NONLINEAR OPTICAL PHENOMENA

PACS numbers: 42.65.Ky, 42.65.Yj, 42.65.Tg, 05.45.-a
DOI: 10.1070/QE2009v039n12ABEH014146

Effective cubic nonlinearity, photoinduced anisotropy, and elliptically polarised cnoidal waves upon frequency doubling

V.M. Petnikova, V.V. Shuvalov

***Abstract*. The peculiarities of the effective cubic nonlinearity are analysed upon second harmonic generation in a medium with a quadratic nonlinearity. It is shown that in this case, the polarisation state of the pump wave changes during its propagation due to additional effective photoinduced anisotropy of propagation constants of two orthogonally polarised pump components. Depending on the initial conditions, either an elliptically polarised cnoidal wave is produced (the polarisation state of pump radiation periodically changes from linear to elliptic and vice versa) or a passage occurs to an aperiodic polarisation 'chaos'.**

Keywords: *frequency doubling on the quadratic nonlinearity, effective cubic nonlinearity and photoinduced anisotropy of propagation constants, elliptically polarised cnoidal wave and aperiodic polarisation 'chaos'.*

1. Introduction

The authors of [1] showed that in the collinear interaction of three plane monochromatic waves (modes), the problem of the stationary parametric frequency conversion on the quadratic nonlinearity [2] is reduced to three independent stationary nonlinear Schrödinger equations (NSEs). Each of them is related to the others only through boundary conditions and describes a cnoidal wave either with a real or complex amplitude. This passage to the NSEs was interpreted as a possibility to describe the result of competition of processes of merging ($\omega_1 + \omega_2 \to \omega_3$) and decay ($\omega_3 \to \omega_1 + \omega_2$) of photons with the frequencies ω_{1-3}, simultaneously proceeding on the second-order nonlinearity, in terms of the effective cascade cubic nonlinearity [3].

We will analyse below the character and peculiarities of this effective nonlinearity for the type II (oee interaction) SHG process ($\omega_1 = \omega_2$) [2] and will show that the proper description of its polarisation peculiarities is not reduced to a trivial account for the well-studied cubic nonlinearity of Kerr type. In this process, both periodic (elliptically polarised cnoidal waves [4]) and aperiodic regimes of the pump wave propagation can be realised due to additional effective photoinduced anisotropy of the dielectric constant of the medium.

V.M. Petnikova, V.V. Shuvalov International Laser Center, M.V. Lomonosov Moscow State University, Vorob'evy gory, 119992 Moscow, Russia; e-mail: vsh@vsh.phys.msu.ru

Received 20 July 2009
Kvantovaya Elektronika **39** (12) 1137 – 1142 (2009)
Translated by I.A. Ulitkin

2. Effective cubic nonlinearity and photoinduced anisotropy

Consider the collinear interaction of three plane linearly polarised monochromatic waves: two waves – at the fundamental frequency (the amplitudes $\boldsymbol{A}_{1,2}$, frequencies $\omega_{1,2} = \omega$, wave vectors $\boldsymbol{k}_{1,2}$) and one wave – at the second harmonic frequency (the amplitude $\boldsymbol{A}_3$, frequency $\omega_3 = 2\omega$, wave vector $\boldsymbol{k}_3$) propagating from the plane $z = 0$ along the z axis in a medium with the quadratic nonlinearity. We assume that the medium occupies the half-space $z \geqslant 0$ in which the type-II parametric process (for example, oee interaction) is realised [2]. By directing now the x and y axes of the laboratory system of coordinates along the orthogonal polarisation vectors $\boldsymbol{e}_{1,2}$ of the pump waves with the amplitudes $\boldsymbol{A}_{1,2}$ and by neglecting the spatial dispersion of the medium and absorption, we will describe the process in question by the known [2] system of equations:

$$\frac{\mathrm{d}A_1}{\mathrm{d}z} = -\mathrm{i}\beta A_2^* A_3 \exp(-\mathrm{i}\varDelta z), \tag{1a}$$

$$\frac{\mathrm{d}A_2}{\mathrm{d}z} = -\mathrm{i}\beta A_1^* A_3 \exp(-\mathrm{i}\varDelta z), \tag{1b}$$

$$\frac{\mathrm{d}A_3}{\mathrm{d}z} = -\mathrm{i}2\beta A_1 A_2 \exp(+\mathrm{i}\varDelta z). \tag{1c}$$

