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STIMULATED CERENKOV RADIATION

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Quantum theory of stimulated Cerenkov radiation of transverse electromagnetic waves by a low-density electron beam in a medium

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Abstract**. The quantum theory of stimulated Cerenkov radiation of transverse electromagnetic waves by an electron beam in an anisotropic medium is presented. Relativistic quantum nonlinear equations of the Cerenkov beam instability are obtained. In the linear approximation, the quantum dispersion equation is derived and the instability growth increments are determined. The nonlinear problem of the saturation of the quantum Cerenkov beam instability is solved.**

Keywords*: simulated Cerenkov radiation, transverse electromagnetic waves, low-density electron beam.*

The quantum theory of the Vavilov – Cerenkov effect was first considered by V.L. Ginzburg based on the laws of conservation of energy and momentum during the interaction of an electron and photon in a medium [1]. The electron was described classically, while the relation between the photon and its momentum and energy was established taking into account the influence of the medium. In fact, paper [1] considered spontaneous radiation because the individual electron was described, neglecting the influence of other radiating electrons on it. In the quantum consideration of stimulated Cerenkov radiation of an electron beam, another approach, developed in the theory of plasma and classic microwave electronics, is justified, where stimulated Cerenkov radiation is treated as a resonance beam instability. Using this approach, the authors of papers [2, 3] considered quantum stimulated Cerenkov radiation of longitudinal waves in plasma. In this paper, we study Cerenkov radiation of transverse electromagnetic waves stimulated in a medium by a relativistic monoenergetic electron beam.

It is known that in the absence of collisions, the most general quantum description of a system of charged particles, including an electron beam, is performed with the help of a density matrix [4, 5]. If the velocity spread of the particles is absent, the density matrix is expressed by the product of the wave functions and the equation for the matrix is reduced to the Schrödinger equation. Therefore, we will describe the electrons of the beam by the equation for the wave function supplemented by an equation for the self-consistent electromagnetic field. Because we will consider a relativistic electron beam, the Klein – Gordon – Fock equation is used as an equation for the wave function.

Taking all the above into account, we will use the following system of equations for the vector $[\boldsymbol{A}(t,\boldsymbol{r})]$ and scalar $[\varphi(t,\boldsymbol{r})]$ potentials of the electromagnetic field in the Coulomb gauge $(\nabla\boldsymbol{A}=0)$:

$$\Delta\boldsymbol{A}-\frac{\varepsilon}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}=-\frac{4\pi}{c}\boldsymbol{j}_{\rm b}+\frac{\varepsilon}{c}\nabla\frac{\partial\varphi}{\partial t},$$
$$\Delta\varphi=-\frac{4\pi}{\varepsilon}\rho_{\rm b}. \qquad (1)$$

Here, ε is the dielectric constant operator, and the current $(\boldsymbol{j}_{\rm b})$ and charge $(\rho_{\rm b})$ densities of electrons in the beam are expressed via the wave function of an electron by the expression [6]

$$\boldsymbol{j}_{\rm b}=-{\rm i}\frac{e\hbar}{2m}[\psi^*\nabla\psi-(\nabla\psi^*)\psi]-\frac{e^2}{mc}\psi\psi^*\boldsymbol{A},$$
$$\rho_{\rm b}={\rm i}\frac{e\hbar}{2mc^2}\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right)-\frac{e^2}{mc^2}\psi\psi^*\varphi. \qquad (2)$$

The wave function $\psi(t,\boldsymbol{r})$ is defined from the Klein – Gordon – Fock equation linearized with respect to the potentials:

$$\hbar^2\frac{\partial^2\psi}{\partial t^2}-\hbar^2c^2\Delta\psi+m^2c^4\psi$$
$$=-2{\rm i}e\hbar\left[\varphi\frac{\partial\psi}{\partial t}+\frac{1}{2}\frac{\partial\varphi}{\partial t}\psi+c(\boldsymbol{A}\nabla)\psi\right]. \qquad (3)$$

Because in the unperturbed state the spread of the electrons in the beam is absent, the unperturbed wave function of each electron can be determined by the expressions

$$\psi(t,z)=N\exp(-{\rm i}\omega_0t+{\rm i}k_0z),\quad k_0=\frac{mu\gamma}{\hbar},$$
$$\omega_0=\left(k_0^2c^2+\frac{m^2c^4}{\hbar^2}\right)^{1/2}=\frac{mc^2\gamma}{\hbar}, \qquad (4)$$

where u is the electron velocity in the beam; $\gamma=(1-u^2/c^2)^{-1/2}$ is the relativistic factor of electrons in the

