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# Efficient cascade quasi-synchronous parametric generation with up-conversion

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***Abstract*. We report efficient cascade up-conversion generation due to two simultaneous quasi-synchronous processes of parametric decay $\omega_3 \to \omega_1 + \omega_2$ of pump quanta at the frequency $\omega_3$ and up-conversion of one of the generated waves $\omega_1 + \omega_3 \to \omega_4 > \omega_3$ at the frequency $\omega_1$ in a medium with a quadratic nonlinearity. It is found that the necessary condition for this generation is the requirement $|\gamma_1|^2 > (\omega_2/\omega_1)|\gamma_1|^2$, where $\gamma_{1,2}$ are the averaged constants of the nonlinear coupling for the processes $\omega_1 + \omega_{2,3} \to \omega_{3,4}$, respectively. If this requirement is fulfilled, the plane monochromatic pump wave is completely depleted, while the limiting (the noise seed intensity is $I_{10,20} \to 0$ at the input) efficiency of the energy conversion into radiation at the frequency $\omega_4$ is independent of $I_{10,20}$ and determined only by the relations between $|\gamma_{1,2}|^2$ and the frequencies of the interacting waves.**

***Keywords***: *quadratic nonlinearity, quasi-synchronous interaction, efficient cascade parametric generation with up-conversion.*

## 1. Introduction

The authors of paper [1] showed that the problem of interaction of three plane collinear monochromatic waves (modes) in a medium with a quadratic nonlinearity [2] can be described in terms of the effective cubic nonlinearity [3]. In this case, the initial problem is reduced to three independent stationary nonlinear Schrödinger equations (NSEs) relative to the wave amplitudes participating in the processes. Later, the authors of paper [4] managed to reduce, in a similar way, the problem of cascade [5] quasi-synchronous [6] quadratic nonlinear conversion to a system of two stationary NSEs with respect to the amplitudes involved in both nonlinear processes. It was found in [4] that this system is transformed into two identical independent NSEs, which determines its solution in the form of a sum and difference of two identical solutions of the same NSE with shifted arguments and allows optimisation of the process efficiency in any concrete situation.

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By using the approach developed in [1, 4], we will consider the problem of efficient cascade quasi-synchronous parametric generation with up-conversion (with respect to the pump wave frequency) and the peculiarities of the analytic solutions corresponding to this process. Generation of this type can proceed in a quadratic nonlinear medium due to two simultaneous processes: parametric decay $\omega_3 \to \omega_1 + \omega_2$ of pump quanta at the frequency $\omega_3$ and up-conversion $\omega_{1,2} + \omega_3 \to \omega_4 > \omega_3$ of the frequency $\omega_{1,2}$ of one of the waves generated owing to this decay. According to numerical calculations performed in paper [7] for the process of the degenerate decay (photons at the frequencies $\omega_{1,2}$ are indiscernible), conversion of the pump energy flux into radiation at the frequency $\omega_4$ can be almost complete in this case. However, because the degeneration regimes correspond to singularities of the system phase space in which the properties of its evolution change qualitatively, the possibility to account only for the degenerate modes in interactions for the generation problem casts some doubt. Therefore, we will be interested below in the non-degenerate ($\omega_1 \neq \omega_2$) case and in the search, among the variants of its realisation, for the analogues of 'soft' excitation regime [8] in which the conversion efficiency of the pump energy flux at the frequency $\omega_4$ can be large at any arbitrarily small input noise seeds at the frequencies $\omega_{1,2}$.

## 2. Cascade parametric generation

Consider collinear propagation of four (subscripts $i = 1, \ldots, 4$) plane monochromatic waves (modes) with multiple (for simplicity of notation) frequencies $\omega_1$, $\omega_2 = 2\omega_1$, $\omega_3 = \omega_1 + \omega_2 = 3\omega_1$ and $\omega_4 = \omega_1 + \omega_3 = 4\omega_1$, wave vectors $\boldsymbol{k}_{1-4}$, and complex amplitudes $A_{1-4}$ along the $z$ axis in a medium with a non-resonance quadratic nonlinearity. We assume that their interaction is caused by two nonlinear processes, $\omega_1 + \omega_{2,3} \to \omega_{3,4}$, with the wave detunings $\Delta\boldsymbol{k}_{1,2} = \boldsymbol{k}_1 + \boldsymbol{k}_{2,3} - \boldsymbol{k}_{3,4}$. Suppose that in a nonlinear medium (half-space $z \geqslant 0$) there is a spatial structure in which the signs of the nonlinear coupling constants $\beta_{1,2}$ for the processes $\omega_1 + \omega_{2,3} \to \omega_{3,4}$ change periodically along the $z$ axis, and in analogy with [4 – 7], we will write in the quasi-synchronous interaction regime

$$\frac{dA_1}{dz} = -i\gamma_1 A_2^* A_3 - i\gamma_2 A_3^* A_4, \tag{1a}$$

$$\frac{dA_2}{dz} = -i2\gamma_1 A_1^* A_3, \tag{1b}$$

$$\frac{dA_3}{dz} = -i3\gamma_1^* A_1 A_2 - i3\gamma_2 A_1^* A_4, \quad (1c)$$

$$\frac{dA_4}{dz} = -i4\gamma_2^* A_1 A_3. \quad (1d)$$

Here, $\gamma_{1,2} = \langle \beta_{1,2} \exp(-i\Delta k_{1,2} z)\rangle_z$ are the averaged nonlinear coupling constants for the processes $\omega_1 + \omega_{2,3} \rightarrow \omega_{3,4}$ respectively.