Here, β is the nonlinear coupling constant; $\varDelta = k_1 + k_2 - k_3$ is the wave detuning. System (1) has two integrals of motion:

$$I_1(z) + I_2(z) + I_3(z) = I_{10} + I_{20} + I_{30} = I_0, \tag{2a}$$

$$I_1(z) - I_2(z) = I_{10} - I_{20} = \Delta I_0, \tag{2b}$$

where $I_i(z) = A_i(z) A_i^*(z)$ is a variable proportional to the energy flux density of the ith (hereafter, $i = 1 - 3$) wave, which we will call the intensity; $I_{i0} = I_i|_{z=0}$. Integral (2a) describes the conservation law of the total energy flux density, while (2b) reflects the Manly – Row relation [2].

The authors of [1] showed that system (1) after the substitution of variables

$$A_{1-3}(z) = \tilde{A}_{1-3}(z)\exp(-\mathrm{i}\alpha_{1-3}z), \tag{3}$$

where $\alpha_{1-3} = \text{const}$, is reduced to three independent equations describing the nonlinear self-consistent oscillations of the amplitudes $A_{1-3}(z)$ in terms of the effective local cubic nonlinearity:

$$\frac{\mathrm{d}^2\tilde{A}_1}{\mathrm{d}z^2} - \beta^2\left[\left(I_0 + 3\Delta I_0 - \frac{\Delta^2}{4\beta^2}\right) - 4\tilde{A}_1\tilde{A}_1^*\right]\tilde{A}_1 = 0, \tag{4a}$$

$$\frac{\mathrm{d}^2\tilde{A}_2}{\mathrm{d}z^2} - \beta^2\left[\left(I_0 - 3\Delta I_0 - \frac{\Delta^2}{4\beta^2}\right) - 4\tilde{A}_2\tilde{A}_2^*\right]\tilde{A}_2 = 0, \tag{4b}$$

$$\frac{\mathrm{d}^2\tilde{A}_3}{\mathrm{d}z^2} + \beta^2\left[\left(2I_0 + \frac{\Delta^2}{4\beta^2}\right) - 2\tilde{A}_3\tilde{A}_3^*\right]\tilde{A}_3 = 0. \tag{4c}$$

In this case, a set of values α_{1-3} and boundary conditions in the form

$$\alpha_1 = \Delta/2, \quad \alpha_2 - \alpha_3 = \Delta/2, \tag{5a}$$

$$\tilde{A}_1|_{z=0} = A_{10}, \quad \left.\frac{\mathrm{d}\tilde{A}_1}{\mathrm{d}z}\right|_{z=0} = \mathrm{i}\frac{\Delta}{2}A_{10} - \mathrm{i}\beta A_{20}^* A_{30} \tag{5b}$$

correspond to equation (4a), while

$$\alpha_2 = \Delta/2, \quad \alpha_1 - \alpha_3 = \Delta/2, \tag{6a}$$

$$\tilde{A}_2|_{z=0} = A_{20}, \quad \left.\frac{\mathrm{d}\tilde{A}_2}{\mathrm{d}z}\right|_{z=0} = \mathrm{i}\frac{\Delta}{2}A_{20} - \mathrm{i}\beta A_{10}^* A_{30} \tag{6b}$$

and

$$\alpha_3 = -\Delta/2, \quad \alpha_1 + \alpha_2 = \Delta/2, \tag{7a}$$

$$\tilde{A}_3|_{z=0} = A_{30}, \quad \left.\frac{\mathrm{d}\tilde{A}_3}{\mathrm{d}z}\right|_{z=0} = -\mathrm{i}\frac{\Delta}{2}A_{30} - \mathrm{i}2\beta A_{10}A_{20} \tag{7b}$$

correspond to equations (4b) and (4c), respectively. In this case, because all the equations in (4) are independent, the initial problem is reformulated so that it describes the independent propagation of three waves having different propagation constants due to the effective cubic nonlinearity of the medium. The solutions of all the three equations are still related with each other but only through their boundary conditions (5b)–(7b) and constants entering Eqn (4).

