be rewritten in the unified form

$$\frac{X_1}{|A_{10}|} = \mathrm{cn}(\alpha z)\left[1 - \frac{\sqrt{1+\varepsilon N} - 1}{2\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)\right]^{-1}, \tag{5a}$$

$$\frac{X_2}{|A_{10}|} = N^{-1/2}\left[1 - \frac{\sqrt{1+\varepsilon N} - 1 + N}{2\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)\right] \times \left[1 - \frac{\sqrt{1+\varepsilon N} - 1}{2\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)\right]^{-1}, \tag{5b}$$

$$\frac{X_3}{|A_{10}|} = \sqrt{2}(1+\varepsilon N)^{-1/4}\mathrm{sn}(\alpha z)\mathrm{dn}(\alpha z) \times \left[1 - \frac{\sqrt{1+\varepsilon N} - 1}{2\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)\right]^{-1}, \tag{5c}$$

$$\frac{X_4}{|A_{10}|} = \frac{1}{2}\left[\frac{3N(1+\varepsilon)}{1+\varepsilon N}\right]^{1/2}\mathrm{sn}^2(\alpha z) \times \left[1 - \frac{\sqrt{1+\varepsilon N} - 1}{2\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)\right]^{-1}, \tag{5d}$$

$$k = \frac{[2(\sqrt{1+\varepsilon N} - 1) + N(2+\varepsilon)]^{1/2}}{2(1+\varepsilon N)^{1/4}},$$

$$\alpha = \sqrt{\frac{2}{N}}(1+\varepsilon N)^{1/4}|\gamma_1||A_{10}|. \tag{5e}$$

Expressions (26), (33) and (45) in [1] are responsible for the domain located to the right of separatrices $\varepsilon_\pm(N)$ over the separatrix $\varepsilon_0(N)$ for $\varepsilon > -2/3$, in which the unified notation

$$\frac{X_1}{|A_{10}|} = \mathrm{dn}(\alpha z)\left[1 - 2\frac{\varepsilon}{2 + 2\varepsilon + (2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)\right]^{-1}, \tag{6a}$$

$$\frac{X_2}{|A_{10}|} = N^{-1/2}\left[1 - 2\frac{1 + \varepsilon + \sqrt{1+\varepsilon N}}{2 + 3\varepsilon + (2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)\right] \times \left[1 - 2\frac{\varepsilon}{2 + 3\varepsilon + (2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)\right]^{-1}, \tag{6b}$$

$$\frac{X_3}{|A_{10}|} = 2N^{-1/2}\left[\frac{2(1+\sqrt{1+\varepsilon N})}{2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}}\right]^{1/2} \times \frac{\mathrm{sn}(\alpha z)\mathrm{cn}(\alpha z)}{1-2\varepsilon\left[2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}\right]^{-1}\mathrm{sn}^2(\alpha z)}, \quad (6c)$$

$$\frac{X_4}{|A_{10}|} = 2N^{-1/2}\frac{\sqrt{3(1+\varepsilon)}(1+\sqrt{1+\varepsilon N})}{2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}} \times \frac{\mathrm{sn}^2(\alpha z)}{1-2\varepsilon\left[2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}\right]^{-1}\mathrm{sn}^2(\alpha z)}, \quad (6d)$$

$$k = 2\left[\frac{\varepsilon\sqrt{1+\varepsilon N}}{\left[2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}\right]\left(\sqrt{1+\varepsilon N}-1\right)}\right]^{1/2},$$

$$\alpha = \sqrt{\frac{2}{N}}\frac{(1+\varepsilon N)^{1/4}}{k}|\gamma_1||A_{10}| \quad (6e)$$

can be also used.

Expressions (44) from [1], shifted by the quarter of the period along the z axis, are responsible for the domain located to the left of the separatrix $\varepsilon_0(N)$ under the separatrix $\varepsilon_-(N)$ for $\varepsilon < -2/3$, where the corresponding solution can be written in the form

$$\frac{X_1}{|A_{10}|} = \frac{\mathrm{cn}(\alpha z)\mathrm{dn}(\alpha z)}{1+(2+\varepsilon)\left(\sqrt{1+\varepsilon N}-1\right)\left[2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}\right]^{-1}\mathrm{sn}^2(\alpha z)}, \quad (7a)$$

$$\frac{X_2}{|A_{10}|} = N^{-1/2}\left[1+\varepsilon\frac{\sqrt{1+\varepsilon N}-1}{2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)\right] \times \left[1+(2+\varepsilon)\frac{\sqrt{1+\varepsilon N}-1}{2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)\right]^{-1}, \quad (7b)$$