System (1) has five second-order integrals, $J_{0-4} = \text{const}$, corresponding to the law of conservation of the total energy flux and the so-called Manly–Row relations [2]

$$J_0 = I_1 + I_2 + I_3 + I_4, \quad (2a)$$

$$J_1 = I_1 - I_2 - \frac{1}{3} I_3, \quad J_2 = I_1 - \frac{1}{2} I_2 + \frac{1}{4} I_4,$$
$$J_3 = I_1 + \frac{1}{3} I_3 + \frac{1}{2} I_4, \quad J_4 = I_2 + \frac{2}{3} I_3 + \frac{1}{2} I_4. \quad (2b)$$

However, only two of these integrals are independent, and we can write, for example, that

$$I_2 - I_{20} = (I_1 - I_{10}) - \frac{1}{3}(I_3 - I_{30}),$$
$$I_4 - I_{40} = -2(I_1 - I_{10}) - \frac{2}{3}(I_3 - I_{30}), \quad (3)$$

where $I_i = A_i A_i^*$ are proportional to the wave intensities and $I_{i0} = A_{i0} A_{i0}^* = A_i A_i^*|_{z=0}$.

It follows from (3) that within the framework of the formulated problem, it is sufficient to consider only the situation in which the mode at the frequency $\omega_3$ plays the role of the pump, i.e. the case $I_{30} \neq 0$ and $I_{10,20,40} = 0$ (more exact, $I_{10,20,40} \leqslant I_{30}$). Indeed, this role cannot be assigned to the modes at the frequencies $\omega_{1,2}$, because $I_2 \leqslant 0$ at $I_{10} \neq 0$, $I_{20} = I_{30} = I_{40} = 0$ and $I_4 \leqslant 0$ at $I_{20} \neq 0$, $I_{10} = I_{30} = I_{40} = 0$ respectively. If the highest-frequency mode (with the frequency $\omega_4$) is the pump, the increase in its frequency due to two above-mentioned nonlinear processes is impossible.

Further passage from (1) to second-order equations yields, in analogy with [4], a closed system of equations for $A_{1,3}$ in the form

$$\frac{d^2 A_1}{dz^2} = -3G_+ |A_1|^2 A_1 + 3\sigma |G_-| |A_3|^2 A_1 + 3J_{13} A_1, \quad (4a)$$

$$\frac{d^2 A_3}{dz^2} = -9G_+ |A_1|^2 A_3 + \sigma |G_-| |A_3|^2 A_3 + 3J_{13} A_3 \quad (4b)$$

with the boundary conditions

$$A_1|_{z=0} = A_{10}, \quad \left.\frac{dA_1}{dz}\right|_{z=0} = -i\gamma_1 A_{20}^* A_{30} - i\gamma_2 A_{30}^* A_{40}, \quad (5a)$$

$$A_3|_{z=0} = A_{30}, \quad \left.\frac{dA_3}{dz}\right|_{z=0} = -i3\gamma_1^* A_{10} A_{20} - i3\gamma_2 A_{10}^* A_{40}, \quad (5b)$$

where $G_\pm = |\gamma_1|^2 \pm 2|\gamma_2|^2$; $J_{13} = |\gamma_1|^2 J_1 + 2|\gamma_2|^2 J_3$; $\sigma = \text{sign}(G_-)$ is a sign-alternating function taking the values $\pm 1$ at $|\gamma_1|^2 > 2|\gamma_2|^2$ and $|\gamma_1|^2 < 2|\gamma_2|^2$, respectively. In this case, the wave intensities $I_{2,4}$ can be found from relations (3).

It is easy to verify that in the situation of interest to us, when $I_{30} \neq 0$, $I_{10} = I_{20} = I_{40} = 0$, $3J_{13} = -\sigma|G_-| I_{30}$, and taking into account the boundary conditions

$$A_1|_{z=0} = 0, \quad \left.\frac{dA_1}{dz}\right|_{z=0} = 0, \quad (6a)$$

$$A_3|_{z=0} = A_{30}, \quad \left.\frac{dA_3}{dz}\right|_{z=0} = 0 \quad (6b)$$

system (4) has the only trivial solution:

$$A_1(z) \equiv 0, \quad A_3(z) \equiv A_{30}. \quad (7)$$

However, this solution is unstable with respect to small perturbations. We will assume below that small ($I_{30} \geqslant I_{10,20,40} \neq 0$) noise (see below) seeds $A_{10,20,40} \neq 0$ for the amplitudes $A_{1,2,4}$ of those modes, which should be finally generated in a nonlinear medium, play the role of such perturbations. As will be shown below, this allows one to realise efficient cascade parametric generation with up-conversion.

