REVIEW

PACS numbers: 42.65.Tg; 42.65.Jx; 42.50.Gy; 42.81.Dp
DOI: 10.1070/QE2010v040n09ABEH014396

Solitons in nonlinear optics

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Contents

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Received 16 July 2010
Kvantovaya Elektronika **40** (9) 756 – 781 (2010)
Translated by I.A. Ulitkin

***Abstract*. The classic examples of optical phenomena resulting in the appearance of solitons are self-focusing, self-induced transparency, and parametric three-wave interaction. To date, the list of the fields of nonlinear optics and models where solitons play an important role has significantly expanded. Now long-lived or stable solitary waves are called solitons, including, for example, dissipative, gap, parametric, and topological solitons. This review considers nonlinear optics models giving rise to the appearance of solitons in a**

narrow sense: solitary waves corresponding to the solutions of completely integrable systems of equations basic for the models being discussed.

Keywords: *optical solitons, inverse scattering transform, ultrashort pulses, dispersion, modulation, multiwave interaction, Raman scattering, optical fibres, Kerr nonlinearity*.

1. Introduction

Nonlinear waves exist in many fundamental natural phenomena. They can be found in hydrodynamics and aerodynamics, in solid-state physics and plasma physics, in optics and field theory, in chemical reaction kinetics and population dynamics, in atomic physics and gravitation theory. All nonlinear waves can be divided into two types: waves in a dispersion medium and waves in a dissipative dispersion medium. A special place among the waves of the first type is occupied by solitons. The term 'solitons' was long referred to as nonlinear solitary waves, which retain both their shape and velocity during propagation and collision with other solitary waves. However, many examples have been accumulated by now where solitons do not retain their shape while propagating, move at an accelerated speed, decompose, or form coupled states during their interaction. Nevertheless, they all have a specific feature which consists in the fact that the equations describing their evolution can be completely integrable.

The history of solitons began in 1834 when J.S. Russel observed and described an unusual type of waves in water, propagating without dispersion broadening. To study long waves on the liquid surface in channels, in 1872 J.V. Bussinesq derived an equation whose solution corresponded to solitary waves propagating in any of two possible directions. These waves could propagate through each other, thereby retaining their initial shape. A simpler equation for the waves on the water surface, which travel only in one direction, was derived by D. Korteweg and G. de Vries in 1895. In 1964 N. Zabuski and M. Kruskal found out that the Korteweg – de Vries equation has solutions in the form of solitary waves, which posses the properties of particles: retain their shape during propagation and after collisions with each other. This made it possible to call these waves the solitons (i.e., 'particles of a solitary wave', in analogy with the terms: phonon, photon, electron, magnon, etc.). Construction of the inverse scattering transform (IST) in 1968 [1] allowed the reason for the soliton longevity to be understood. Now in many parts of physics solitons are an important element for describing different effects. We can say that they play the same role in nonlinear physics as harmonic waves in linear physics. Note that due to the creation of lasers, nonlinear optics – the field in which the main features of solitons are most conspicuous – appeared and developed.

It was predicted in 1962 [2] that a positive correction to the refractive index, proportional to the radiation intensity, can lead to suppression of the diffraction divergence of the beam and to its collapse. This phenomenon was called self-focusing and was actively studied elsewhere.

Studying the propagation of a ruby laser pulse through a ruby rod, McCall and Hahn [3] found out that under certain conditions an electromagnetic pulse propagates through a resonant absorbing medium without energy losses, resulting in the emergence of the self-induced transparency phenomenon. The pulse duration should be markedly smaller than the times of the medium polarisation relaxation and the difference of resonance level populations. In this case, all the atoms respond in-phase under the action of the electric field of a pulse, accompanied by absorption (strong, resonant) and stimulated emission. At a sufficiently large pulse amplitude, all the atoms first undergo transition to the excited state, then – to the ground state, thus returning the energy absorbed due to the stimulated emission back to the pulse.

Self-focusing of light beams and self-induced transparency are the first bright examples of the important role of solitons in nonlinear optics. At the same time, most modern investigations in this field of physics are devoted to studying propagation of nonlinear waves in nonresonant media. Here, due to modulation instability a cw wave can produce a chain of separate pulses, which under certain conditions can evolve into solitons. Since 1979, when formation of solitons in an optical fibre was experimentally demonstrated, their propagation in fibres and fibreoptic communication lines has attracted much attention.

Another classic problem of nonlinear optics is the parametric wave interaction. Harmonic generation, stimulated Raman scattering, parametric amplification, four-wave mixing still attract attention of the researchers. Of special interest is three-wave interaction because it is an example of soliton emergence in a dispersionless nonlinear system, while there is an opinion that a soliton appears due to balance between dispersion broadening and nonlinear compression of a solitary wave.

1.1 Self-focusing

To describe self-focusing, the authors of [3, 4] considered a paraxial wave beam in a nonlinear medium, whose dielectric constant has a positive addition, proportional to the intensity. The self-focusing theory [5] is based on a three-dimensional nonlinear parabolic equation, which is called in many applications the $(2+1)$-dimensional nonlinear Schrödinger equation. This model made it possible to calculate the focusing threshold, the focal length and to obtain a dependence of the intensity on the distance to the focusing point. Waves in a nonlinear dispersion medium and the decay of the plane wave into separate wave packets in the case of modulation instability were studied in [6] with the help of a nonlinear parabolic equation. Reviews of the geometrical and wave optics of nonlinear media, self-focusing, and nonlinear optical effects in the field of self-focusing beams are presented in [7, 8]. Main fundamental works on self-focusing and results of modern studies in this field are given in monograph [9].

The authors of paper [10], important for the soliton theory, showed that the one-dimensional nonlinear Schrödinger equation (NLSE)

$$\mathrm{i}e_{,\zeta} + e_{,\xi\xi} + |e|^2 e = 0, \tag{1}$$

describing plane self-focusing (in this case, ζ is the normalised coordinate directed along the beam axis and ξ is the normalised transverse coordinate) and self-modulation (ζ is the normalised time and ξ is the normalised coordinate in the direction of the wave propagation), can be exactly solved by the inverse scattering transform. (Hereafter, partial derivatives are designated by a comma in the subscript.) Within the framework of the NLSE model, the problem related to diffraction of an electromagnetic wave from an opaque slotted screen behind which nonlinear (Kerr) medium is located was solved [11, 12]. The shift of the diffraction maxima and minima, proportional to the square of the field strength, was obtained. As the threshold intensity increases, a waveguide narrow channel develops. The authors of paper [13] analysed the stabilities of this phenomenon within the IST. It was found that a plane self-induced waveguide and soliton are unstable with respect to transverse perturbations ('S-turn' and 'waist' instabilities). On the whole, a soliton is unstable with respect to perturbation bending its front. The IST allows one to solve the NLSE, which takes into account the linear spatial inhomogeneity [14]. The N-soliton solution was found and the recurrent expression for the conserved quantities was derived.