$$\frac{X_3}{|A_{10}|} = 2N^{-1/2}\left[\frac{2(\sqrt{1+\varepsilon N}-1)}{2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}}\right]^{1/2} \times \frac{\mathrm{sn}(\alpha z)}{1+(2+\varepsilon)\left(\sqrt{1+\varepsilon N}-1\right)\left[2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}\right]^{-1}\mathrm{sn}^2(\alpha z)}, \quad (7c)$$

$$\frac{X_4}{|A_{10}|} = 2N^{-1/2}\frac{\sqrt{3(1+\varepsilon)}\left(\sqrt{1+\varepsilon N}-1\right)}{2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}} \times \frac{\mathrm{sn}^2(\alpha z)}{1+(2+\varepsilon)\left(\sqrt{1+\varepsilon N}-1\right)\left[2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}\right]^{-1}\mathrm{sn}^2(\alpha z)}, \quad (7d)$$

$$k = \left\{\frac{\left(1-\sqrt{1+\varepsilon N}\right)\left[2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}\right]}{\left(1+\sqrt{1+\varepsilon N}\right)\left[2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}\right]}\right\}^{1/2},$$

$$\alpha = \left\{\frac{\left(1+\sqrt{1+\varepsilon N}\right)\left[2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}\right]}{2\varepsilon N}\right\}^{1/2}|\gamma_1||A_{10}|. \quad (7e)$$

Expressions (47) from [1] are responsible for the domain located below the separatrix $\varepsilon_0(N)$, in which the indicated solution can be rewritten in the form

$$\frac{X_1}{|A_{10}|} = \frac{2\eta}{\eta+1}\frac{\mathrm{dn}(\alpha z)}{1+[(\eta-1)/(\eta+1)]\mathrm{cn}(\alpha z)}, \quad (8a)$$

$$\frac{X_2}{|A_{10}|} = N^{-1/2}\frac{(2+\varepsilon)(\eta+1)-2}{(2+\varepsilon)(\eta+1)} \times \frac{1+\left\{[(2+\varepsilon)(\eta-1)+2]/[(2+\varepsilon)(\eta+1)-2]\right\}\mathrm{cn}(\alpha z)}{1+[(\eta-1)/(\eta+1)]\mathrm{cn}(\alpha z)}, \quad (8b)$$

$$\frac{X_3}{|A_{10}|} = N^{-1/2}\frac{2}{\eta+1}\sqrt{\frac{2\eta}{2+\varepsilon}}\frac{\mathrm{sn}(\alpha z)}{1+[(\eta-1)/(\eta+1)]\mathrm{cn}(\alpha z)}, \quad (8c)$$

$$\frac{X_4}{|A_{10}|} = 2N^{-1/2}\frac{\sqrt{3(1+\varepsilon)}}{(2+\varepsilon)(\eta+1)} \times \frac{1-\mathrm{cn}(\alpha z)}{1+[(\eta-1)/(\eta+1)]\mathrm{cn}(\alpha z)}, \quad (8d)$$

$$k = \left[\frac{2+(2+\varepsilon)(\eta-1)N}{2(2+\varepsilon)\eta N}\right]^{1/2}, \quad \alpha = \sqrt{2(2+\varepsilon)\eta}\,|\gamma_1||A_{10}|,$$

$$\eta = \left[1-8\frac{1+\varepsilon}{N(2+\varepsilon)^2}\right]^{1/2}. \quad (8e)$$

And finally, the solution at the singularity ($\varepsilon = -2/3, N = 3/2$) can be obtained as the corresponding limit of the expressions written above, which gives the rather unusual aperiodic soliton-like solution

$$\frac{X_1}{|A_{10}|} = \frac{1}{1+\lambda z^2}, \quad (9a)$$

$$\frac{X_2}{|A_{10}|} = \frac{1}{\sqrt{6}}\frac{2-\lambda z^2}{1+\lambda z^2}, \quad (9b)$$

$$\frac{X_3}{|A_{10}|} = \frac{2\sqrt{\lambda}z}{1+\lambda z^2}, \quad (9c)$$

$$\frac{X_4}{|A_{10}|} = \sqrt{\frac{3}{2}}\frac{\lambda z^2}{1+\lambda z^2}, \quad (9d)$$

$$\lambda = \frac{2}{3}|\gamma_1|^2 I_{10}. \quad (9e)$$

Note that we determined $X_{2,4}(z)$ in (5)–(9) by using (3) and took into account the sign of derivatives $\mathrm{d}A_{2,4}/\mathrm{d}z$ in the vicinity of points $X_{2,4} = 0$.