Below, we will restrict our consideration to the analysis of constant and optimal (from the point of view of the generation development rate) initial phases $\varphi_i = \varphi_{i0}$ (see below) of all the four interacting modes. In fact, this is equivalent to the assumption that the source of the seeds ensuring the development of generation is the input noise in which the necessary spectral components with such optimal initial phases are always present. In this case, selecting by the substitution

$$A_i(z) = X_i(z) \exp(i\varphi_{i0}) \quad (8)$$

real and non-negative [in the input plane ($X_i|_{z=0} = X_{i0} \geqslant 0$)] amplitudes $X_i$ of the interacting modes for the optimal relation for $\varphi_{i0}$, specified by the conditions

$$\varphi_{10} + \varphi_{20,30} - \varphi_{30,40} - \varphi_{\gamma_1,\gamma_2} \pm \frac{\pi}{2} = 0, \quad (9)$$

which provide the maximum growth rate of $X_{2,4}$ due to transformation of equations (1b) and (1d) into relations

$$\frac{dX_2}{dz} = 2|\gamma_1| X_1 X_3, \quad \frac{dX_4}{dz} = 4|\gamma_2| X_1 X_3, \quad (10)$$

we will reduce problem (4) to the system

$$\frac{d^2 X_1}{dz^2} = -3G_+ X_1^3 + 3\sigma |G_-| X_3^2 X_1 + 3J_{13} X_1, \quad (11a)$$

$$\frac{d^2 X_3}{dz^2} = -9G_+ X_1^2 X_3 + \sigma |G_-| X_3^3 + 3J_{13} X_3 \quad (11b)$$

with the boundary conditions

$$X_1|_{z=0} = X_{10}, \quad \left.\frac{dX_1}{dz}\right|_{z=0} = +|\gamma_1| X_{20} X_{30} - |\gamma_2| X_{30} X_{40}, \quad (12a)$$

$$X_3|_{z=0} = X_{30}, \quad \left.\frac{dX_3}{dz}\right|_{z=0} = -3|\gamma_1| X_{10} X_{20} - 3|\gamma_2| X_{10} X_{40}. \quad (12b)$$

In (9), the phases $\varphi_{1,2}$ are determined by the relations $\gamma_{1,2} = |\gamma_{1,2}|\exp(\mathrm{i}\varphi_{\gamma_{1,2}})$.

It is easy to check that the further analysis can be restricted by two situations only in which either $I_{30} \neq 0$, $I_{10} \neq 0$, $I_{20} = I_{40} = 0$ (case I), or $I_{30} \neq 0$, $I_{20} \neq 0$, $I_{10} = I_{40} = 0$ (case II). Indeed, at $I_{30} \neq 0$, $I_{10} = I_{20} = 0$ the trivial solution of (7) proves stable with respect to small fluctuations of $I_{40}$ (at least, in the first approximation) because

$$X_{10,20} = \left.\frac{\mathrm{d}X_{1,2}}{\mathrm{d}z}\right|_{z=0} \equiv 0,\ X_{3,4}|_{z=0} = X_{30,40}, \text{ and } \left.\frac{\mathrm{d}X_{3,4}}{\mathrm{d}z}\right|_{z=0} \equiv 0.$$

## 3. Exact analytic solutions

Consider first case I when $X_{30} \neq 0$, $X_{10} \neq 0$, $X_{20} = X_{40} = 0$. In this case, $3J_{13} = 3G_+I_{10} - \sigma|G_-|I_{30}$, and system (11) has the boundary conditions

$$X_{1,3}|_{z=0} = X_{10,30} = \sqrt{I_{10,30}},\ \left.\frac{\mathrm{d}X_{1,3}}{\mathrm{d}z}\right|_{z=0} = 0. \qquad (13)$$

Although the authors of paper [4] did not consider the situation with the boundary conditions of type (13), the exact solutions for this case can be found by using the same method of separation of variables. At $\sigma = -1$ (i.e. $|\gamma_1|^2 < 2|\gamma_2|^2$) system (11), by using substitutions $Y_1(z) = X_1(z) \mp X_2(z)$ and $Y_2(z) = X_1(z) \pm X_2(z)$, is still reduced to two identical independent NSEs with the nonlinearity of focusing type. Therefore, $Y_{1,2}(z)$ should be proportional to the same fundamental solution of the NSE [4, 9] (i.e. one of two possible elliptic Jacobi functions: $\mathrm{cn}(\gamma z, k)$ or $\mathrm{dn}(\gamma z, k)$ [10]) but with the arguments shifted relative to each other (see [4]). The proportionality coefficient, parameter $\gamma$, and modulus $1 \geqslant k \geqslant 0$ of this elliptic Jacobi function are determined by the coefficients of the NSE obtained in this way, and the shift of the arguments $Y_{1,2}(z)$, taking into account (13), should be set by the conditions

$$Y_{1,2}|_{z=0} = Y_{10,20} \neq 0,\ Y_{10} \neq Y_{20} \text{ and } \left.\frac{\mathrm{d}Y_{1,2}}{\mathrm{d}z}\right|_{z=0} = 0.$$

These three conditions can be fulfilled only when $Y_{1,2}(z) \propto \mathrm{dn}(\gamma z, k)$, and $Y_{2,1}(z) \propto \mathrm{dn}(\gamma z + K, k) = k' \times \mathrm{dn}^{-1}(\gamma z, k)$, where $k' = \sqrt{1-k^2}$; $K(k)$ is the complete elliptic integral of the first kind [10]. Therefore, at $|\gamma_1|^2 < 2|\gamma_2|^2$, system (11) with boundary conditions (13) has two types of solutions, which will be written by using the identities $\mathrm{dn}(z,k) + \mathrm{dn}(z+K,k) \equiv (1+k')\mathrm{dn}(\tilde{z},\tilde{k})$ and $\mathrm{dn}(z,k) - \mathrm{dn}(z+K,k) \equiv (1-k')\mathrm{cn}(\tilde{z},\tilde{k})$, which follow from the so-called ascending Landen transformation [11]. Here, $\tilde{z} = (1+k')z$, $\tilde{k} = (1-k')/(1+k')$.