Self-focusing and self-modulation of a light wave in a Kerr medium, taking into account arbitrary polarisation, were considered in detail in [15]. The resultant system of equations for the electromagnetic field vector projections (the waves e_1, e_2) is a system of coupled NLSEs (the system of Manakov equations):

$$\begin{aligned} &\mathrm{i}e_{1,\zeta} + e_{1,\xi\xi} + (|e_1|^2 + |e_2|^2)e_1 = 0, \\ &\mathrm{i}e_{2,\zeta} + e_{2,\xi\xi} + (|e_1|^2 + |e_2|^2)e_2 = 0. \end{aligned} \tag{2}$$

It was found that while crossing the channels (solitons) their polarisations change only if the initial polarisations were not collinear or orthogonal.

It is worth noting that both while taking int account the spatial inhomogeneity and solving the vector NLSE, the solitons are nonstationary and some of their parameters change in collisions.

1.2 Optical solitons in optical fibres

One of the limitations on the data transfer rate in fibreoptic communication lines (FOCLs) in the case of pulse-code modulation results from broadening of optical pulses during their propagation in the fibre due to group velocity dispersion. The dispersion broadening of the pulse can be suppressed if rather high-power light pulses are used. Here, the analogy with the self-focusing phenomenon is appropriate, when the diffraction broadening of the pulse is compensated for by its compression due to the nonlinear properties of the medium in which it propagates. In an optical fibre, instead of a spatial soliton, a temporal soliton should be formed. The authors of papers [16, 17] suggested using optical solitons to transmit information via FOCLs. The authors of papers [18, 19] derived an equation describing propagation of optical pulses in a single-mode fibre, taking into account the second-order group velocity dispersion, which formally coincides with the NLSE.

Experiments performed in [20] showed that a soliton is actually produced from a 5-ps pulse in a $\sim$ 1-km-long fibre (use was made of the fibres 0.76 – 2.5 km in length to make certain that dispersion broadening is suppressed) when the power threshold equal to $\sim$ 1 W is exceeded. A mode-locked colour centre laser (F centres in KF) was used in the experiment. The laser was pumped by a 5-W Nd laser and had an output power of 1–2.5 W at 1.23 – 1.46 μm. Later the authors of papers [21, 22] demonstrated propagation of 6 – 7-ps pulses in a negative dispersion fibre to a distance of 700 m. Initial pulses experienced changes typical of a nonlinear solitary wave [nonlinear Schrödinger (NLS) soliton] at their peak power of $\sim$ 1.24 W (theoretical prediction – 1.0 W). At a lower power, the pulse experienced dispersion broadening and at a power of 5 W it was compressed by 3.5 times. The authors also observed the pulse splitting into several subpulses as well as their further merging into one pulse. This phenomenon, called the recurrence, is typical of some class of solitons, in particular, NLS solitons.

Because the higher-order group velocity dispersion, transverse distribution dispersion of the electric field in the fibre, nonlinear susceptibility dispersion, optical losses, and other effects violate the dynamic balance between the nonlinear compression of the pulse and its dispersion broadening, solitons in real FOCLs do not exist in a strict sense. But distances propagated by an optical soliton in the fibre significantly exceed the dispersion length or the length, which could be propagated by a weak pulse. Thus, we can treat a soliton as a good approximation for real nonlinear pulses in FOCLs, i.e., pulses with durations of the order of or higher than 10 ps.

Let e designate the normalised complex slowly varying envelope of an optical pulse so that the electric field strength is given by the expression

$$\begin{aligned} E(t,x,y,z) &= A_0 e(t,z)\Psi(x,y)\exp(-\mathrm{i}\omega_0 t + \mathrm{i}\beta_0 z) \\ &+ A_0 e^*(t,z)\Psi^*(x,y)\exp(\mathrm{i}\omega_0 t - \mathrm{i}\beta_0 z), \end{aligned}$$

where β_0 is a propagation constant, depending on the carrier wave frequency ω_0; $\Psi(x,y)$ is the mode function determining the transverse distribution of the electric field in the fibre; A_0 is a material normalisation amplitude (maximal value of the electric field strength E). This normalised envelope is described by the NLSE [18, 19] in the form:

$$\mathrm{i}e_{,\zeta} + se_{,\tau\tau} + \hat{\mu}|e|^2 e = 0. \tag{3}$$

Here, $\zeta = z/L_\mathrm{d}$ and $\tau = (t - z/v_\mathrm{g})/t_{\mathrm{p}0}$ are the normalised independent variables: the coordinate along the fibre axis and time, respectively; $t_{\mathrm{p}0}$ is the pulse duration at $z = 0$; v_g is the group velocity of the pulse. The term in (3) with the second derivative in time describes the dispersion pulse broadening ($s = -1$ corresponds to normal dispersion and $s = +1$ – anomalous dispersion). The dispersion length is $L_\mathrm{d} = 4\beta_0 t_{\mathrm{p}0}^2 |\partial^2\beta_0/\partial\omega^2|^{-1}$. The third term in (3) takes into account the self-action effect or self-modulation. The coefficient $\hat{\mu}$ is equal to the ratio of the dispersion length to the length $L_\mathrm{k} = \beta_0 c^2 (2\pi\omega_0^2 A_0^2 |\chi_\mathrm{eff}|)^{-1}$. Here, χ_eff is the effective third-order nonlinear susceptibility describing the high-frequency Kerr effect.

The complete integrability of the NLSE was found in [11, 23]. A one-soliton solution of the NLSE (in the case of $s = 1$) has the form:

$$e_\mathrm{s}(\zeta,\tau) = \frac{2\mathrm{i}\eta\exp[\mathrm{i}F(\zeta,\tau)]}{\cosh[2\eta(\tau + 4\theta\zeta - \tau_0)]},$$

where $F(\zeta,\tau) = -2\theta\tau - 4(\theta^2 - \eta^2)\zeta - \varphi_0$; η, θ, τ_0 and φ_0 are the constants determined by the initial conditions. Such a soliton is sometimes called a bright soliton (Fig. 1). The NLS solitons, as follows from this expression, propagate at

$\theta = 0$ at a velocity of a weak (linear) pulse in the medium. However, their velocity can differ from the linear pulse velocity because of the initial phase modulation due to which $\theta \neq 0$. Multisoliton pulses behave similarly to breathers (Fig. 2); however, when some threshold of the phase modulation depth of the initial pulse is exceeded, such a multisoliton pulse is transformed into some separate solitons.