The specific features of analytic solutions (5)–(8) are illustrated in Fig. 2, which shows the transformation of the dependences of amplitudes X_{1-4} (normalised to $|A_{10}|$) on the coordinate z (normalised to α^{-1}) with changing parameters ε and N. These dependences correspond to expressions (25) [expressions (5) of this paper for $\varepsilon = 0$], (26) [(6) for $\varepsilon = 0$], (32) [(5) for $\varepsilon > 0$], (33) [(6) for $\varepsilon > 0$], (45) [(6) for $\varepsilon < 0$], and (47) [(8) of this paper] and to shifted solutions (43) [(5) for $\varepsilon < 0$] and (44) [(7) of this paper] from [1].

It is easy to verify that the most drastic changes in the type of dependences $X_{1-4}(z)$ occur near separatrices, which

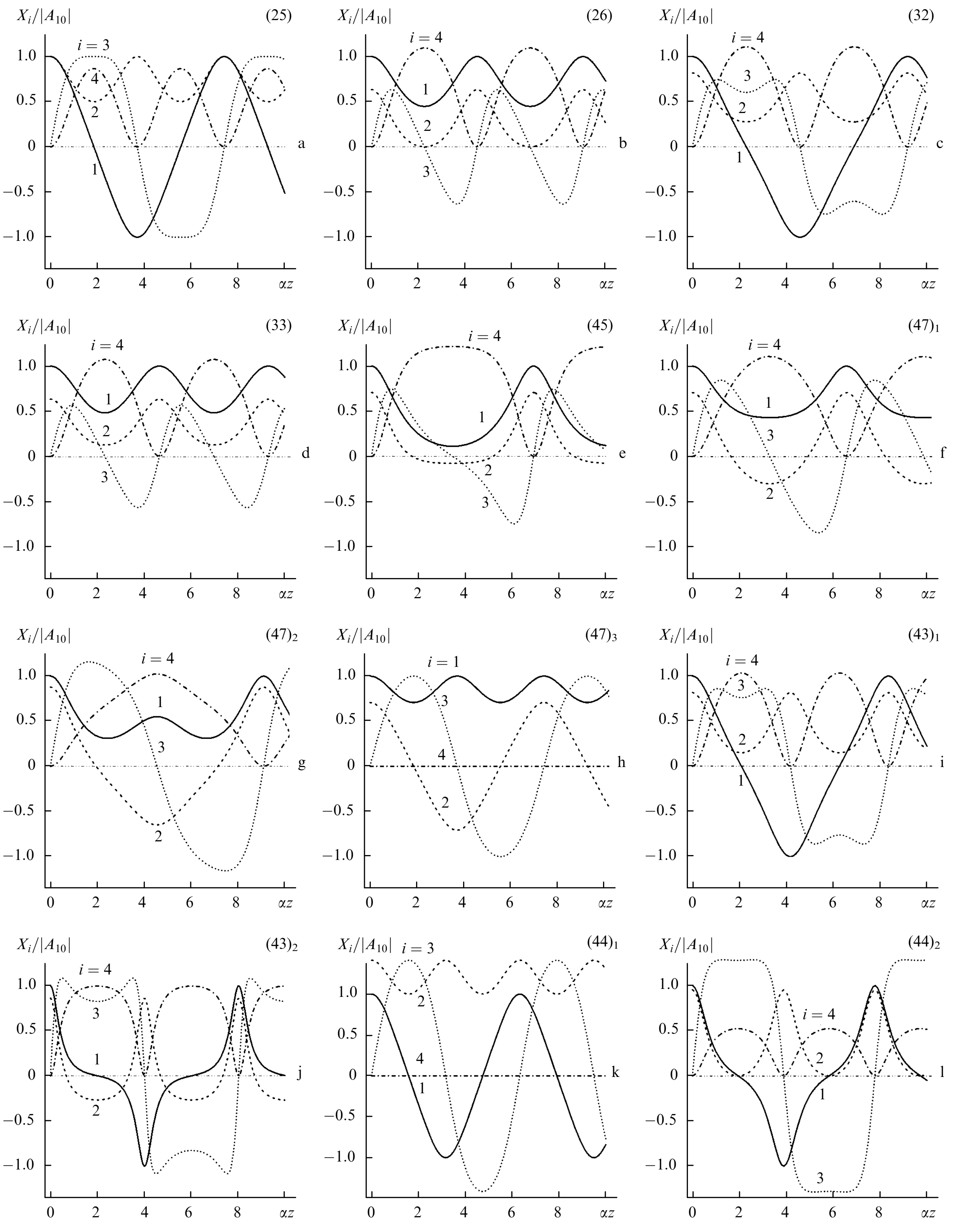

Figure 2. Evolution of $X_{1-4}(z)$ with changing parameters ε and N. The dependences correspond to expressions (25) (a), (26) (b), (32) (c), (33) (d), (45) (e), and (47) (f–h) and to shifted solutions (43) (i, j) and (44) (k, l) from [1]. The values of ε and N for all curves are indicated in the (ε, N) plane by points with the numbers of the corresponding solutions (see Fig. 1).