Note that conditions (5a)–(7a) are compatible in pairs. It means that any pair of equations in (4) can be considered as a close system. By selecting, for example, equations (4a) and (4b) as such a pair (i.e. setting $\alpha_1 = \alpha_2 = \Delta/2$ and $\alpha_3 = 0$) and using equalities (2), the corresponding system of equations can be written in different forms. Indeed, by substituting the identity

$$\Delta I_0 \equiv q\Delta I_0 + (1-q)(\tilde{A}_1\tilde{A}_1^* - \tilde{A}_2\tilde{A}_2^*), \tag{8}$$

where q is an arbitrary constant, to equations (4a) and (4b), we obtain the general form for the system of interest:

$$\frac{\mathrm{d}^2\tilde{A}_1}{\mathrm{d}z^2} - \beta^2\left[\left(I_0 + 3q\Delta I_0 - \frac{\Delta^2}{4\beta^2}\right) - (1+3q)\tilde{A}_1\tilde{A}_1^* - 3(1-q)\tilde{A}_2\tilde{A}_2^*\right]\tilde{A}_1 = 0, \tag{9a}$$

$$\frac{\mathrm{d}^2\tilde{A}_2}{\mathrm{d}z^2} - \beta^2\left[\left(I_0 - 3q\Delta I_0 - \frac{\Delta^2}{4\beta^2}\right) - 3(1-q)\tilde{A}_1\tilde{A}_1^* - (1+3q)\tilde{A}_2\tilde{A}_2^*\right]\tilde{A}_2 = 0. \tag{9b}$$

Considering now (9) as an analogue of the pair of equations of motion, we can easily ascertain that in those cases when $\Delta I_0 \neq 0$, the position of the point of static equilibrium

$$I_{1,2}^{(0)} = \frac{1}{4}\left(I_0 \pm \frac{6q}{3q-1}\Delta I_0 - \frac{\Delta^2}{4\beta^2}\right), \tag{10}$$

i.e. the point in which the intensities $I_{1,2}^{(0)}$ zero the second terms ('mechanical forces') in the left-hand sides of (9a) and (9b), proves incompatible with condition (2b) at any q because

$$I_1^{(0)} - I_2^{(0)} = \frac{3q}{3q-1}\Delta I_0 \neq \Delta I_0. \tag{11}$$

A similar incompatibility is also observed in the only singular [with respect to computations performed in deriving (11)] point $q = -1/3$, where system (9) takes the form

$$\frac{\mathrm{d}^2\tilde{A}_{1,2}}{\mathrm{d}z^2} - \beta^2\left[\left(I_0 \mp \Delta I_0 - \frac{\Delta^2}{4\beta^2}\right) - 4\tilde{A}_{2,1}\tilde{A}_{2,1}^*\right]\tilde{A}_{1,2} = 0, \tag{12}$$

for which

$$I_{1,2}^{(0)} = \frac{1}{4}\left(I_0 \pm \Delta I_0 - \frac{\Delta^2}{4\beta^2}\right), \tag{13}$$

and at $\Delta I_0 \neq 0$

$$I_1^{(0)} - I_2^{(0)} = \frac{1}{2}\Delta I_0 \neq \Delta I_0. \tag{14}$$

This means that the system of equations under analysis has stationary (with respect to z) solutions satisfying conservation laws (2) only in those cases, when $\Delta I_0 = 0$.

One can easily see that the notation (4a) and (4b) stands separate from the class of notations (9) because only this notation corresponds to diagonalization (absence of combinations $\tilde{A}_i\tilde{A}_j$ at $i \neq j$) of expression for the potential (free) energy

$$U = U^{(1)} + U^{(2)}, \tag{15a}$$

where

$$U^{(1)} = \beta^2\left(I_0 - \frac{\Delta^2}{4\beta^2}\right)(\tilde{A}_1\tilde{A}_1^* + \tilde{A}_2\tilde{A}_2^*) + 3\beta^2\Delta I_0(\tilde{A}_1\tilde{A}_1^* - \tilde{A}_2\tilde{A}_2^*), \tag{15b}$$

$$U^{(2)} = -2\beta^2[(\tilde{A}_1\tilde{A}_1^*)^2 + (\tilde{A}_2\tilde{A}_2^*)^2] \tag{15c}$$

in the selected coordinate system and only in this case, the dependence of the variables $I_{1,2}$ on each other (2) is not principal. Note also that because equations (4a) and (4b) coincide in their form with the stationary (with respect to time) wave equations in the presence of linear and cubic (with respect to the external field) components to polarisation, relations (15) describe actually photoinduced anisotropy, which is characterised by contributions both to the tensor components of the dielectric constant

$$\Delta\varepsilon_{xx,yy} \propto \frac{\partial^2 U^{(1)}}{\partial \tilde{A}_{1,2}\partial \tilde{A}^*_{1,2}} = \beta^2\left(I_0 \pm 3\Delta I_0 - \frac{\Delta^2}{4\beta^2}\right),$$
$$\Delta\varepsilon_{xy,yx} \propto \frac{\partial^2 U^{(1)}}{\partial \tilde{A}_{2,1}\partial \tilde{A}^*_{1,2}} = 0, \tag{16}$$