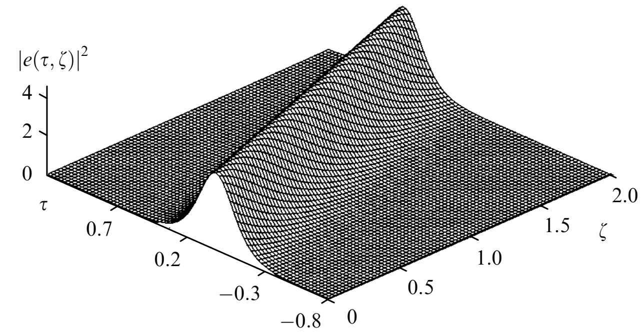

Figure 1. Bright NLS soliton (corresponds to a stationary electromagnetic pulse).

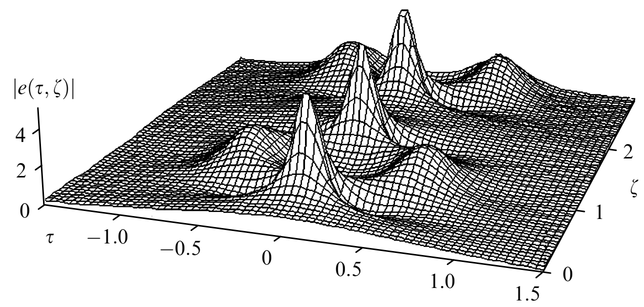

Figure 2. Pulse corresponding to a three-soliton NLSE solution (demonstrates the recurrence phenomenon).

The higher the multiplicity N of a multisoliton pulse, the more complex the pattern of its spatial evolution; however, it is important that at some path length the N-soliton is composed into one peak whose width is smaller than the initial signal by N times. This fact was used to compress an optical picosecond pulse down to femtosecond durations.

If $s = -1$, the NLS solitons with a zero asymptotic are absent. However, if $|e(\zeta,\tau)| \to e_0$ at $\tau \to \pm\infty$, equation (3) has the solution

$$e_s(\zeta,\tau) = e_0\{\cos\phi \tanh$$
$$\times[e_0\cos\phi(\tau - e_0\zeta\sin\phi) + \mathrm{i}\sin\phi]\}\exp(-\mathrm{i}e_0^2\zeta),$$

which is called a grey soliton. In a particular case, when the parameter ϕ is equal to zero, it transforms into the solution called a dark soliton (Fig. 3):

$$e_s(\zeta,\tau) = e_0\tanh(e_0\tau)\exp(-\mathrm{i}e_0^2\zeta).$$

Apart from multisoliton solutions, the NLSE has also another type of a solitary solution called a multipole soliton [24]. This solution corresponds to multiple points of a discrete spectrum of the spectral Zakharov–Shabat problem in the IST. In fact, they are difficult to realise because any small perturbation of the initial condition removes the degeneracy of the discrete spectrum points. This means that only in some exceptional cases the initial optical pulse can be transformed into a multipole soliton. Different properties of NLS solitons are described in many papers and reviews, for example, in [25–28]. Note also wonderful books [29–34] whose authors made a noticeable contribution to the development of the soliton theory.

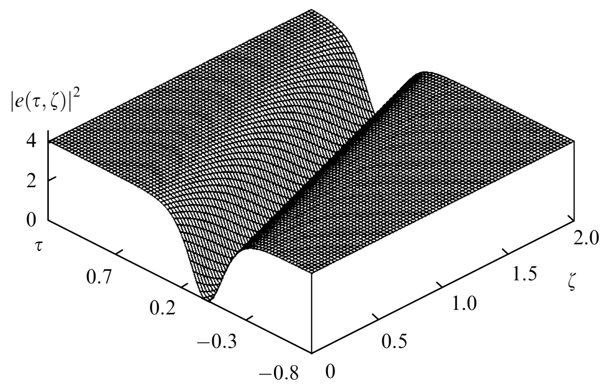

Figure 3. Dark NLS soliton (looks like a moving dip against the background of a wave of constant intensity).

1.3 Self-induced transparency

The self-induced transparency (SIT) phenomenon consists in propagation of a sufficiently high-power ultrashort light pulse in a resonant medium without the pulse shape distortion and energy losses [35–37]. Here, an ultrashort pulse is a pulse whose duration is much smaller than the polarisation relaxation times and population differences of resonance energy levels. In this case, the light–medium interaction consists in stimulated absorption and emission of electromagnetic radiation by resonance atoms of the medium. When both processes are ideally balanced, the state of the medium, after an ultrashort pulse traverses it, coincides with its initial state, and in this sense, the medium is transparent. The group velocity of such a stationary ultrashort pulse, called a 2π-pulse or a SIT soliton, is smaller than the phase velocity of light in a medium and depends on the pulse duration: the shorter the pulse, the greater its propagation velocity [35–38]. When pulses with different velocities propagate in a medium under SIT conditions, one of them catches up with the second pulse and, having colliding it propagates through this second pulse. The shape and group velocity of 2π-pulses, as is typical of solitons, do not change. Depending on the ratio between durations t_{p1} and t_{p2} of colliding 2π-pulses, their interaction pattern is different. If $t_{p1}/t_{p2} < (3-\sqrt{5})/2 \approx 0.382$, the pulse trajectories do not intersect. Otherwise, the collision resembles a repulsive interaction of solid balls which exchange energy during collisions, – the amplitudes of 2π-pulses change (Fig. 4). The SIT phenomenon itself and the behaviour of SIT solitons are described in detail in [37–39].

From the mathematical point of view, the properties of SIT pulses (2π-pulses) follow from the completely integrable system of reduced Maxwell–Bloch (RMB) equations describing the SIT by using the model of two-level atoms with nondegenerate energy levels as was shown in [40, 41]

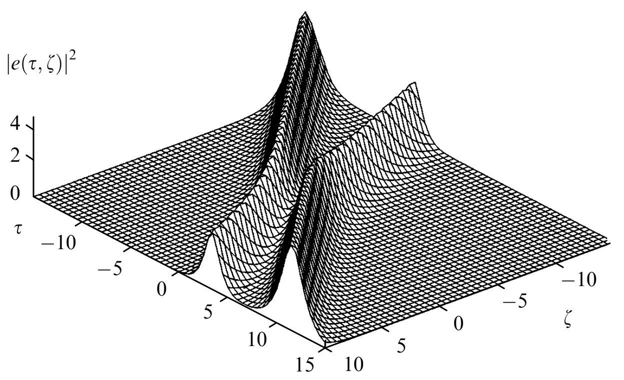

Figure 4. Collision of two SIT solitons leading to an energy exchage while preserving the pulse area.

and later thoroughly studied in papers [42–46]; 2π-pulses correspond to single-soliton solutions of these equations, and the collision process reflects the evolution of the two-soliton solution – it asymptotically transforms into a pair of solitons [42, 47, 48].