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beam; $N = n_b^{1/2}\gamma^{-1/2}$ is the normalised multiplier; n_b is the electron concentration in the beam. Wave function (4) is an initial condition (at $t = 0$ or $t \to \infty$) for the Klein–Gordon–Fock equation. Note that if the electrons in the beam were spread with respect to momenta, the quantum description would be more difficult: in solving equation (3) with initial condition (4), the momentum $mu\gamma$ would be a free parameter, which, taking into account the distribution function relative to the momenta in expressions (2), would be used in integration. In the nonrelativistic case, this description would be completely equivalent to the description by a single-particle density matrix [5]. The momentum spread is insignificant when the inequality $|\delta\omega/\omega| \gg \Delta p \times (mu\gamma)^{-1}$ is fulfilled, where $\delta\omega$ is the increment, ω is the radiation frequency, and Δp is the width of the distribution function of electrons in the beam with respect to momenta.

We will show below that Cerenkov radiation in the form of transverse waves is possible only at an angle to the direction of unperturbed motion of electrons in the beam; hence, we will represent the potentials and the wave function in the form

$$\boldsymbol{A}(t,\boldsymbol{r}) = \frac{1}{2}[\tilde{\boldsymbol{A}}(t)\exp(\mathrm{i}\boldsymbol{kr}) + \tilde{\boldsymbol{A}}^*(t)\exp(-\mathrm{i}\boldsymbol{kr})],$$

$$\varphi(t,\boldsymbol{r}) = \frac{1}{2}[\tilde{\varphi}(t)\exp(\mathrm{i}\boldsymbol{kr}) + \tilde{\varphi}^*(t)\exp(-\mathrm{i}\boldsymbol{kr})], \tag{5}$$

$$\psi = H_0(t)\exp(\mathrm{i}\boldsymbol{k}_0\boldsymbol{r}) + H_-(t)\exp[\mathrm{i}(\boldsymbol{k}_0 - \boldsymbol{k})\boldsymbol{r}] + H_+(t)\exp[\mathrm{i}(\boldsymbol{k}_0 + \boldsymbol{k})\boldsymbol{r}].$$

Here, $\boldsymbol{k}_0 = \{0, 0, k_0\}$; $\boldsymbol{k} = \{k_\perp, 0, k_\parallel\}$. By substituting (5) in expressions (2) and equations (1), after rather cumbersome computations we obtain the equations:

$$\frac{\varepsilon}{c^2}\frac{\mathrm{d}^2\tilde{\boldsymbol{A}}}{\mathrm{d}t^2} + k^2\tilde{\boldsymbol{A}} + \mathrm{i}\,\frac{\varepsilon}{c}\,\boldsymbol{k}\,\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t} + \frac{4\pi e^2}{mc^2}(H_0H_0^* + H_-H_-^* + H_+H_+^*)\tilde{\boldsymbol{A}} = \frac{8\pi}{c}\frac{e\hbar}{2m}[(2\boldsymbol{k}_0 - \boldsymbol{k})H_0H_-^* + (2\boldsymbol{k}_0 + \boldsymbol{k})H_0^*H_+],$$

$$k^2\tilde{\varphi} + \frac{4\pi e^2}{\varepsilon mc^2}(H_0H_0^* + H_-H_-^* + H_+H_+^*)\tilde{\varphi} = \mathrm{i}\,\frac{8\pi}{\varepsilon}\,\frac{e\hbar}{2mc^2}\left[\left(\dot{H}_0H_-^* - H_0\dot{H}_-^*\right) + \left(H_0^*\dot{H}_+ - \dot{H}_0^*H_+\right)\right], \tag{6}$$

$$\hbar^2\frac{\mathrm{d}^2H_0}{\mathrm{d}t^2} + (\hbar^2k_0^2c^2 + m^2c^4)H_0 = -\mathrm{i}e\hbar\Big\{\left(\tilde{\varphi}\dot{H}_- + \tilde{\varphi}^*\dot{H}_+\right) + \frac{1}{2}\left(\dot{\tilde{\varphi}}H_- + \dot{\tilde{\varphi}}^*H_+\right) + \mathrm{i}c\left[(\boldsymbol{k}_0 - \boldsymbol{k})\tilde{\boldsymbol{A}}H_- + (\boldsymbol{k}_0 + \boldsymbol{k})\tilde{\boldsymbol{A}}^*H_+\right]\Big\},$$