determined out choice of points for calculations. The amplitude $X_4(z)$ vanishes at the boundary $\varepsilon = -1$, and expressions (7) and (8) pass to classical analytic formulas describing the generation of the wave A_3 by waves $A_{1,2}$ [3].

Note also that, although all the solutions shown in Fig. 2 are constructed by using the fundamental solutions of the first-order Lame equation (see [4]), apart from the period doubling $2K \to 4K$, which always occurs in passing from the function $\mathrm{dn}(z,k)$ with a constant sign to functions $\mathrm{sn}(z,k)$ and $\mathrm{cn}(z,k)$ with alternating signs, the period doubling is also observed for alternating components. Here, $K = K(k)$ is the total elliptic integral of the first kind, which determines both the period of fundamental solutions of the first-order Lame equation (i.e. elliptic Jacobi functions [2]) and the period of analytic solutions described above.

As pointed out above, the two low-frequency modes $A_{1,2}$ in the case under study play the role of two-component pumping, which is used to generate two high-frequency modes $A_{3,4}$. This means that the question on the possibility of realisation of energy-exchange regimes with the depletion of pump waves becomes foremost. It is easy to see that in the case under study the intensity of at least one of the pump components almost always vanishes. The only exception is solution (6) for $\varepsilon > 0$ (Fig. 2d) for which the minimal intensities of the pump components are determined by the expressions

$$I_1^{\min} = I_{10}\left[1 - \frac{8(\varepsilon+1)}{N(\varepsilon+2)^2}\right],\ I_2^{\min} = I_{20}\left(\frac{\varepsilon}{2+\varepsilon}\right)^2. \quad (10)$$

The minimal intensity $I_1^{\min}$ of the first component in solutions (6) for $\varepsilon \leqslant 0$ and (8) for $-2/3 < \varepsilon \leqslant 0$ is determined by the same expression [see (10)] and corresponds to the point at which $I_2 = 0$ (Figs 2e, f). However, solution (8) for $\varepsilon < -2/3$ has already two minima $I_1^{\min} = I_{10}(\varepsilon N + 1)/(\varepsilon N)$ shifted with respect to each other, which are located at points for which the intensities I_{2-4} of other modes are neither minimal nor maximal (Fig. 2g). Note also that the dependence $I_3(z)$ has similar extrema almost for all obtained solutions (Figs 2b–g, i, j). In solutions (5) and (7), we have $I_1^{\min} = 0$ because $X_1(z)$ is an alternating function in this case (see Figs 2a, c, i–l). The minimal intensity of the second pump component is determined by the expression

$$I_2^{\min} = I_{20}\left(\frac{1+\varepsilon-\sqrt{1+\varepsilon N}}{\varepsilon}\right)^2. \quad (11)$$

Obviously due to the complete overlap of the domain of possible variations of boundary conditions, the obtained analytic solutions can provide the optimisation of the conversion efficiency in any particular situation. For example, the proper choice of values of ε and N according to solution (45) from [1] near the separatrix $\varepsilon_-(N)$ gives the maximum conversion efficiency to the frequency ω_4 (Fig. 2e). As a whole, we can state that the region corresponding to solutions (5) corresponds to the possibility of the efficient generation of two modes $A_{3,4}$. The change in the values of parameters ε and N, resulting in the passage to the regions corresponding to solutions (6) and (7), allows one to obtain the predominant generation of only one of the two high-frequency modes (A_4 and A_3, respectively). In solutions of type (8), the generation of three modes ($A_{1,2,4}$) occurs simultaneously.