The simplest theory of the SIT phenomenon was developed by McCall and Hahn [35, 36]. In the general case, the interaction of radiation with an ensemble of two-level atoms is described by Bloch equations for atoms and by Maxwell equations for a classical electromagnetic field. In an isotropic dielectric the system of Maxwell equations is reduced to one wave equation for the electric field $\boldsymbol{E} = E\boldsymbol{l}$. For plane waves with a constant polarisation vector $\boldsymbol{l}$, we can write a system of complete Maxwell–Bloch equations (MB):

$$E_{,zz} - c^{-2}E_{,tt} = (4\pi n_{\mathrm{a}} d/c^2)\langle r_{1,tt}\rangle, \tag{4}$$

$$r_{1,t} = -\omega_{\mathrm{a}} r_2, \quad r_{2,t} = \omega_{\mathrm{a}} r_1 + (2d/\hbar)Er_3,$$
$$r_{3,t} = -(2d/\hbar)Er_2, \tag{5}$$

where d is the projection to the direction of the vector $\boldsymbol{l}$ of the matrix element of the dipole transition operator; n_{a} is the concentration of resonance atoms. Note that the components of the Bloch vector, r_1, r_2, and r_3 depend on the transition frequency ω_{a}. The angle brackets in (4) indicate averaging over the distribution of these frequencies.

In a linear homogeneous medium an electromagnetic wave can propagate in one of two possible directions. The reflected wave emerges due to scattering from macroscopic inhomogeneities, after propagating the interface between homogeneous media or in a graded-index medium. In nonlinear media the refractive index inhomogeneities can be induced by the wave itself. Thus, the applicability of the unidirectional wave approximation requires additional substantiation [49]. The authors of [50, 51] showed that if the atomic concentration is so small that the parameter $4\pi n_{\mathrm{a}} d^2/\hbar\omega_{\mathrm{a}} < 1$, we can take into account only the wave propagating in one of the directions and neglect the wave propagating in the opposite direction. It was found that for typical values of the resonance system parameters $d \sim 1$ D, $\omega \sim 10^{15}$ s^{-1}, and $n_{\mathrm{a}} \ll 10^{23}$ cm^{-3} the effect of the backward wave can be neglected. In this case, the MB equations are reduced to a system of RMB equations:

$$E_{,z} + c^{-1}E_{,t} = -(2\pi n_{\mathrm{a}} d/c)\langle r_{1,t}\rangle, \tag{6}$$

$$r_{1,t} = -\omega_{\mathrm{a}} r_2, \quad r_{2,t} = \omega_{\mathrm{a}} r_1 + (2d/\hbar)Er_3,$$
$$r_{3,t} = -(2d/\hbar)Er_2. \tag{7}$$

It is pertinent to note that both in the MB and RMB equations, E is a real value of the electric field strength of an electromagnetic wave. No limitations on the pulse duration, except the fact that equations of macroscopic electrodynamics are used, are imposed.

The next step in constructing the approximate theory of resonance interaction of the electromagnetic radiation with the medium is the quasi-harmonic wave concept. It means that the electric field of the wave propagating along the z axis can be represented as a field $\mathscr{E}$ of a harmonic wave but with a variable amplitude and phase:

$$E(z,t) = 2A(z,t)\cos[k_0 z - \omega_0 t + \varphi(z,t)]$$
$$= \mathscr{E}(z,t)\exp[\mathrm{i}(k_0 z - \omega_0 t)] + \text{c. c.} \tag{8}$$

Here, ω_0 is the carrier (harmonic) wave frequency; k_0 is the wavenumber corresponding to this frequency; the real envelope (the instantaneous harmonic wave amplitude) $A(z,t)$ and and phase $\varphi(z,t)$ are assumed to be the functions slowly varying in space and time such as

$$|A_{,t}| \ll \omega_0|A|, \ |A_{,z}| \ll k_0|A|,$$

$$|\varphi_{,t}| \ll \omega_0|\varphi|, \ |\varphi_{,z}| \ll k_0|\varphi|.$$

In addition, the ultrashort pulse amplitude is usually so small that the instantaneous Rabi frequency $[(d/\hbar)\max A]$ is much smaller than the resonance transition frequency. In the approximation of a slowly varying real envelope of a pulse e and its phase φ, system (6), (7) is replaced by:

$$e_{,z} + c^{-1}e_{,t} = -\alpha'\langle p\rangle, \tag{9}$$

$$e(\varphi_{,z} + c^{-1}\varphi_{,t}) = -\alpha'\langle q\rangle, \tag{10}$$

$$q_{,t} = (\Delta\omega + \varphi_{,t})p,$$
$$p_{,t} = -(\Delta\omega + \varphi_{,t})q + er, r_{,t} = -ep, \tag{11}$$

where $\Delta\omega = (\omega_{\mathrm{a}} - \omega_0)$ is the detuning from resonance; $\alpha' = (2\pi n_{\mathrm{a}} d^2/\hbar c)$; $e = (d/2\hbar)A$ is the normalised pulse envelope; the quantities p, q and r are related to the initial components of the Bloch vector by the expressions

$$r_1 = -p(z,t)\sin[k_0 z - \omega_0 t + \varphi(z,t)] + q(z,t)$$
$$\times\cos[k_0 z - \omega_0 t + \varphi(z,t)], \quad r_3 = -r(z,t).$$

The self-induced transparency theory proposed by McCall and Hahn is based on equations (9)–(11). But if we restrict our consideration to the situation when the initial pulse lacks the phase modulation and the shape of the inhomogeneously broadened line is specified by the symmetric function $\Delta\omega$, it follows from (9)–(11) that the phase modulation does not appear. In this case, equations (9)–(11) are reduced to a system of SIT equations:

$$e_{,z} + c^{-1}e_{,t} = -\alpha'\langle p\rangle,\ q_{,t} = \Delta\omega p,$$
$$p_{,t} = -\Delta\omega q + er,\ r_{,t} = -ep. \tag{12}$$

When the absorption line is homogeneously broadened, the SIT equations in the case of an exact resonance are reduced to the well-known sine-Gordon equation

$$\phi_{,\xi\tau} + \sin\phi = 0, \tag{13}$$

where $\tau = (t - z/c)$, $\xi = \alpha' z$, and $\phi_{,\tau} = e$.