$$\hbar^2\frac{\mathrm{d}^2H_-}{\mathrm{d}t^2} + [\hbar^2(\boldsymbol{k}_0 - \boldsymbol{k})^2c^2 + m^2c^4]H_- = -\mathrm{i}e\hbar\left(\dot{H}_0\tilde{\varphi}^* + \frac{1}{2}\,H_0\dot{\tilde{\varphi}}^* + \mathrm{i}cH_0\boldsymbol{k}_0\tilde{\boldsymbol{A}}*\right), \tag{7}$$

$$\hbar^2\frac{\mathrm{d}^2H_+}{\mathrm{d}t^2} + [\hbar^2(\boldsymbol{k}_0 + \boldsymbol{k})^2c^2 + m^2c^4]H_+ = -\mathrm{i}e\hbar\left(\dot{H}_0\tilde{\varphi} + \frac{1}{2}\,H_0\dot{\tilde{\varphi}} + \mathrm{i}cH_0\boldsymbol{k}_0\tilde{\boldsymbol{A}}\right).$$

Equations (6) and (7) are basic for the presented relativistic quantum theory of Cerenkov radiation of transverse electromagnetic waves in a medium.

In the linear approximation, the right-hand side of the first equation of system (7) is equal to zero; hence, taking (4) into account, we have $H_0(t) = N\exp(-\mathrm{i}\omega_0 t)$. In this case, after linearization other equations of systems (6) and (7) take the form

$$\frac{\mathrm{d}^2H_-}{\mathrm{d}t^2} + \omega_-^2H_- = -\frac{en_b^{1/2}}{\hbar\gamma^{1/2}}\left(\omega_0\tilde{\varphi}^* + \frac{1}{2}\,\mathrm{i}\dot{\tilde{\varphi}}^* - c\boldsymbol{k}_0\tilde{\boldsymbol{A}}^*\right) \times \exp(-\mathrm{i}\omega_0 t),$$

$$\frac{\mathrm{d}^2H_+}{\mathrm{d}t^2} + \omega_+^2H_+ = -\frac{en_b^{1/2}}{\hbar\gamma^{1/2}}\left(\omega_0\tilde{\varphi} + \frac{1}{2}\mathrm{i}\dot{\tilde{\varphi}} - c\boldsymbol{k}_0\tilde{\boldsymbol{A}}\right)\exp(-\mathrm{i}\omega_0 t),$$

$$\frac{\varepsilon}{c^2}\,\frac{\mathrm{d}^2\tilde{\boldsymbol{A}}}{\mathrm{d}t^2} + \left(k^2 + \frac{\omega_b^2\gamma^{-1}}{c^2}\right)\tilde{\boldsymbol{A}} + \mathrm{i}\,\frac{\varepsilon}{c}\,\boldsymbol{k}\,\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t} = \frac{8\pi}{c}\,\frac{e\hbar n_b^{1/2}}{2m\gamma^{1/2}} \times[(2\boldsymbol{k}_0 - \boldsymbol{k})H_-^*\exp(-\mathrm{i}\omega_0 t) + (2\boldsymbol{k}_0 + \boldsymbol{k})H_+\exp(\mathrm{i}\omega_0 t)], \tag{8}$$

$$\left(k^2 + \frac{\omega_b^2\gamma^{-1}}{\varepsilon c^2}\right)\tilde{\varphi} = -\mathrm{i}\,\frac{8\pi}{\varepsilon}\,\frac{e\hbar n_b^{1/2}}{2mc^2\gamma^{1/2}} \times\left[\left(\mathrm{i}\omega_0H_-^* + \dot{H}_-^*\right)\exp(-\mathrm{i}\omega_0 t) + \left(\mathrm{i}\omega_0H_+ - \dot{H}_+\right)\exp(\mathrm{i}\omega_0 t)\right].$$

Here, ω_b is the Langmuir frequency of electrons in the beam;

$$\omega_\mp = \left[(\boldsymbol{k}_0 \mp \boldsymbol{k})^2c^2 + \frac{m^2c^4}{\hbar^2}\right]^{1/2}. \tag{9}$$