and to the cubic nonlinear susceptibility

$$\chi_{xxxx,\,yyyy} \propto \frac{\partial^4 U^{(2)}}{\partial \tilde{A}_{1,2}\partial \tilde{A}^*_{1,2}\partial \tilde{A}_{1,2}\partial \tilde{A}^*_{1,2}} = -4\beta^2,$$
$$\chi_{xxyy,\,yyxx} \propto \frac{\partial^4 U^{(2)}}{\partial \tilde{A}_{1,2}\partial \tilde{A}^*_{1,2}\partial \tilde{A}_{2,1}\partial \tilde{A}^*_{2,1}} = 0 \tag{17}$$

(see [5]). In this case, the use of the term 'dielectric constant' is purely conditional because we deal with the contributions whose quantities, first, depending on the boundary conditions, can be both larger and smaller than zero, and, second, they explicitly depend on the input intensities, and when $I_{10-30} = 0$, are absent. Note again that it is impossible to use identity (8) to change somehow the character and symmetry of expressions (16) and (17) because the diagonalization of tensors corresponding to (15b) and (15c), taking into account the dependences of the variables $I_{1,2}$ on each other [integrals (2)], is of fundamental importance.

3. Elliptically polarised cnoidal waves

Consider now the situation, in which $I_{30} = 0$ and the SH frequency is absent at the input plane of the nonlinear medium. In this case, we can restrict our consideration to the analysis of the propagation process of two orthogonally polarised pump waves with the amplitudes $A_{1,2}$ in a medium with a suitable photoinduced anisotropy of the dielectric constant and cubic nonlinearity. The process of the mode generation with the amplitude A_3 can be 'forgotten' in this case, because we consider it as a physical mechanism responsible for the appearance of the corresponding nonlinearity. In this case, system (4) takes the form

$$\frac{\mathrm{d}^2\tilde{A}_{1,2}}{\mathrm{d}z^2} - \beta^2\left[\left(4I_{10,20} - 2I_{20,10} - \frac{\Delta^2}{4\beta^2}\right) - 4\tilde{A}_{1,2}\tilde{A}^*_{1,2}\right]\tilde{A}_{1,2} = 0 \tag{18}$$

at boundary conditions

$$\tilde{A}_{1,2}|_{z=0} = A_{10,20}, \left.\frac{\mathrm{d}\tilde{A}_{1,2}}{\mathrm{d}z}\right|_{z=0} = \mathrm{i}\frac{\Delta}{2}A_{10,20}, \tag{19}$$

the photoinduced anisotropy of the dielectric constant is determined by the expression

$$\Delta\varepsilon_{xx,\,yy} \propto \beta^2\left(4I_{10,20} - 2I_{20,10} - \frac{\Delta^2}{4\beta^2}\right),$$
$$\Delta\varepsilon_{xy} = \Delta\varepsilon_{yx} = 0, \tag{20}$$

while the effective cubic nonlinearity – by expressions (17). It is easy to notice that the main peculiarity of the problem is independent (with the accuracy to boundary conditions and numerical parameters) nonlinear propagation of two waves with different ($\Delta\varepsilon_{xx} \neq \Delta\varepsilon_{yy}$) propagation constants at $I_{10} \neq I_{20}$.

Although it follows from (19) that at $\Delta \neq 0$, the amplitudes $\tilde{A}_{1,2}$ are complex, and, therefore, using the replacement

$$\tilde{A}_{1,2} = X_{1,2}\exp(\mathrm{i}\varphi_{1,2}) \tag{21}$$

it is necessary to introduce their real amplitudes $X_{1,2}$ and phases $\varphi_{1,2}$, it is convenient to seek for the solution of problem (18) via the complex mode amplitude $\tilde{A}_3$ whose phase is a constant when $I_{30} = 0$ [1]. The solution of equation (4c), taking into account (7), yields

$$\tilde{A}_3(z) = -\mathrm{i}\,\mathrm{sn}(\gamma z, k)\times \frac{2(I_{10}I_{20})^{1/2}\exp[\mathrm{i}(\varphi_{10}+\varphi_{20})]}{\{I_{10}+I_{20}+\Delta^2/8\beta^2+[(I_{10}+I_{20}+\Delta^2/8\beta^2)^2-4I_{10}I_{20}]^{1/2}\}^{1/2}}, \tag{22a}$$

$$\gamma = \beta\{I_{10}+I_{20}+\Delta^2/8\beta^2+[(I_{10}+I_{20}+\Delta^2/8\beta^2)^2 - 4I_{10}I_{20}]^{1/2}\}^{1/2}, \tag{22b}$$

$$k = \frac{2(I_{10}I_{20})^{1/2}}{I_{10}+I_{20}+\Delta^2/8\beta^2+[(I_{10}+I_{20}+\Delta^2/8\beta^2)^2-4I_{10}I_{20}]^{1/2}}. \tag{22c}$$