Taking into account the above assumptions, at

$$|\gamma_1|^2 < 2|\gamma_2|^2 \text{ and } I_{10} < \frac{1}{3}\frac{2|\gamma_2|^2 - |\gamma_1|^2}{2|\gamma_2|^2 + |\gamma_1|^2} I_{30}$$

system (11) has the solution

$$X_1 = \sqrt{I_{10}}\mathrm{cn}(\gamma z, k),\quad X_3 = \sqrt{I_{30}}\mathrm{dn}(\gamma z, k), \qquad (14\mathrm{a})$$

$$I_2 = \frac{1}{3}I_{30}[1 - \mathrm{dn}^2(\gamma z, k)] - I_{10}\mathrm{sn}^2(\gamma z, k),$$
$$I_4 = \frac{2}{3}I_{30}[1 - \mathrm{dn}^2(\gamma z, k)] + 2I_{10}\mathrm{sn}^2(\gamma z, k). \qquad (14\mathrm{b})$$

Here, the values of the parameters $k$ and $\gamma$ are determined by the relations

$$k^2\gamma^2 = 6(2|\gamma_2|^2 + |\gamma_1|^2)I_{10},\ \ \gamma^2 = 2(2|\gamma_2|^2 - |\gamma_1|^2)I_{30}. \qquad (15)$$

It is easy to verify that the positions of the maxima $z = z_{\max}$ on the dependence $I_4(z)$ correspond to the condition $\mathrm{cn}(\gamma z_{\max}, k) = 0$; from whence,

$$z_{\max} = (2m-1)\gamma^{-1}K(k), \qquad (16)$$

where $m = 1, 2, \ldots$ is an arbitrary positive integer. In this case,

$$I_{4\max} = I_4(z_{\max}) = 8\frac{|\gamma_2|^2}{2|\gamma_2|^2 - |\gamma_1|^2}I_{10} \to 0 \text{ for } I_{10} \to 0, \qquad (17)$$

which is unsuitable for creating an efficient generator operating in the soft excitation regime.

At

$$|\gamma_1|^2 < 2|\gamma_2|^2 \text{ and } I_{10} > \frac{1}{3}\frac{2|\gamma_2|^2 - |\gamma_1|^2}{2|\gamma_2|^2 + |\gamma_1|^2} I_{30}$$

the functions, which are fulfilled by the modes with the frequencies $\omega_1$ and $\omega_3$, in fact interchange their places. This yields at once

$$X_1 = \sqrt{I_{10}}\mathrm{dn}(\gamma z, k),\quad X_3 = \sqrt{I_{30}}\mathrm{cn}(\gamma z, k), \qquad (18\mathrm{a})$$

$$I_2 = \frac{1}{3}I_{30}[1 - \mathrm{dn}^2(\gamma z, k)] - I_{10}\mathrm{sn}^2(\gamma z, k),$$
$$I_4 = \frac{2}{3}I_{30}[1 - \mathrm{dn}^2(\gamma z, k)] + 2I_{10}\mathrm{sn}^2(\gamma z, k). \qquad (18\mathrm{b})$$

Here, the values of the parameters $k$ and $\gamma$ are determined by the relations

$$k^2\gamma^2 = 2(2|\gamma_2|^2 - |\gamma_1|^2)I_{30},\ \gamma^2 = 6(2|\gamma_2|^2 + |\gamma_1|^2)I_{10}. \qquad (19)$$

In this case, solution (18) corresponds to the hard excitation regime of the system because the seed intensity $I_{10}$ at the nonlinear medium input should be large. Moreover, analysing the character of the dependences of $I_1(z)$ specified in (14a) and (18a), we can conclude that in these two solutions the noise seed itself, $I_{10}$, in fact plays the role of the second component of the pump because $I_1(z) \leqslant I_{10}$.

At first glance, solutions (14) and (18) are new and were not derived in paper [4]. However, this is not so and at $\mathrm{sn}(\beta\tilde{z}_0, k') = (1+k')^{-1}$ ($\beta\tilde{z}_0 = K/2$, see notations in [4]), both these solutions, taking into account the possible permutation of subscripts $1 \leftrightarrow 3$, can be obtained from the relations similar to expressions (33) from paper [4] with the help of the Landen transformation [11] and the shift of the transformation result by a quarter of the period $K$ along the $z$ axis. Moreover, by all appearances, many stationary solutions of a system consisting of two NSEs, known from the literature, can be presented due to the Landen transformation and the mentioned identities in a simpler form.