We will represent the solution of equations (8) in the form

$$\tilde{\boldsymbol{A}} = \boldsymbol{a}\exp(-\mathrm{i}\omega t), \quad \tilde{\varphi} = b\exp(-\mathrm{i}\omega t),$$
$$H_- = a_-\exp[-\mathrm{i}(\omega_0 - \omega)t], \quad H_+ = a_+\exp[-\mathrm{i}(\omega_0 + \omega)t], \tag{10}$$

where $\boldsymbol{a}$, b, a_-, a_+ are the constants. By substituting expressions (10) in equations (8) and eliminating the constants, we obtain the dispersion equations to determine the complex frequency $\omega(\boldsymbol{k})$:

$$D_\perp(\omega,\boldsymbol{k})D_\parallel(\omega,\boldsymbol{k}) = k_\perp^2u^2\omega_b^2\gamma^{-1}(\varepsilon - 1) \times\left[(\omega - \boldsymbol{ku})^2 - \frac{\hbar^2\left(\omega^2 - k^2c^2\right)^2}{4m^2c^4\gamma^2}\right]^{-1}, \tag{11}$$

where

$$D_\perp(\omega,\boldsymbol{k}) = \omega^2\varepsilon - k^2c^2 - \omega_b^2\gamma^{-1};$$
$$D_\parallel(\omega,\boldsymbol{k}) = \varepsilon - \omega_b^2\gamma^{-1}\left[1 - \frac{u^2}{c^2} - \frac{\hbar^2\left(\omega^2 - k^2c^2\right)}{4m^2c^4\gamma^2}\right] \times\left[(\omega - \boldsymbol{ku})^2 - \frac{\hbar^2\left(\omega^2 - k^2c^2\right)^2}{4m^2c^4\gamma^2}\right]^{-1}. \tag{12}$$

In the classical limit $\hbar \to 0$, equation (11) is transformed into known dispersion equation [7]

$$\left(\omega^2\varepsilon - k^2c^2 - \omega_b^2\gamma^{-1}\right)\left[\varepsilon - \frac{\omega_b^2\gamma^{-3}}{(\omega - \boldsymbol{ku})^2}\right] = \frac{\omega_b^2\gamma^{-1}}{(\omega - \boldsymbol{ku})^2}\, k_\perp^2 u^2(\varepsilon - 1). \qquad (13)$$

The equation $D_\perp(\omega, \boldsymbol{k}) = 0$ determines the frequencies of transverse electromagnetic waves in a medium with the dielectric constant ε, while the equation $D_\parallel(\omega, \boldsymbol{k}) = 0$ – the frequencies of quantum longitudinal waves of a relativistic electron beam. At $k_\perp = 0$, i.e. during the propagation in the direction of the beam motion, the transverse waves do not interact with the beam and Cerenkov radiation proves possible only in longitudinal waves. Cerenkov radiation of longitudinal waves in plasma ($\varepsilon = 1 - \omega_p^2/\omega^2$, ω_p is the plasma frequency) in the nonrelativistic case was considered in paper [3] and now presents no interest to us. Of no interest is also the case of interaction of the electron beam with the transverse waves in plasma. Indeed, in this case, we obtain from the equation $D_\perp(\omega, \boldsymbol{k}) = 0$ at $\varepsilon = 1 - \omega_p^2/\omega^2$ the transverse wave frequency $\omega = (k^2c^2 + \omega_p^2 + \omega_b^2\gamma^{-1})^{1/2}$. The phase velocity of this wave is greater that the velocity of light and Cerenkov radiation is impossible both in the classical and quantum cases. Therefore, we consider here stimulated Cerenkov radiation in an isotropic dispersionless dielectric, i.e. we assume that $\varepsilon = \text{const} > 1$.