The beneficial properties of the systems of equations (6)–(7), (9–(11), (12), and equation (13) consist in the fact that they all can be represented as an integrability condition for the pair of linear equations, which makes it possible to use the IST to solve these equations. If we assume that before the arrival of ultrashort pulses all the atoms are in the ground state and after their propagation the atoms again return to the ground state, all the mentioned equations can be supplemented by additional boundary conditions

$$\lim_{\tau\to\pm\infty} r_3 = -1,\quad \lim_{\tau\to\pm\infty} r_{1,2} = 0.$$

Both the RMB equations and the SIT equations with such boundary conditions are solved by the IST method in a standard way [41, 42, 44–48]. In the general case, the solution is an N-soliton wave and a wave spreading due to dispersion. The single-soliton solution of the system of equations (12) is given by the expression

$$e_{\mathrm{s}}(z,t) = 2t_{\mathrm{p}}^{-1}\operatorname{sech}[t_{\mathrm{p}}^{-1}(t - z/\upsilon_{\mathrm{s}})],$$

where the group velocity υ_{s} of solitons is related to the pulse duration t_{p} and the resonance absorption length $l_{\mathrm{a}}^{-1} = \alpha'$ by the expression

$$\upsilon_{\mathrm{s}}^{-1} = c^{-1}(1 + ct_{\mathrm{p}}^2 l_{\mathrm{a}}^{-1}).$$

In the general case, an N-soliton ultrashort pulse transforms (during its propagation) into L_1 isolated solitons and L_2 breathers (in this case, the condition $N = L_1 + 2L_2$) should be met). The breather is a stable solitary wave (as an ordinary soliton) but with internal amplitude oscillations. The authors of paper [37] showed that the breather of RMB equations, being a real space- and time-localised pulse with internal oscillations, resembles the 0π-pulse of McCall and Hahn. If the frequency of internal oscillations increases, the breather envelope can be described by a soliton solution of the SIT equations with a high accuracy. Therefore, the 2π-pulse of McCall and Hahn is the limiting case of the breather described by RMB equations.

1.4 Three-wave interaction

One of the well-studied nonlinear optical phenomena is the frequency conversion of electromagnetic radiation in optical nonlinear media. The classic examples of these phenomena are the harmonic generation of the fundamental (pump) wave, parametric frequency summation and subtraction, Raman scattering [52]. At a sufficiently high pump intensity, the medium polarisation nonlinearly depends on the electric field strength of the wave. If the electromagnetic wave frequencies are not in resonance with the atomic transitions, this dependence can be obtained by using the conventional perturbation theory. This gives a series expansion for polarisation $\boldsymbol{P}$ in powers of the electric field strength. The coefficients of this series, which are in the general case nth-rank tensors $\chi^{(n)}$, are called the nonlinear susceptibilities. They describe different processes of nonresonant interaction of electromagnetic waves in a medium.

Consider a quadratically nonlinear medium characterised by the tensor $\chi^{(2)}$. Let two harmonic (or quasi-harmonic) waves with the frequencies ω_1 and ω_2 propagate along the z axis. Because the medium polarisation is a quadratic function of the electric field strength of these waves, the medium can generate waves with the carrier frequencies $\omega_1 \pm \omega_2$, $2\omega_1$, and $2\omega_2$. These waves in turn generate new waves with the frequencies $2\omega_1 \pm \omega_2$, $\omega_1 \pm 2\omega_2$, etc. In dispersion media these processes are not equally effective. There exists the phase-matching condition due to which, at certain types of three-wave interactions, the wave amplitudes change markedly while all other interactions remain ineffective. In some cases, the phase-matching condition can take place for the waves propagating in the same direction. In this case we speak of collinear parametric interaction, which means that the distance at which the waves effectively interact can be quite large and, thus, a high frequency conversion coefficient can be obtained. If the phase matching is achieved for the waves propagating in different directions, these waves interact only in the region where the wave beams overlap. The noncollinear parametric three-wave interaction is an interesting example, when the corresponding system of equations turns out to be completely integrable in the two- or three-dimensional case [53, 54]. The multidimensional integrable systems, having a physical content, are exceptionally rare examples in the soliton theory.

Let e_1, e_2, and e_3 be slowly varying normalised envelopes of the pulses of interacting waves with the carrier wave frequencies ω_1, ω_2, and ω_3 and wavenumbers k_1, k_2, and k_3, respectively. Consider the case when only one wave with the sum and difference frequency is generated during the collinear propagation with the pump and idler waves. In the slowly varying envelope and phase approximation the system of equations describing the three-wave interaction can be written in the form [55, 56]:

$$e_{1,z} + \upsilon_1^{-1}e_{1,t} = \mathrm{i}\sigma e_2^* e_3^* \mathrm{e}^{\mathrm{i}\Delta kz},$$
$$e_{2,z} + \upsilon_2^{-1}e_{2,t} = \mathrm{i}\sigma e_3^* e_1^* \mathrm{e}^{-\mathrm{i}\Delta kz}, \tag{14}$$
$$e_{3,z} + \upsilon_3^{-1}e_{3,t} = -\mathrm{i}\sigma e_1^* e_2^* \mathrm{e}^{-\mathrm{i}\Delta kz},$$

where for $\omega_3 = \omega_1 + \omega_2$ and $\Delta k = k_3 - k_2 - k_3$ the fields e_1, e_2, e_3 are the envelopes of the pump and idler waves and the complex-conjugate envelope of the signal wave, respectively; for $\omega_3 = \omega_1 - \omega_2$ and $\Delta k = k_3 + k_2 - k_3$ the fields e_1, e_2, e_3 are the envelopes of the signal wave and the complex-conjugate envelopes of the idler and pump waves, respectively; σ is the coupling constant; $\upsilon_{1,2,3}$ are the group velocities of the corresponding waves. The group velocity dispersion is neglected.