At $\sigma = +1$ (i.e. at $|\gamma_1|^2 > 2|\gamma_2|^2$), the only possible solution of system (11) is also determined by the expressions (47) [4] shifted along $z$ axis by $K$ at $\mathrm{sn}(2\beta\tilde{z}_0, k') = 0$ ($\beta\tilde{z}_0 = K/2$, see notations in [4]), which allows one to write it in the form

$$X_1 = \sqrt{I_{10}}\frac{1}{\mathrm{dn}(\gamma z, k)}, \quad X_3 = \sqrt{I_{30}}\frac{\mathrm{cn}(\gamma z, k)}{\mathrm{dn}(\gamma z, k)}, \tag{20a}$$

$$I_2 = \frac{1-\mathrm{dn}^2(\gamma z, k)}{\mathrm{dn}^2(\gamma z, k)} I_{10} + \frac{1}{3}\frac{\mathrm{dn}^2(\gamma z, k) - \mathrm{cn}^2(\gamma z, k)}{\mathrm{dn}^2(\gamma z, k)} I_{30},$$
$$I_4 = -2\frac{1-\mathrm{dn}^2(\gamma z, k)}{\mathrm{dn}^2(\gamma z, k)} I_{10} + \frac{2}{3}\frac{\mathrm{dn}^2(\gamma z, k) - \mathrm{cn}^2(\gamma z, k)}{\mathrm{dn}^2(\gamma z, k)} I_{30}. \tag{20b}$$

Here, the values of the parameters $k$ and $\gamma$ are determined by the relations

$$k^2\gamma^2 = 2(|\gamma_1|^2 - 2|\gamma_2|^2)I_{30},$$
$$\gamma^2 = 2[3(|\gamma_1|^2 + 2|\gamma_2|^2)I_{10} + (|\gamma_1|^2 - 2|\gamma_2|^2)I_{30}]. \tag{21}$$

In this case, localisation of the maxima $z = z_{\max}$ on the dependence $I_4(z)$ corresponds to the condition $\mathrm{cn}(\gamma z_{\max}, k) = 0$, i.e. is set by expression (16) in which $\gamma$ and $k$ are determined by relations (21). It is easy to see that now

$$I_{4\max} = I_4(z_{\max}) = \frac{8}{3}\frac{|\gamma_2|^2}{|\gamma_1|^2 + 2|\gamma_2|^2} I_{30} \tag{22}$$

does not change with varying the seed intensity $I_{10}$ at the nonlinear medium input and, hence, at $I_{10} \to 0$ the quantity $I_{4\max}|_{2|\gamma_2|^2 \to |\gamma_1|^2} \to \frac{2}{3} I_{30}$ can be large.

Note that, unlike numerical calculations [7] performed for the case of the degenerate decay (photons at the frequencies $\omega_{1,2}$ are indiscernible), the conversion of the pump energy flux into radiation at the frequency $\omega_4$ in non-degenerate situations is never complete. At points $z = z_{\max}$, where the pump wave is completely depleted, a part of its energy flux $J_0 - I_{4\max}$ is redistributed between two other generated modes

$$I_1(z_{\max}) = I_{10} + \frac{1}{3}\frac{|\gamma_1|^2 - 2|\gamma_2|^2}{|\gamma_1|^2 + 2|\gamma_2|^2} I_{30},$$
$$I_2(z_{\max}) = \frac{2}{3}\frac{|\gamma_1|^2}{|\gamma_1|^2 + 2|\gamma_2|^2} I_{30}, \quad I_3(z_{\max}) = 0. \tag{23}$$

Note also that although in the case under study $I_{4\max}$ is independent of $I_{10}$, the presence of the seed for the generation process is crucial. The matter is that the position of those points $z = z_{\max}$, at which the maximum energy efficiency is realised, depends on $I_{10}$ and, because

$$k^2 \approx 1 - 3\frac{|\gamma_1|^2 + 2|\gamma_2|^2}{|\gamma_1|^2 - 2|\gamma_2|^2}\frac{I_{10}}{I_{30}} \to 1,$$
$$\gamma^2 \to 2(|\gamma_1|^2 - 2|\gamma_2|^2)I_{30} \text{ for } I_{10} \to 0, \tag{24}$$

$z_{\max}$ and the nonlinear medium length $L \approx z_{\max}$ required for efficient conversion grow infinitely at $I_{10} \to 0$.

Consider now case II when $X_{30} \neq 0$, $X_{20} \neq 0$, $X_{10} = X_{40} = 0$. In this case, $3J_{13} = -3|\gamma_1|^2 I_{20} - \sigma|G_-|I_{30}$, and system (11) has the boundary conditions

$$X_1|_{z=0} = 0, \quad \left.\frac{\mathrm{d}X_1}{\mathrm{d}z}\right|_{z=0} = |\gamma_1|X_{20}X_{30} = |\gamma_1|\sqrt{I_{20}I_{30}}, \tag{25a}$$

$$X_3|_{z=0} = X_{30} = \sqrt{I_{30}}, \quad \left.\frac{\mathrm{d}X_3}{\mathrm{d}z}\right|_{z=0} = 0, \tag{25b}$$

corresponding to the situations considered previously in [4]. Therefore, all the searched-for solutions are reduced to expressions (32), (33), and (44) from paper [4] corrected taking into account the concrete choice of relations between the frequencies $\omega_{1-4}$ of the interacting modes.