When the electron density in equation (11) tends to zero, we obtain the quantum condition of the Cerenkov wave – particle resonance:

$$\omega = \boldsymbol{ku} \pm \frac{\hbar\left(\omega^2 - k^2c^2\right)}{2mc^2\gamma}. \qquad (14)$$

Let us explain that we define the Cerenkov resonance condition as the poles of the beam terms in the dispersion equation. At $\varepsilon = \text{const}$ and $\omega_b \to 0$, the transverse wave frequency is $\omega = kc/\sqrt{\varepsilon}$; therefore, expression (14) can be transformed to the form

$$\omega = \boldsymbol{ku} \pm \frac{\varepsilon - 1}{\varepsilon}\frac{\hbar k^2}{2m\gamma}. \qquad (15)$$

Equalities (15) should be considered as the quantum conditions of Cerenkov radiation (the 'minus' sign) and Cerenkov absorption (the 'plus' sign) in the isotropic dielectric. It is condition (15) with the 'minus' sign that was first derived based on the conservation laws in paper [1]. If the quantum member is small, condition (14) can be written in another form:

$$\omega = \boldsymbol{ku} \pm \frac{\hbar\left(k_\perp^2 + k_\parallel^2\gamma^{-2}\right)}{2m\gamma}. \qquad (16)$$

In the case of a low-density electron beam, when the inequality is $\omega_b^2\gamma^{-1} \ll k_\perp^2 c_0^2$ ($c_0 = c/\sqrt{\varepsilon}$ is the speed of light in the medium), for the frequencies close to the transverse wave frequency, dispersion equation (11) is substantially simplified:

$$\left(\omega^2 - k^2c_0^2\right)\left[(\omega - \boldsymbol{ku})^2 - \left(\frac{\varepsilon - 1}{\varepsilon}\frac{\hbar k^2}{2m\gamma}\right)^2\right] = k_\perp^2 u^2\,\frac{\omega_b^2\gamma^{-1}}{\varepsilon}\frac{\varepsilon - 1}{\varepsilon}. \qquad (17)$$

We will restrict here our consideration to the analysis of dispersion equation (17). Let us assume that the transverse component $k_\perp$ of the electromagnetic wave vector is fixed (as in a waveguide), while the longitudinal component $k_\parallel$ can take any values. In addition, we assume that the speed of light c_0 in the medium is smaller than the electron beam velocity u. The points with the coordinates $k_\parallel$, ω of single-particle Cerenkov resonance are determined from the system of equations

$$\omega^2 = k_\perp^2 c_0^2 + k_\parallel^2 c_0^2, \qquad \omega = k_\parallel u - \frac{\hbar\left(k_\perp^2 + k_\parallel^2\right)}{2m\gamma}\frac{\varepsilon - 1}{\varepsilon}. \qquad (18)$$

If the inequality

$$\frac{\hbar k_\perp}{m(u - c_0)\gamma}\frac{\varepsilon - 1}{\varepsilon} \ll 1 \qquad (19)$$

is fulfilled, the coordinates of the resonance points are given by the expressions:

$$k_{\parallel 1} = \frac{\omega_1}{u}, \quad \omega_1 = \frac{k_\perp c_0}{\left(1 - c_0^2/u^2\right)^{1/2}}, \qquad k_{\parallel 2} = \frac{2m(u - c_0)\gamma}{\hbar}\frac{\varepsilon}{\varepsilon - 1}, \quad \omega_2 = k_{\parallel 2}c_0, \qquad (20)$$

where $\omega_2 \gg \omega_1$. The resonance at point $k_{\parallel 1}, \omega_1$ is also present in the classical case, while the resonance at point $k_{\parallel 2}, \omega_2$ is purely quantum because at $\hbar \to 0$, the point $k_{\parallel 2}, \omega_2$ tends to infinity. At $u > c_0$, the position of the resonance points in the plane $\omega k_\parallel$ is shown in Fig. 1.

At the resonance points, dispersion equation (17) can be written in the form

$$(\delta\omega)_{1,2}^2\left[(\delta\omega)_{1,2} - (\varepsilon - 1)\frac{\hbar\omega_{1,2}^2}{mc^2\gamma}\right] = \frac{1}{2}k_\perp^2 u^2\,\frac{\omega_b^2\gamma^{-1}}{\omega_{1,2}\varepsilon}\frac{\varepsilon - 1}{\varepsilon}, \qquad (21)$$

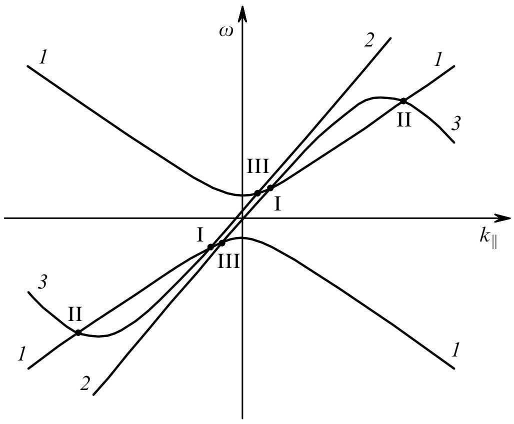

Figure 1. Cerenkov resonance lines and resonance points in the relativistic quantum theory of stimulated Cerenkov radiation: dispersion dependences for the transverse electromagnetic waves in a dielectric (*1*) as well as the resonance lines of the Cerenkov absorption (*2*) and Cerenkov radiation (*3*).