The resonance Raman scattering (when the populations of the energy levels of atoms and molecules in the medium weakly change) and the scattering of optical waves on the sound wave can be treated as special cases of three-wave interaction. For the normalised envelope of the incident wave pulse e_1, the envelope of the scattered wave e_2, and the envelope of the acoustic wave w the authors of [57] used the equations

$$\begin{aligned} &\mathrm{i}e_{1,\zeta} = e_2 w \mathrm{e}^{\mathrm{i}\delta\zeta}, \\ &\mathrm{i}e_{2,\zeta} = e_1 w^* \mathrm{e}^{-\mathrm{i}\delta\zeta}, \\ &\mathrm{i}w_{,\tau} = -\epsilon w + e_1 e_2^* \mathrm{e}^{-\mathrm{i}\delta\zeta}, \end{aligned} \tag{15}$$

where ϵ is the normalised frequency detuning; δ is the normalised wave detuning. Under conditions of the exact resonance $\epsilon = 0$, and when the phase-matching conditions are met, $\delta = 0$. It is assumed that the group velocities of the incident and scattered waves are identical. We can show that under certain conditions equation (15) is rewritten in the form of a system of RMB equations. Due to this, Raman scattering can be studied, as is the case of the SIT, with the help of the IST.

Equations (14) describing the three-wave interaction have an infinite number of conservation laws with the Bäcklund transforms having place for these waves [55]. The authors of [58] showed that these equations withstand the Painlevé test and there exists a class of their self-model solutions expressed by P-V and P-VI Painlevé transcendentals. Because the dispersion of the phase and group velocities is absent, the soliton part of the solution, describing the three-wave interaction, is not separated from the nonsoliton part, which is often called radiation. This makes it difficult to study the process of the three-wave interaction by analytic methods and makes us restrict our consideration to some special cases. The noncollinear SHG is the example of such a case where we can find a particular exact solution without the IST. In this case, the obtained solution demonstrates the inseparability of solitons and radiation (nonsoliton part of the solution of these equations).

The authors of paper [59] studied the stimulated Brillouin backscattering with the help of the IST in an amplifying medium for two initial rectangular pulses. Stimulated Raman scattering and stimulated Brillouin scattering in the quasi-stationary regime were considered in [60] as a three-wave interaction, which made it possible to use the IST with the spectral Zakharov – Shabat problem.

The phase matching can be achieved by using anisotropic crystals, which allow one to select the direction in which the ordinary and extraordinary waves will have equal phase velocities. Another way to achieve the phase matching is to select the angle at which the beams of the interacting waves intersect so that the vector equality $\boldsymbol{k}_3 = \boldsymbol{k}_1 \pm \boldsymbol{k}_2$ be fulfilled. The system of the equations describing the noncollinear SHG (in the normalised form) has the form:

$$\begin{aligned} &\eta_z e_{1,z} + \eta_x e_{1,x} + \upsilon_1^{-1} e_{1,t} = \mathrm{i}\sigma e_2^* e_3, \\ &\eta_z e_{2,z} - \eta_x e_{2,x} + \upsilon_1^{-1} e_{2,t} = \mathrm{i}\sigma e_1^* e_3, \\ &e_{3,z} + \upsilon_3^{-1} e_{3,t} = 2\mathrm{i}\sigma e_1 e_2, \end{aligned} \tag{16}$$

where $\eta_z = k_{1z}/k_1 = k_{2z}/k_2$; $\eta_x = k_{1x}/k_1 = -k_{2x}/k_2$ are the direction cosines of the pump wave beams; e_1, e_2 are the normalised electric fields of the pump beams intersecting at an angle $2\theta_{\mathrm{m}}$; e_3 is the normalised electric second-harmonic field. The phase-matching condition is met if the angle θ_{m} is selected so that the condition $n(2\omega) = n(\omega)\cos\theta_{\mathrm{m}}$ is fulfilled for the refractive index at different frequencies. This condition is realised only in the anomalous dispersion region when $n(2\omega) < n(\omega)$.

1.5 Reasons for searching for new, completely integrable systems

With the appearance of such a powerful tool for studying nonlinear problems, as the IST, new models and theories began emerging, based on completely integrable equations and describing phenomena in nonlinear optics. Moreover, along with the development of the known theories, new problems have been considered and new means have been developed to describe the propagation of optical waves in nonlinear media.

Nanosecond and picosecond optical pulses contain many $(10^6 - 10^3)$ electromagnetic field oscillations. To describe the evolution of such signals it is sufficient to consider only the pulse envelope and phase. In the case of multiwave interaction, pulses with well-separated carrier wave frequencies were considered. To this end, the representation of electromagnetic signals as quasi-harmonic waves is quite acceptable. Passage to the femtosecond (10^{-15} s), attosecond (10^{-18} s), zeptosecond (10^{-21} s) durations makes it impossible to use the quasi-harmonic wave approximation (or the slowly varying amplitude and phase approximation). The above-mentioned NLSE is unacceptable and its generalisation leads to some model equations among which there are completely integrable equations.

If an optical quasi-harmonic pulse is characterised by some carrier-wave frequency, in resonant media we can restrict our consideration to two energy states, with the transition between the states occurring under the action of a pulse, – this is the two-level atom model. Many carrier-wave frequencies and resonant medium levels participating in the response to such a multifrequency pulse require generalisation of the two-level model. Among the generalisations we have some which are based on systems of completely integrable equations and in these cases solitons are possible.

An electromagnetic field is a vector one; therefore, it may be needed to generalise the scalar models considered above. In this direction we managed to find some examples of emergence of the vector (two-component or three-component) solitons.

1.6 A few words about the inverse scattering transform

There exist hamiltonian systems for which canonical transforms can be found to transform the original equations of motion to new systems of equations describing an ensemble of independent harmonic oscillators. After such a substitution of the variables, the equations of motion corresponding to them are trivially integrated. It is believed that the Hamiltonian system in this case allows the action – angle variables, the system itself being called completely integrable. The discovery of Gardner, Green, Kruskal, and Miura in 1967 armed the researchers with a powerful tool for studying completely integrable systems, i.e., the IST. Later the IST was elaborated by V.E. Zakharov, L.D. Fadeev, A.B. Shabat, and S.V. Manakov as well as by Ablowitz, Kaup, Newell, and Segur (AKNS).

The essence of the IST consists in the following. Let $\hat{L} = \hat{L}(\partial_x, q)$ and $\hat{A} = \hat{A}(\partial_x, q)$ be two linear operators, where $q(x,t)$ is the potential – either scalar, or vector, or matrix function of the variables x and t. The compatibility condition of the pair of equations

$$\hat{L}\psi = 0, \qquad \hat{T}\psi = (\partial_t - \hat{A})\psi = 0 \tag{17}$$

means that the operators $\hat{L}$ and $\hat{T}$ commute, i.e.,

$$\frac{\partial}{\partial t}\hat{L} = [\hat{L}\ , \hat{A}]. \tag{18}$$

This operator equation is a differential equation (system of equations) with respect to the potential (potentials) $q(x,t)$. It is believed that for this equation expression (18) is the Lax representation or the zero curvature representation and the operators $\hat{L}$ and $\hat{A}$ are called a Lax pair. Quite often the operator $\hat{L}$ is used in the form $\hat{L} = (\partial_x - \hat{U})$.