At

$$|\gamma_1|^2 < 2|\gamma_2|^2 \text{ and } I_{20} < \frac{1}{24}\frac{(|\gamma_1|^2 - 2|\gamma_2|^2)^2}{|\gamma_1|^2|\gamma_2|^2} I_{30}$$

the form of the solution of system (11) corresponds to expressions (33) from paper [4] and

$$X_1 = \frac{2\gamma\sqrt{|\gamma_1|^2 I_{20} I_{30}}\,\mathrm{sn}(\gamma z, k)\mathrm{cn}(\gamma z, k)}{2\gamma^2 - (\gamma^2 k^2 - 3|\gamma_1|^2 I_{20})\mathrm{sn}^2(\gamma z, k)},$$
$$X_3 = \frac{2\gamma^2\sqrt{I_{30}}\,\mathrm{dn}(\gamma z, k)}{2\gamma^2 - (\gamma^2 k^2 - 3|\gamma_1|^2 I_{20})\mathrm{sn}^2(\gamma z, k)}, \tag{26a}$$

$$I_2 = I_{20}\left[1 + \frac{2|\gamma_1|^2 I_{30}\mathrm{sn}^2(\gamma z, k)}{2\gamma^2 - (\gamma^2 k^2 - 3|\gamma_1|^2 I_{20})\mathrm{sn}^2(\gamma z, k)}\right]^2,$$
$$I_4 = \frac{16|\gamma_1|^2|\gamma_2|^2 I_{20} I_{30}^2 \mathrm{sn}^4(\gamma z, k)}{[2\gamma^2 - (\gamma^2 k^2 - 3|\gamma_1|^2 I_{20})\mathrm{sn}^2(\gamma z, k)]^2}. \tag{26b}$$

Here, the values of the parameters $k$ and $\gamma$ are determined by the relations

$$k^2\gamma^2 = \sqrt{3|\gamma_1|^2 I_{20}}\sqrt{3|\gamma_1|^2 I_{20} + 2(|\gamma_1|^2 + 2|\gamma_2|^2)I_{30}},$$
$$\gamma^2 = \frac{1}{2}\left\{(2|\gamma_2|^2 - |\gamma_1|^2)I_{30} + \sqrt{3|\gamma_1|^2 I_{20}} \times\left[\sqrt{3|\gamma_1|^2 I_{20} + 2(|\gamma_1|^2 + 2|\gamma_2|^2)I_{30}} - \sqrt{3|\gamma_1|^2 I_{20}}\right]\right\}. \tag{27}$$

It is easy to check that the positions of the maxima $z = z_{\max}$ on the dependence $I_4(z)$ still correspond to condition (16) at $\gamma$ and $k$ specified by expressions (27). In this case,

$$I_{4\max} = I_4(z_{\max}) = 16\frac{|\gamma_1|^2|\gamma_2|^2}{(2|\gamma_2|^2 - |\gamma_1|^2)^2} I_{20} \to 0 \text{ for } I_{20} \to 0, \tag{28}$$

which, as in the previously considered case of solution (14), is not suitable for realising an efficient generator operating in the soft excitation regime.

The case when

$$I_{20} > \frac{1}{24}\frac{(|\gamma_1|^2 - 2|\gamma_2|^2)^2}{|\gamma_1|^2|\gamma_2|^2} I_{30}$$

at any relation $|\gamma_1|^2$ and $2|\gamma_2|^2$ is reduced, in the form, to solution (32) from paper [4], and

$$X_1 = \frac{2\gamma\sqrt{|\gamma_1|^2 I_{20} I_{30}}\mathrm{sn}(\gamma z, k)\mathrm{dn}(\gamma z, k)}{\gamma^2 + 3|\gamma_1|^2 I_{20} + (\gamma^2 - 3|\gamma_1|^2 I_{20})\mathrm{cn}^2(\gamma z, k)},$$

(29a)

$$X_3 = \frac{2\gamma^2\sqrt{I_{30}}\mathrm{cn}(\gamma z, k)}{\gamma^2 + 3|\gamma_1|^2 I_{20} + (\gamma^2 - 3|\gamma_1|^2 I_{20})\mathrm{cn}^2(\gamma z, k)},$$

$$I_2 = I_{20}\left[1 + \frac{2|\gamma_1|^2 I_{30}\mathrm{sn}^2(\gamma z, k)}{\gamma^2 + 3|\gamma_1|^2 I_{20} + (\gamma^2 - 3|\gamma_1|^2 I_{20})\mathrm{cn}^2(\gamma z, k)}\right]^2,$$

(29b)

$$I_4 = \frac{16|\gamma_1|^2|\gamma_2|^2 I_{30}\mathrm{sn}^4(\gamma z, k)}{[\gamma^2 + 3|\gamma_1|^2 I_{20} + (\gamma^2 - 3|\gamma_1|^2 I_{20})\mathrm{cn}^2(\gamma z, k)]^2}.$$

The values of the parameters $k$ and $\gamma$ are set by the relations

$$k^2\gamma^2 = \frac{1}{2}[\gamma^2 - 3|\gamma_1|^2 I_{20} - (|\gamma_1|^2 - 2|\gamma_2|^2)I_{30}],$$

(30)

$$\gamma^2 = \sqrt{3|\gamma_1|^2 I_{20}[3|\gamma_1|^2 I_{20} + 2(|\gamma_1|^2 + 2|\gamma_2|^2)I_{30}]}.$$

It is obvious that solution (29) also corresponds to the hard excitation regime of the system because the seed intensity $I_{20}$ at the nonlinear medium input should be large.