In this case, it follows directly from conservation laws (2) that

$$X^2_{1,2}(z) = I_{10,20}\times \left[1 - \frac{2I_{20,10}}{I_{10}+I_{20}+\Delta^2/8\beta^2+[(I_{10}+I_{20}+\Delta^2/8\beta^2)^2-4I_{10}I_{20}]^{1/2}} \times \mathrm{sn}^2(\gamma z, k)\right]. \tag{23}$$

Here, $\mathrm{sn}(\xi, k)$ is the elliptic Jacobi function; ξ and $1 \geqslant k \geqslant 0$ is its argument and modulus [6]. For the phases $\varphi_{1,2}$ of the searched-for solutions $\tilde{A}_{1,2}$ of system (18), taking into account (19) and expression (7) from paper [7], we can write at once

$$\varphi_{1,2}(z) = \varphi_{10,20} + \frac{\Delta}{2}I_{10,20}\int_0^z \frac{\mathrm{d}z'}{X^2_{1,2}(z')}. \tag{24}$$

After that expressions (23) and (24) determine the solutions $\tilde{A}_{1,2}(z)$ of the system of equations (18) necessary to us for the further analysis.

It is obvious directly from the character of the above expressions that the division of the contributions, whose appearance is associated with the effective photoinduced anisotropy of the dielectric constant (20) and cubic nonlinearity (17), is not a trivial problem in the general case. Therefore, we will pass to a simpler situation with a linearly polarised input pump radiation and will assume that at $z = 0$, the vector of its polarisation $\boldsymbol{e}_0$ lies in the plane xy and is turned by the angle ψ_0 with respect to the y axis due to which two orthogonally polarised field components at the frequency ω with the amplitudes $\boldsymbol{A}_{1,2}$ are formed at the medium input. Taking into account the fact that $\varphi_{10} = \varphi_{20} = \varphi_0$, the boundary conditions for system (18) are written in the form

$$\tilde{A}_{10} = \sqrt{I_0}\sin\psi_0\exp(\mathrm{i}\varphi_0), \tag{25a}$$

$$\tilde{A}_{20} = \sqrt{I_0}\cos\psi_0\exp(\mathrm{i}\varphi_0). \tag{25b}$$

To illustrate clearly the role of the photoinduced processes, we will neglect now the initial anisotropy of the medium by setting $\boldsymbol{k}_1 = \boldsymbol{k}_2$. In practice, this can correspond, for example, to introducing, to the converter scheme, an additional wave plate compensating for the difference in the refractive indices of two pump components. It follows from (20) that in this case, the photoinduced anisotropy of the dielectric constant $\Delta\varepsilon = \Delta\varepsilon_{yy} - \Delta\varepsilon_{xx} \propto 6\beta^2 I_0\cos 2\psi_0$ does not vanish only when $\psi_0 \neq \psi_n = (2n+1)\pi/4$, where n is an arbitrary integer. The current polarisation state of the pump wave can thus be determined by two parameters, $\tan\psi(z) = X_1(z)/X_2(z)$ and $\Delta\varphi(z) = \varphi_1(z) - \varphi_2(z)$, describing its evolution during the radiation propagation [8]. Taking into account (23)–(25), it is easy to see that under those conditions when $\psi_0 \neq \psi_n$, the polarisation state of the wave at the frequency ω really changes. Thus, at $\psi_0 \neq \psi_n$ and $\varDelta = 0$ (i.e. when the phase-matching condition is exactly fulfilled), the pump radiation polarisation remains linear because

$$\Delta\varphi(z) = \varphi_1 - \varphi_2 \equiv 0, \tag{26}$$

although the orientation of the polarisation vector $\boldsymbol{e}$ in the plane xy changes periodically during the propagation because