If for some specific nonlinear evolution equation one manages to find the Lax representation, its solution can be obtained following the algorithm given below. As a direct spectral problem, use is made of the scattering problem or eigenvalue problem

$$\hat{L}\psi = \lambda\psi, \quad \partial\lambda/\partial t = 0$$

plus the condition of the $q(x)$ behaviour at $|x| \to \infty$. The solution of this problem yields the eigenvalues $\{\lambda_j\}$, eigenfunctions $\{\psi_j\}$, and the scattering matrix. The set of these quantities is the scattering data $C(0)$ at $q(x,0) = q_0(x)$. According to the second equation from (17), the scattering data $C(t)$ can be found for all $t > 0$. The potential recovery $q(x,t)$ by the scattering data, i.e., the solution of the inverse spectral problem, leads to the solution of the initial nonlinear equation for which (18) is the Lax representation.

Note that the transition from the equation $\hat{L}\psi = 0$ to the equation $\hat{L}\psi = \lambda\psi$ retains the Lax representation (18). Because it was assumed here that the change (deformation) in the potential $q(x,t)$ with time t does not change the spectrum of the operator $\hat{L}$, the nonlinear equation for $q(x,t)$ is called the isospectral deformation equation.

For many nonlinear equations with the Lax representation, there was developed a way to construct the action–angle variables, which makes it possible to call them completely integrable.

2. Beyond the model of two-level atoms

The development of the SIT theory is associated with a passage outside the model of two-level atoms and with consideration of multifrequency ultrashort pulses. The interaction of multilevel resonant media with radiation characterised by some carrier-wave frequencies is studied. In addition, direct interaction between the resonance atoms, nonlinear properties of the dielectric into which the resonance atoms are submerged, and polarisation (vector behaviour) of radiation itself were taken into account. Figure 5 shows two configurations of the energy states of three-level atoms, which represent a very popular recent model for the resonance coherent optics. Figure 6 presents the resonance transitions between the two-level atom states, degenerate in the angular moment projections, when it is important to take into account the vector character of the electromagnetic radiation of ultrashort pulses. The difference between the double and two-photon resonances is illustrated in Fig. 7.

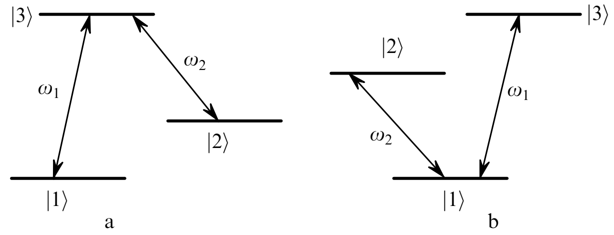

Figure 5. (a) Λ- and (b) V-configurations of the energy levels of a three-level atom.

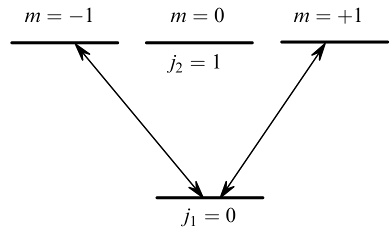

Figure 6. Resonance transitions between the states of a two-level atom with $j_1 = 0$ and $j_2 = 1$.

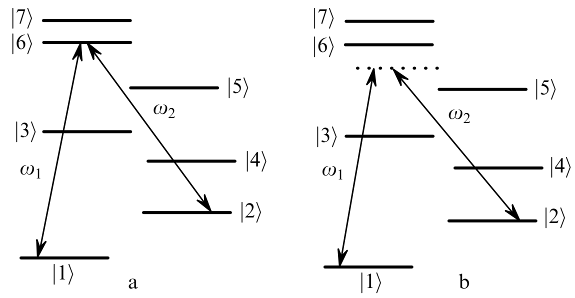

Figure 7. Double (a) and two-photon (b) resonances.

2.1 Double resonance

Let the optical pulse propagate along the z axis and its electric field strength be

$$\boldsymbol{E}(z,t) = \boldsymbol{A}(z,t)\exp[\mathrm{i}(k_0 z - \omega_0 t)] + \text{c. c.} \tag{19}$$

The carrier frequency ω_0 is close to the frequency ω_{21} of the $j_2 \leftrightarrow j_1$ transition between the energy levels $|1\rangle$ and $|2\rangle$, degenerate with respect to the projections m_1 and m_2 of the total angular momenta j_1 and j_2. In the general case, the evolution of the ultrashort pulse envelope and the resonant medium state is described by a system of equations for which the exact solution in the case of arbitrary j_1 and j_2 is unknown. But for the transitions $j_2 = 0 \leftrightarrow j_1 = 1$, $j_2 = 1 \leftrightarrow j_1 = 1$ and $j_2 = 1/2 \leftrightarrow j_1 = 1/2$, the corresponding systems of the generalised RMB (GRMB) equations are completely integrable. As is shown in [61–63], their soliton solutions can be obtained by using the IST. For the transitions $j_2 = 0 \leftrightarrow j_1 = 1$ and $j_2 = 1 \leftrightarrow j_1 = 1$, the GRMB equations can be written in the unified form:

$$\boldsymbol{e}_{,\zeta} = -\mathrm{i}\sum_{a=1,2}\beta_a\langle \boldsymbol{p}^{(a)}\rangle,$$

$$\boldsymbol{p}^{(a)}_{,\tau} = \mathrm{i}\delta\boldsymbol{p}^{(a)} - \mathrm{i}\boldsymbol{e}\cdot\hat{m}^{(a)} + \mathrm{i}\boldsymbol{e}n^{(a)},$$

$$\hat{m}^{(a)}_{,\tau} = -\mathrm{i}(\boldsymbol{e}^*\otimes\boldsymbol{p}^{(a)} - \boldsymbol{p}^{(a)*}\otimes\boldsymbol{e}),$$

$$n^{(a)}_{,\tau} = -\mathrm{i}(\boldsymbol{p}^{(a)*}\cdot\boldsymbol{e} - \boldsymbol{e}^*\cdot\boldsymbol{p}^{(a)}), \tag{20}$$

where the symbol $\otimes$ denotes the tensor multiplication, i.e., $(\boldsymbol{a}\otimes\boldsymbol{b})_{ik} = a_i b_k$, and the dimensionless variables are $\tau = (t - z/c)$ and $\zeta = \alpha' z$.