And finally, at

$$|\gamma_1|^2 > 2|\gamma_2|^2 \text{ and } I_{20} < \frac{1}{24}\frac{(|\gamma_1|^2 - 2|\gamma_2|^2)^2}{|\gamma_1|^2|\gamma_2|^2} I_{30}$$

system (11) has one more solution, which is reduced to expressions (44) from paper [4]. This solution, as solution (20) considered above, provides the possibility of realising an efficient cascade generator operating in the soft excitation regime,

$$X_1 = \frac{2\sqrt{\gamma^2|\gamma_1|^2 I_{20} I_{30}}\mathrm{sn}(\gamma z, k)}{2\gamma^2 - (|\gamma_1|^2 - 2|\gamma_2|^2)I_{30}\mathrm{sn}^2(\gamma z, k)},$$

(31a)

$$X_3 = \sqrt{I_{30}}\frac{2\gamma^2\mathrm{cn}(\gamma z, k)\mathrm{dn}(\gamma z, k)}{2\gamma^2 - (|\gamma_1|^2 - 2|\gamma_2|^2)I_{30}\mathrm{sn}^2(\gamma z, k)},$$

$$I_2 = I_{20}\frac{[2\gamma^2 + (|\gamma_1|^2 + 2|\gamma_2|^2)I_{30}\mathrm{sn}^2(\gamma z, k)]^2}{[2\gamma^2 - (|\gamma_1|^2 - 2|\gamma_2|^2)I_{30}\mathrm{sn}^2(\gamma z, k)]^2},$$

(31b)

$$I_4 = \frac{16|\gamma_1|^2|\gamma_2|^2 I_{20} I_{30}^2\mathrm{sn}^4(\gamma z, k)}{[2\gamma^2 - (|\gamma_1|^2 - 2|\gamma_2|^2)I_{30}\mathrm{sn}^2(\gamma z, k)]^2}.$$

The values of the parameters $k$ and $\gamma$ are determined here by the relations

$$k^2\gamma^2 = \frac{1}{2}\Bigg\{(|\gamma_1|^2 - 2|\gamma_2|^2)I_{30} - \sqrt{3|\gamma_1|^2 I_{20}}$$

$$\times\left[\sqrt{3|\gamma_1|^2 I_{20} + 2(|\gamma_1|^2 + 2|\gamma_2|^2)I_{30}} - \sqrt{3|\gamma_1|^2 I_{20}}\right]\Bigg\},$$

(32)

$$\gamma^2 = \frac{1}{2}\Bigg\{(|\gamma_1|^2 - 2|\gamma_2|^2)I_{30} + \sqrt{3|\gamma_1|^2 I_{20}}$$

$$\times\left[\sqrt{3|\gamma_1|^2 I_{20} + 2(|\gamma_1|^2 + 2|\gamma_2|^2)I_{30}} + \sqrt{3|\gamma_1|^2 I_{20}}\right]\Bigg\}.$$

In this case, the condition $\mathrm{cn}(\gamma z_{\max}, k) = 0$ also corresponds to localisation of the maxima $z = z_{\max}$; therefore their positions are still determined by expression (16) into which we should substitute $\gamma$ and $k$ specified by relations (32). It is easy to verify that although

$$I_{4\max} = I_4(z_{\max}) = \frac{16}{3}$$

$$\times\frac{|\gamma_2|^2 I_{30}^2}{\left[\sqrt{3|\gamma_1|^2 I_{20} + 2(|\gamma_1|^2 + 2|\gamma_2|^2)I_{30}} - \sqrt{3|\gamma_1|^2 I_{20}}\right]^2} \quad (33)$$

and depends on the seed intensity $I_{20}$ at the nonlinear medium input, all the limiting ($I_{20} \to 0$) characteristics of the converter remain the same as those for solution (20). Retained in this case is both limiting ($I_{20} \to 0$) conversion efficiency (20) and those fraction in which, unlike the numerical results [7], the remaining part of the pump energy flux $J_0 - I_{4\max}$ is redistributed between other generated modes (23).

As in the case of solution (20), the presence of the seed at the nonlinear medium input is also crucial because the position of those points on the $z$ axis, at which the maximum conversion efficiency is realised, depends on $I_{20}$, and because at $I_{20} \to 0$

$$k^2 \approx 1 - \sqrt{\frac{6|\gamma_1|^2}{|\gamma_1|^2 + 2|\gamma_2|^2}\frac{I_{20}}{I_{30}}} \to 1,$$

(34)

$$\gamma^2 \to \frac{1}{2}(|\gamma_1|^2 - 2|\gamma_2|^2)I_{30},$$

$z_{\max}$ and the nonlinear medium length $L \approx z_{\max}$ required for efficient conversion also grow infinitely at $I_{20} \to 0$.

## 4. Conclusions

By using the approach of paper [4], we have considered analytically the problem of realisation of the efficient cascade (simultaneous processes of the decay $\omega_3 \to \omega_1 + \omega_2$ of the pump quanta at the frequency $\omega_3$ and up-conversion $\omega_{1,2} + \omega_3 \to \omega_4 > \omega_3$ of one of the waves generated in quadratic nonlinear medium at the frequency $\omega_{1,2}$) quasi-synchronous parametric generation with up-conversion (amplification) (with respect to the pump wave frequency).