Of special interest is unusual aperiodic soliton-like solution (9) (Fig. 3). The two components $A_{1,3}$ of this solution [see (9a) and (9c)] are proportional to the real and imaginary parts of the complex Lorentzian dependence, which usually describes the dispersion of the contribution of any resonance transition to the complex permittivity in the case of a homogeneously broadened line. From the mathematical point of view, the appearance of the solution of type (9) in the problem under study is not so unexpected because it is known that an ordinary differential equation of the type $\mathrm{d}^2A/\mathrm{d}z^2 = aA^3$, which is quite similar to the NSE, apart from solutions expressed in terms of elliptic integrals, has also a particular solution $A = \sqrt{2/a}(z-C)^{-1}$, where $a = \mathrm{const}$ is a parameter of the problem and C is an arbitrary constant, which can be purely imaginary [5]. Therefore, the necessary condition for obtaining solution of this type for system (2) is the absence of the linear term ($|\gamma_1|^2 J_1 + 3|\gamma_2|^2 J_3 = 0$) in equations (2a) and (2b), which is reduced in our conditions to the requirement

$$\varepsilon = 2(N^{-1} - 1). \quad (12)$$

Condition (12) is fulfilled on dotted curve (*3*) in Fig. 1. This curve passes, of course, through the singular point ($\varepsilon = -2/3, N = 3/2$) at which the unusual solution is realised. However, it is much interesting that at all the other points (ε, N) in the plane, where this condition is also fulfilled, solutions (5) and (8) are realised [in regions located to the left of separatrices $\varepsilon_\pm(N)$ and below the separatrix $\varepsilon_0(N)$] for $k = 1/2$. In this case, solutions (5), (8), and (9) prove to be analytically sewed together at the point ($\varepsilon = -2/3, N = 3/2$) due to the matched variation in the values of parameters k, α and η in this passage to the limit.

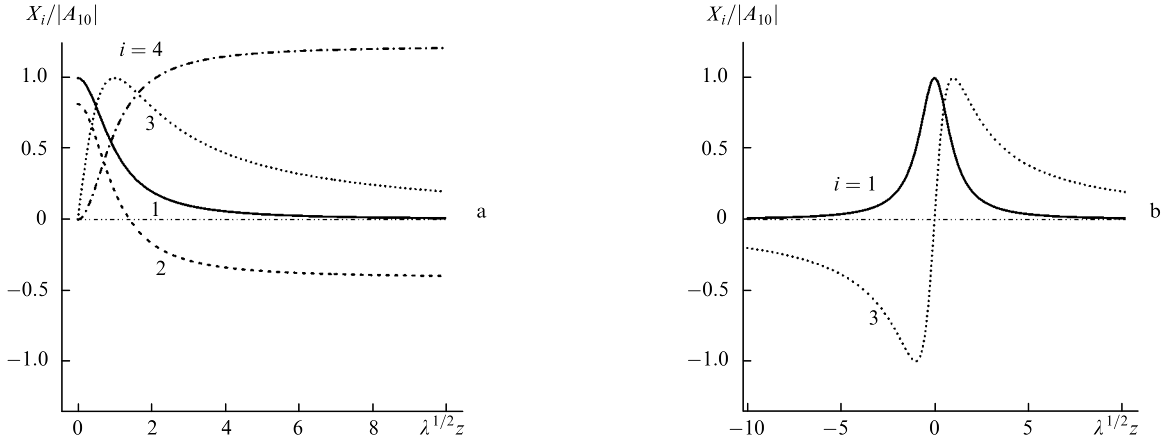

Figure 3. Dependences $X_{1-4}(z)$ at the singular point ($\varepsilon = -2/3, N = 3/2$) (a) and two components $X_{1,3}$ proportional to the real and imaginary parts of the complex Lorentzian dependence along the entire z axis (b).

3. Conclusions

We have analysed the domains of existence and peculiarities of analytic solutions obtained in [1] for the problem of quasi-synchronous interaction of four plane collinear monochromatic waves – modes in a quadratically nonlinear medium during the cascade frequency up-conversion. It has been shown that unusual types of multicomponent cnoidal waves and solitary soliton-like solutions are realised. Two of the four components of the latter are proportional to the real and imaginary parts of the classical Lorentzian dependence which is commonly used to describe the dispersion of contributions from resonance transitions to the complex permittivity in the case of homogeneously broadened lines.

It has been found that due to a complete overlap of the domain of possible variations in boundary conditions, the analytic solutions obtained in [1] can provide the optimisation of the conversion efficiency in any particular situation. Thus, the proper choice of the parameters (ε, N) of the problem in the region corresponding to solution (5) allows the efficient generation of two high-frequency modes $A_{3,4}$. At the same time, the change in the values of these parameters corresponding to the passage to regions corresponding to solutions (6) and (7) allows the predominant generation of only one of these modes (A_4 or A_3, respectively). The solution of type (8) corresponds to the simultaneous generation of three modes $A_{1,2,4}$.

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