where $(\delta\omega)_{1,2} = \omega - \omega_{1,2}$ is the complex increment. When the inequality

$$\frac{|(\delta\omega)_{1,2}|}{\omega_{1,2}} \gg (\varepsilon - 1)\,\frac{\hbar\omega_{1,2}}{mc^2\gamma} \tag{22}$$

is fulfilled, we find from (21) the next increment of the instability development:

$$(\delta\omega)_1 = \frac{-1 + \mathrm{i}\sqrt{3}}{2}\left(\frac{1}{2}\,k_\perp^2 u^2\,\frac{\omega_b^2\gamma^{-1}}{\omega_1\varepsilon}\,\frac{\varepsilon - 1}{\varepsilon}\right)^{1/3}. \tag{23}$$

The instability with increment (23) is caused by the ordinary classical single-particle stimulated Cerenkov effect [2], and inequality (22) is written in the form

$$\left(\frac{k_\perp^2 u^2 \omega_b^2 \gamma^{-1}}{2\omega_{1,2}^4 \varepsilon^2}\right)^{1/3} \gg (\varepsilon - 1)^{2/3}\,\frac{\hbar\omega_{1,2}}{mc^2\gamma}. \tag{24}$$

Inequality (24) can be fulfilled only for $\omega = \omega_1$, which is taken into account in (23).

When the beam density decreases, inequality (24) is violated and the instability character becomes different. Thus, if the inequality inverse to (22) is fulfilled, the increment is determined by the expression

$$(\delta\omega)_{1,2} = \mathrm{i}\left(\frac{1}{2}\,\frac{mc^2\gamma}{\hbar}\,\frac{k_\perp^2 u^2 \omega_b^2 \gamma^{-1}}{\varepsilon^2 \omega_{1,2}^3}\right)^{1/2}. \tag{25}$$

The instability with increment (25) is a purely quantum effect. At point $k_{\|1}, \omega_1$, both the classical and quantum instabilities are possible depending on the beam density. At point $k_{\|2}, \omega_2$, because $\omega_2 \propto \hbar^{-1}$, only the quantum instability with increment (25) proportional to $\hbar$ develops. The typical dispersion curves $\omega(k_\|)$ of equation (17) are presented in Figs 2 and 3 (only the regions closest to points I and II are shown, see Fig. 1). In the case of Fig. 2, inequality (25) is fulfilled, while in the case of Fig. 3, the inverse inequality takes place.

Based on the wave representations, we can give an interesting interpretation of the quantum stimulated Cerenkov effect. As follows from the third expression in Eqn (5) and two last expressions in Eqn (1), the wave function of the electron beam has the form

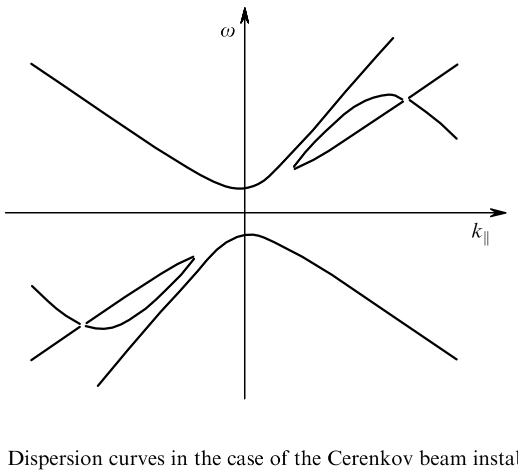

Figure 2. Dispersion curves in the case of the Cerenkov beam instability in a dielectric when inequality (22) is fulfilled (description of the curves is given in the caption to Fig. 1).