$$\tan\psi(z) = X_1/X_2$$

$$= \tan\psi_0\left[\frac{1 + |\cos 2\psi_0| - 2\cos^2\psi_0\,\mathrm{sn}^2(\gamma z, k)}{1 + |\cos 2\psi_0| - 2\sin^2\psi_0\,\mathrm{sn}^2(\gamma z, k)}\right]^{1/2} \tag{27}$$

periodically changes (see [8]). In this case, $\tan\psi(z) = \tan\psi_0$, and the directions $\boldsymbol{e}$ and $\boldsymbol{e}_0$ coincide only at the points $z = z_n = 2nK/\gamma$, i.e. at distances from the input plane $z = 0$ multiple of the period $2K/\gamma$ of oscillations $X_{1,2}$. Here, $K(k)$ is the complete elliptic integral [6].

At $\psi_0 \neq \psi_n$ and $\varDelta \neq 0$, the situation becomes significantly more complicated. The polarisation of the pump wave continuously changes during its propagation, thereby transforming constantly from linear to elliptic and vice versa. Taking into account (24) and (25), pump radiation is linearly polarised only at those points $z = z_j$ of the z axis, where

$$\Delta\varphi(z_j) = \frac{\varDelta}{2} I_0 \int_0^{z_j}\left[\frac{\sin^2\psi_0}{X_1^2(z')} - \frac{\cos^2\psi_0}{X_2^2(z')}\right]\mathrm{d}z' = j2\pi. \tag{28}$$

Here, j is an arbitrary integer. Because in the general case, $z_j \neq z_n$ at no j and n (see above), the vector $\boldsymbol{e}$ even at $z = z_j$ should be rotated with respect to $\boldsymbol{e}_0$. At other points $z \neq z_j$, the pump wave should be elliptically polarised, the orientation of the polarisation ellipse axes of this wave being different all the time because

$$\tan\psi(z) = \tan\psi_0\Bigg(\Bigg\{1 + \frac{\varDelta^2}{8\beta^2 I_0} + \Bigg[\left(1 + \frac{\varDelta^2}{8\beta^2 I_0}\right)^2 - \sin^2 2\psi_0\Bigg]^{1/2} - 2\cos^2\psi_0\,\mathrm{sn}^2(\gamma z, k)\Bigg\}$$
$$\times\Bigg\{1 + \frac{\varDelta^2}{8\beta^2 I_0} + \Bigg[\left(1 + \frac{\varDelta^2}{8\beta^2 I_0}\right)^2 - \sin^2 2\psi_0\Bigg]^{1/2} - 2\sin^2\psi_0\,\mathrm{sn}^2(\gamma z, k)\Bigg\}^{-1}\Bigg)^{1/2} \tag{29}$$

and $\psi(z) = \psi_0$ only at $z = z_n$.

This rather complicated transformation can be illustrated with the help of the Poincare sphere [8]. To this end, after determining in the standard way the components s_{1-3} of the normalised $(s_1^2 + s_2^2 + s_3^2 = 1)$ Stokes vector $\boldsymbol{s} = \{s_1, s_2, s_3\}$, taking into account substitution (21) and boundary conditions (25), we obtain

$$s_1 = \frac{\tilde{A}_1\tilde{A}_2^* + \tilde{A}_1^*\tilde{A}_2}{\tilde{A}_1\tilde{A}_1^* + \tilde{A}_2\tilde{A}_2^*} = \frac{2X_1X_2}{X_1^2 + X_2^2}\cos\Delta\varphi, \tag{30a}$$

$$s_2 = \mathrm{i}\frac{\tilde{A}_1^*\tilde{A}_2 - \tilde{A}_1\tilde{A}_2^*}{\tilde{A}_1\tilde{A}_1^* + \tilde{A}_2\tilde{A}_2^*} = \frac{2X_1X_2}{X_1^2 + X_2^2}\sin\Delta\varphi, \tag{30b}$$

$$s_3 = \frac{\tilde{A}_1\tilde{A}_1^* - \tilde{A}_2\tilde{A}_2^*}{\tilde{A}_1\tilde{A}_1^* + \tilde{A}_2\tilde{A}_2^*} = -\frac{I_0\cos 2\psi_0}{X_1^2 + X_2^2}. \tag{30c}$$