Unlike the numerical calculations performed in [7] for the case of the degenerate decay (a singularity of the system phase space at which photons at the frequencies $\omega_{1,2}$ are indiscernible and the possibility of taking into account only degenerate mode in the problem of the generation type casts some doubts), we have shown that in non-degenerate cases ($\omega_1 \neq \omega_2$) the conversion of the pump energy flux into radiation at the frequency $\omega_4$ cannot be complete. We have found that the necessary condition for realising the 'soft' excitation regime of the system is the requirement $|\gamma_1|^2 > (\omega_2/\omega_1)|\gamma_2|^2$. If this requirement is fulfilled, the plane monochromatic pump wave is completely depleted, and the limiting (the noise seed intensity $I_{10,20} \to 0$ at the nonlinear medium input) efficiency of conversion of its energy flux into radiation at the frequency $\omega_4$ is independent of $I_{10,20}$ and determined only by the relations between the nonlinear coupling constants $\gamma_{1,2}$ and the frequencies of the interacting waves. It is these parameters that determine those fraction in which the pump energy flux is redistributed between the waves $I_{1,2,4}$ generated in the nonlinear medium. At the same time, the presence of the noise seeds at the nonlinear medium input is obligatory because the nonlinear medium length (required to realise efficient conversion), which depends on $I_{10,20}$ and at $I_{10,20} \to 0$ tends to infinity.

Note that the solutions, describing the ordinary parametric generation on a nonlinear medium (i.e. the process $\omega_3 \to \omega_1 + \omega_2$) [2], which can be obtained from the above expressions at $|\gamma_2|^2 \to 0$, behave in this manner.

## References


1. Petnikova V.M., Shuvalov V.V. *Phys. Rev. E*, **76** (4), 046611 (2007).
2. Armstrong J.A., Bloembergen N., Ducuing J., Pershan P.S. *Phys. Rev.*, **127** (6), 1918 (1962); Akhmanov S.A., Khokhlov R.V. *Problems of Nonlinear Optics* (New York: Gordon and Breach, 1972; Moscow: VINITI, 1964); Bloembergen N. *Nonlinear Optics* (New York: Benjamin, 1965; Moscow: Mir, 1966); Zernike F., Midwinter J.E. *Applied Nonlinear Optics* (New York: John Wiley, 1973; Moscow: Mir, 1976); Shen Y.R. *The Principles of Nonlinear Optics* (New York: Wiley-Interscience, 1984; Moscow: Nauka, 1989).
3. Ostrovskii L.A. *Pis'ma Zh. Eksp. Teor Fiz.*, **5** (9), 331 (1967); Klyshko D.N., Polkovnikov B.F. *Kvantovaya Elektron.*, No. 4 (16), 81 (1973) [*Sov. J. Quantum Electron.*, **3** (4), 324 (1973)]; Meredith G.R *J. Chem. Phys.*, **77** (12), 5863 (1982); Kobyakov A., Lederer F. *Phys. Rev. A*, **54** (4), 3455 (1996).
4. Petnikova V.M., Shuvalov V.V. *Phys. Rev. E*, **79** (2), 026605 (2009).
5. Aleksandrovski A.L., Chirkin A.S., Volkov V.V. *J. Rus. Laser Res.*, **18** (2), 101 (1997); Chirkin A.S., Volkov V.V., Laptev G.D., Morozov E.Yu. *Kvantovaya Elektron.*, **30** (10), 847 (2000) [*Quantum Electron.*, **30** (10), 847 (2000)]; Saltiel M.S., Sukhorukov A.A., Kivshar Yu.S. *Progress in Optics*, **47**, 1 (2005).
6. Somekh S., Yariv A. *Opt. Commun.*, **6** (3), 301 (1972); Yacoby Y., Aggarwal R.L., Lax B. *J. Appl. Phys.*, **44** (7), 3180 (1973); McMullen J.D. *J. Appl. Phys.*, **46** (7), 3076 (1975); Fejer M.M., Magel G.A., Jundt D.H., Byer R.L. *IEEE J. Quantum Electron.*, **28** (11), 2631 (1992).
7. Chirkin A.S., Volkov V.V. *Kvantovaya Elektron.*, **25** (2), 101 (1998) [*Quantum Electron.*, **28** (2), 95 (1998)].
8. Andronov A.A., Vitt A.A., Khaikin S.E. *Theory of Oscillations* (Oxford: Pergamon, 1966; Moscow: Nauka, 1981); Migulin V.V., Medvedev V.I., Mustel E.R., Parygin V.N. *Osnovy teorii kolebanii* (Fundamentals of the Oscillation Theory) (Moscow: Nauka, 1978).
9. Petnikova V.M., Shuvalov V.V., Vysloukh V.A. *Phys. Rev. E*, **60** (1), 1009 (1999).
10. Gradshteyn I.S., Ryzhik I.M. *Tables of Integrals, Series, and Products* (New York: Acad. Press, 2000; Moscow: Nauka, 1971); Kuznetsov D.S. *Spetsial'nye funktsii* (Special Functions) (Moscow: Vyssh. shk., 1965).
11. Abramowitz M., Stegun I.A. (Eds) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (New York: Dover Publications, 1972; Moscow: Nauka, 1979) p. 385.