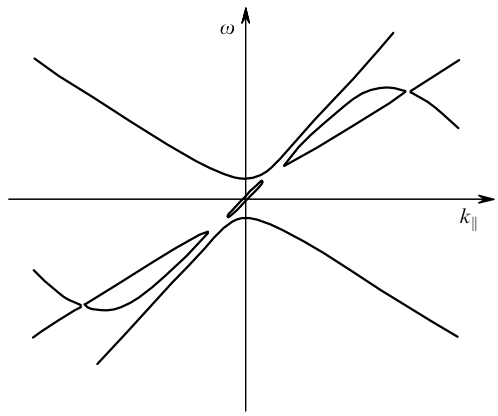

Figure 3. Dispersion curves in the case of the Cerenkov beam instability in a dielectric when the inequality inverse to (22) is fulfilled (description of the curves is given in the caption to Fig. 1)

$$\psi = A\exp(-\mathrm{i}\omega_0 t + \boldsymbol{k}_0\boldsymbol{r}) + a_-\exp[-\mathrm{i}(\omega_0 - \omega)t + \mathrm{i}(\boldsymbol{k}_0 - \boldsymbol{k})\boldsymbol{r}]$$
$$+ a_+\exp[-\mathrm{i}(\omega_0 + \omega)t + (\boldsymbol{k}_0 + \boldsymbol{k})\boldsymbol{r}]. \tag{26}$$

We will call the first term in (26) the primary de Broglie wave, which is the wave function of an electron in the initial state with the energy $\hbar\omega_0$ and momentum $\hbar\boldsymbol{k}_0$. The second and the third terms denote secondary de Broglie waves appearing during scattering on the electromagnetic field potentials. The second term in (26) is the wave function of an electron emitting an electromagnetic photon with the energy $\hbar\omega$ and momentum $\hbar\boldsymbol{k}$, while the third term is the wave function of an electron absorbing a photon.

Let us designate the frequency and the wave vector of any of the secondary de Broglie waves by ω_0' and $\boldsymbol{k}_0'$. Then, in accordance with (26), we can write the expressions

$$\omega_0 = \omega_0' + \omega, \quad \boldsymbol{k}_0 = \boldsymbol{k}_0' + \boldsymbol{k}, \tag{27a}$$

$$\omega_0' = \omega_0 + \omega, \quad \boldsymbol{k}_0' = \boldsymbol{k}_0 + \boldsymbol{k}. \tag{27b}$$

The resonance of the radiation wave and the de Broglie waves means that one the following resonance conditions

$$\omega_0' = \omega_\mp \tag{28}$$

is fulfilled, where the frequencies $\omega_\mp$ are determined in (9). In this case, relations (27) prove to be general conditions for the decay in the three-wave interation [8], and taking into account (28) and (9), they are reduced to the conditions for the Cerenkov resonance (14). Thus, Cerenkov radiation can be treated as a decay of a de Broglie wave into a de Broglie wave and an electromagnetic wave. This process takes place when conditins (27a) are fulfilled. The reverse process – Cerenkov absorption – is the wave coalescence and is realised under conditions (27b).

It is convenient to describe the nonlinear theory of quantum Cerenkov beam instabilities by considering the resonance three-wave interaction. By assuming the potential $\tilde{\varphi}$ to be zero in system (6) and (7) and neglecting in it the terms cubic in the field, we will write the following equations:

$$\frac{d^2H_0}{dt^2} + \omega_0^2 H_0 = \frac{ec}{\hbar}\,\boldsymbol{nk}_0(\tilde{A}H_- + \tilde{A}^* H_+),$$

$$\frac{d^2H_-}{dt^2} + \omega_-^2 H_- = \frac{ec}{\hbar}\,\boldsymbol{nk}_0 H_0 \tilde{A}^*,$$

$$\frac{d^2H_+}{dt^2} + \omega_+^2 H_+ = \frac{ec}{\hbar}\,\boldsymbol{nk}_0 H_0 \tilde{A}, \quad (29)$$

$$\frac{d^2\tilde{A}}{dt^2} + \omega^2 \tilde{A} = \frac{8\pi c}{\varepsilon}\frac{e\hbar}{m}\,\boldsymbol{nk}_0(H_0 H_-^* + H_0^* H_+),$$

where $\tilde{\boldsymbol{A}} = \boldsymbol{n}\tilde{A}$; $\omega = kc_0$ is the electromagnetic wave frequency; $\boldsymbol{n}$ is the polarisation unit vector, which is assumed constant; in this case, $\boldsymbol{nk} = 0$ for the transverse wave. We will represent the solution of equations (29) in the form [see (10)]