It follows from relations (30) that at $\psi_0 = \psi_n$ and $\Delta\varphi \equiv 0$ (i.e. at $\Delta I_0 = 0$ and $\varDelta = 0$), the polarisation state of the pump wave does not change and is described on the Poincare sphere by the point $\{s_1 = 1, s_2 = 0, s_3 = 0\}$ lying in its equatorial plane (Fig. 1a). At $\psi_0 \neq \psi_n$ and $\Delta\varphi \equiv 0$ (i.e. at $\Delta I_0 \neq 0$ and $\varDelta = 0$), the end of the vector $\boldsymbol{s}$, starting its motion along the surface of the Poincare sphere from the point $\{s_1 = \sin 2\psi_0, s_2 = 0, s_3 = -\cos 2\psi_0\}$ and passing along the meridian $s_2 = 0$ (conservation of linear polarisation) through its closest vertex $(s_3 = \pm 1)$, reaches the point $\{s_1 = -\sin 2\psi_0, s_2 = 0, s_3 = -\cos 2\psi_0\}$, where it turns around and starts moving in the backward direction (Fig. 1a, heavy solid curve). This periodic change in the orientation $\boldsymbol{s}$ corresponds to the fact that the polarisation of the pump wave, remaining linear, changes periodically its direction in the xy plane.

At $\psi_0 \neq \psi_n$ and $\Delta \neq 0$ (i.e. at $\Delta I_0 \neq 0$ and $\varDelta \neq 0$), the character of motion of the end of the vector $\boldsymbol{s}$ becomes more complicated. Note, first of all, that it follows from (30) that the polar (φ) and azimuth (θ) angles, characterising the orientation of the vector $\boldsymbol{s}$ in the coordinate system $\{s_1, s_2, s_3\}$ are determined by the expressions

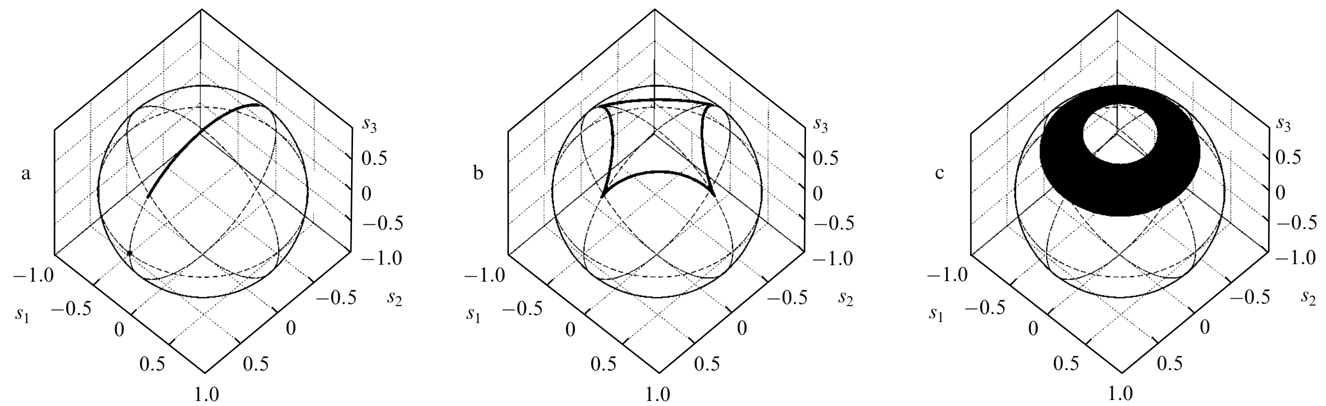

Figure 1. Trajectory of motion (in changing z) of the end of the normalised Stokes vector $\boldsymbol{s} = \{s_1, s_2, s_3\}$ of the pump wave on the Poincare sphere: $\Delta I_0 = 0$ and $\varDelta = 0$ ($\psi_0 = \pi/4$ and $\Omega = 1$, the point is in the equatorial plane), $\Delta I_0 \neq 0$ and $\varDelta = 0$ ($\psi_0 = 2$ and $\Omega = 1$, heavy solid curve) (a); $\Delta I_0 \neq 0$ and $\varDelta \neq 0$ for the radio of the periods of changes in φ and θ equal to 4 ($\psi_0 = 2$ and $\Omega \approx 1.1667$) (b) and $2\sqrt{3}$ ($\psi_0 = 2$ and $\Omega = 1.1065$) (c).

$$\cot\varphi = \frac{s_2}{s_1} = \tan\Delta\varphi, \tag{31a}$$

$$\cot\theta = \frac{s_3}{(s_1^2 + s_2^2)^{1/2}} = -\frac{I_0\cos 2\psi_0}{2X_1X_2}. \tag{31b}$$