$$H_0 = n_b^{1/2}\gamma^{-1/2} a_0(t)\exp(-i\omega_0 t),$$

$$H_\mp = n_b^{1/2}\gamma^{-1/2} a_\mp(t)\exp[-i(\omega_0 \mp \omega)t], \quad (30)$$

$$\tilde{A} = a(t)\exp(-i\omega t),$$

where $a_0, a_\mp, a$ are functions slowly varying in time. Substituting (30) into (29) leads to the equations:

$$\omega_0 \frac{da_0}{dt} = i\,\frac{ec}{2\hbar}\,\boldsymbol{nk}_0(aa_- + a^* a_+),$$

$$(\omega_0 - \omega)\frac{da_-}{dt} - \frac{1}{2}\,i\left[(\omega_0 - \omega)^2 - \omega_-^2\right]a_- = i\,\frac{ec}{2\hbar}\,\boldsymbol{nk}_0 a_0 a^*,$$

$$(\omega_0 + \omega)\frac{da_+}{dt} - \frac{1}{2}\,i\left[(\omega_0 + \omega)^2 - \omega_+^2\right]a_+ = i\,\frac{ec}{2\hbar}\,\boldsymbol{nk}_0 a_0 a, \quad (31)$$

$$\omega\frac{da}{dt} = i\,\frac{\omega_b^2\gamma^{-1}}{\varepsilon}\frac{\hbar c}{e}\,\boldsymbol{nk}_0(a_0 a_-^* + a_0^* a_+).$$

Let condition (27a) and first condition (28) be fulfilled, i.e. Cerenkov radiation takes place. Then, the de Broglie wave with the amplitude a_- is resonant, while the amplitude a_+ of the nonresonant wave can be assumed equal to zero. In this case, equations (31) can be written in the form

$$\omega_0\frac{da_0}{dt} = i\,\frac{ec}{2\hbar}\,k_\perp\frac{mu\gamma}{\hbar k}\,aa_-,$$

$$\omega_-\frac{da_-}{dt} = i\,\frac{ec}{2\hbar}\,k_\perp\frac{mu\gamma}{\hbar k}\,a_0 a^*, \quad (32)$$

$$\omega\frac{da}{dt} = i\,\frac{\omega_b^2\gamma^{-1}}{\varepsilon}\frac{\hbar c}{e}\,k_\perp\frac{mu\gamma}{\hbar k}\,a_0 a_-^*.$$

These equations take into account that $\boldsymbol{nk}_0 = k_\perp mu\gamma/(\hbar k)$. In the linear approximation ($a_0 = 1$), increment (25) follows from (32). Note that neglecting the amplitude a_+ in equations (31), we used the inequality inverse to (22). Otherwise, the amplitudes a_+ and a_- are comparable because when inequality (22) is fulfilled with the accuracy to the increment $\delta\omega$, both responance conditions (28) take place. Therefore, the Cerenkov instability under conditions (22), when it is classical, is analogous to the so-called modified decay [8–10]. The quantum inbstability is a simple decay, which is confirmed by the structure of equations (32).

For adiabatic intial conditions [$a_0(t \to -\infty) = 1$, $a(t \to -\infty) = 0$, $a_-(t \to -\infty) = 0$], the solution of system (32) leads to the relation

$$\frac{1}{8\pi}\,\varepsilon\,\frac{\omega^2}{c^2}|a|^2 = \frac{\hbar\omega n_b}{\cosh^2(|\delta\omega|t)}, \quad -\infty < t < \infty, \quad (33)$$

where $\delta\omega$ is increment (25). The left-hand side of relation (33) is the energy density of the electromagnetic field of the transverse wave excited under conditions of the quantum Cerenkov effect.

Note in conclusion that the possibility of experimental observation of quantum effects studied in this paper is problematic. Lasing at a high frequency ω_2 could serve as a direct check of the quantum condition of the Cerenkov resonance (15). However, the frequency ω_2 proves to be too high; therefore, lasing at this frequency requires a more detailed theroretical investigation taking into account the temporal and spatial dispersion of the dielectric constant ε. As for the easily obtained lasing at a low frequency ω_1, the quantum effects affect the radiation time [inverse to increment (25)] and its power (33). These parameters can be easily measured but the results of measurements are difficult to interpret unambigously.

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