It follows from (31b) taking into account (23) and (25) that at points $z = z_n$ and $z = (z_n + 2K/\gamma)$ for which the values $X_{1,2}^2$ are maximal and minimal, the relations

$$\cot\theta|_{\mathrm{sn}^2(\gamma z,k)=0} = -\frac{\cos 2\psi_0}{|\sin 2\psi_0|}, \tag{32a}$$

$$\cot\theta|_{\mathrm{sn}^2(\gamma z,k)=1} = -\frac{\cos 2\psi_0}{|\sin 2\psi_0|} \times \frac{[\Omega + (\Omega^2 - \sin^2 2\psi_0)^{1/2}]^{1/2}}{[2(\Omega-1)]^{1/2}} \tag{32b}$$

are fulfilled, where $\Omega = 1 + \varDelta^2/(8\beta^2 I_0) \geqslant 1$. Therefore, in the evolution of the polarisation state of the pump wave (during its propagation), the end of the vector $\boldsymbol{s}$ should move along the trajectories localised within the spherical layer surface

$$\theta \in [\theta|_{\mathrm{sn}^2(\gamma z,k)=0},\ \theta|_{\mathrm{sn}^2(\gamma z,k)=1}]. \tag{33}$$

The character of this motion, due to the parameter $\Omega = 1 + \varDelta^2/(8\beta^2 I_0)$, depends on the ratio of the periods of changes in φ and θ. If these periods are multiple, i.e. the condition

$$j\pi = [2(\Omega-1)]^{1/2}[\Omega + (\Omega^2 - \sin^2 2\psi_0)^{1/2}]^{1/2}\cos 2\psi_0 \times \int_0^{2nK} \mathrm{sn}^2(\xi,k)\mathrm{d}\xi[\Omega + (\Omega^2 - \sin^2 2\psi_0)^{1/2} - 2\cos^2\psi_0\mathrm{sn}^2(\xi,k)]^{-1}[\Omega + (\Omega^2 - \sin^2 2\psi_0)^{1/2} - 2\sin^2\psi_0\mathrm{sn}^2(\xi,k)]^{-1} \tag{34}$$

is fulfilled, where j and n are integers, the polarisation state of the pump wave changes periodically (Fig. 1b) and a cnoidal wave is produced in which radiation polarisation during its propagation changes periodically from linear to elliptic and vice versa. If condition (34) is not fulfilled and the ratio of the periods of changes in φ and θ is irrational, the polarisation state of the pump wave during its propagation changes aperiodically, and despite the strict determinancy of the problem, the end of the Stokes vector $\boldsymbol{s}$ during its evolution obligatory passes through each point of the spherical layer surface (33) (Fig. 1c). The vector $\boldsymbol{s}$ behaves in this case as a strange attractor, completely filling the corresponding region of the phase space, during its aperiodic movement.

4. Conclusions

We have shown above that the presence of the effective photoinduced anisotropy of the dielectric constant ($\Delta\varepsilon \neq 0$) at $\Delta I_0 \neq 0$ and $\varDelta \neq 0$ drastically distinguishes the parametric frequency conversion on the quadratic nonlinearity from the well-studied processes of cnoidal wave formation in media with the local cubic nonlinearity. Due to this, the solution of the corresponding problem is not reduced to the trivial account for the ordinary effective Kerr nonlinearity.

In the type-II SHG process, at $\Delta I_0 \neq 0$ and $\varDelta \neq 0$, the evolution character of the polarisation state of pump radiation is determined, due to this effective anisotropy, by the value of the numerical parameter $\Omega = 1 + \varDelta^2/(8\beta^2 I_0)$ and can be completely different. When condition (34) is fulfilled, pump radiation represents a cnoidal wave whose polarisation changes periodically from linear to elliptic and vice versa during its propagation due to a change in the ratios of the amplitudes and phases of its two orthogonally polarised components. When this condition is violated, the Stokes vector of the pump wave behaves as a strange attractor and the polarisation state of this wave changes aperiodically. Note that the formation of solitary and periodical nonlinear waves of a similar type – the so-called elliptically polarised solitons and cnoidal waves – was considered before only for birefrigent optical fibres and gyrotropic media with the Kerr-type nonlinearity [4], i.e. for systems with real but ineffective anisotropy.

Note also that allowance for the above-described photo-induced polarisation peculiarities of the pump radiation propagation in the case of type-II parametric interaction is very important from the practical point of view, especially, in designing efficient intracavity frequency doublers.

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