If the transition $j_1 = 0 \to j_2 = 1$ is considered, equation (20) assumes that the vector $\boldsymbol{e}$ is determined by the components $e_j = dt_{\mathrm{p}0}\hbar^{-1}A_j$, $\beta_1 = 1$ and $\beta_2 = 0$. For the transition $j_1 = 1 \to j_2 = 0$ we have in (20) $e_j = dt_{\mathrm{p}0}\hbar^{-1}A_j$, $\beta_1 = 0$ and $\beta_2 = 1$. Finally, for the transition $j_1 = 1 \to j_2 = 1$, we have $e_j = dt_{\mathrm{p}0}(\sqrt{2}\hbar)^{-1}A_j$ and $\beta_1 = \beta_2 = 1/2$. Hereafter, the subscripts take the values $j = \pm 1$, $\delta = \Delta\omega t_{\mathrm{p}0}$ is the normalised detuning from the resonance, and the superscript is equal to 1, 2. The scalars $n^{(a)}$, vectors $\boldsymbol{p}^{(a)}$, and matrices $\hat{m}^{(a)}$ are expressed by the slowly varying envelopes of the matrix elements of the density matrix $\hat{\rho}$ as:

$$p^{(1)}_j = \langle j_2, 0|\hat{\rho}|j_1, j\rangle,\ p^{(2)}_j = \langle j_2, -j|\hat{\rho}|j_1, 0\rangle,$$

$$n^{(1)} = \langle j_2, 0|\hat{\rho}|j_2, 0\rangle,\ n^{(2)} = -\langle j_1, 0|\hat{\rho}|j_1, 0\rangle,$$

$$m^{(1)}_{jl} = \langle j_1, j|\hat{\rho}|j_1, l\rangle,\ m^{(2)}_{jl} = \langle j_2, -j|\hat{\rho}|j_2, -l\rangle.$$

Generalised RMB equations (20) serve as a condition of the solvability of a pair of linear equations of the inverse scattering transform for the AKNS hierarchy of the high-dimensionality equations. The spectral problem of this type in the IST was first proposed and studied by Manakov [15] while describing self-focusing of a polarised light beam.

If we introduce a unit vector $\boldsymbol{l}$ determining the ultrashort pulse polarisation, the single-soliton solution of the system of equations (20) will correspond to an electromagnetic pulse with the envelope in the form

$$\boldsymbol{e}_{\mathrm{s}}(\tau,\zeta) = -\mathrm{i}2\eta\boldsymbol{l}\,\mathrm{sech}[2\eta(\tau-\tau_0) - b\zeta]\exp(-2\alpha\tau + \mathrm{i}\kappa\zeta),$$

where $\eta, \xi, \alpha, \kappa, b = b(\eta)$ are the parameters determined from the initial conditions, as the polarisation vector. This ultrashort pulse can be called a polarised 2π-pulse.

Let two polarised 2π-pulses, differing in durations ($1/2\eta_1$ and $1/2\eta_2$) and polarisation vectors ($\boldsymbol{l}_1$ and $\boldsymbol{l}_2$), enter the resonant medium at point $\zeta = 0$ one after another with an interval $\tau_2 - \tau_1$:

$$\boldsymbol{e}_{\mathrm{s}}(\tau, 0) = -\mathrm{i}2\eta_1\boldsymbol{l}_1\mathrm{sech}[2\eta_1(\tau-\tau_1)] - \mathrm{i}2\eta_2\boldsymbol{l}_2\mathrm{sech}[2\eta_2(\tau-\tau_2)].$$

For simplicity, we selected the solitons with $\alpha = \kappa = 0$. On the axis of the normalised time τ the first soliton is located to the left from the second one and it is assumed that $\tau_2 - \tau_1 \gg 1/2\eta_1,\ 1/2\eta_2$ so that they do not interact. If $\eta_2 > \eta_1$, the velocity of the second soliton is higher than the velocity of the first soliton, which means that after a while they will collide and then will move apart so that the second pulse will be to the left from the first (slow) pulse on the τ axis at $\zeta \to \infty$. The two-soliton envelope will take the from:

$$\boldsymbol{e}_{\mathrm{s}}(\tau,\zeta) = -\mathrm{i}2\eta_1\boldsymbol{l}'_1\mathrm{sech}[2\eta_1(\tau-\tau'_1) - b_1\zeta]$$
$$-\mathrm{i}2\eta_2\boldsymbol{l}'_2\mathrm{sech}[2\eta_2(\tau-\tau'_2) - b_2\zeta].$$

The possible phase shift is included in the change in the polarisation vectors. The rules of changes in the vectors $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are given by the expressions:

$$\boldsymbol{l}'_1 = \epsilon\left[-\boldsymbol{l}_1 + \frac{2\eta_2(\boldsymbol{l}_1\cdot\boldsymbol{l}^*_2)}{\eta_2-\eta_1}\boldsymbol{l}_2\right],\ \boldsymbol{l}'_2 = \epsilon\left[-\boldsymbol{l}_2 + \frac{2\eta_1(\boldsymbol{l}_2\cdot\boldsymbol{l}^*_1)}{\eta_2-\eta_1}\boldsymbol{l}_1\right], \tag{21}$$

where

$$\epsilon = \left[1 + \frac{4\eta_1\eta_2}{(\eta_2-\eta_1)^2}|\boldsymbol{l}_2\cdot\boldsymbol{l}^*_1|^2\right]^{-1/2};\quad (\boldsymbol{l}_1\cdot\boldsymbol{l}^*_2) = (\boldsymbol{l}'_1\cdot\boldsymbol{l}'^*_2).$$

The formulae show that if the solitons are linearly polarised before the collisions, they will remain linearly polarised after them. But if the polarisation vectors of initial solitons are such that $(\boldsymbol{l}_1\cdot\boldsymbol{l}_2) = \cos\theta$, the collision leads to the rotation of the polarisation vectors by the angles θ_1 and θ_2, determined by the expressions:

$$\boldsymbol{l}'_1\cdot\boldsymbol{l}_1 = \cos\theta_1 = -\frac{1 + B_{12}\cos^2\theta}{(1 - B_{12}B_{21}\cos^2\theta)^{1/2}},$$

$$\boldsymbol{l}'_2\cdot\boldsymbol{l}_2 = \cos\theta_2 = -\frac{1 + B_{21}\cos^2\theta}{(1 - B_{12}B_{21}\cos^2\theta)^{1/2}},$$

where

$$B_{12} = \frac{2\eta_2}{\eta_1-\eta_2};\quad B_{21} = \frac{2\eta_1}{\eta_2-\eta_1}.$$

The expressions for the N-solitons, breathers, and the Bäcklund transform were found in paper [64]. The authors of papers [65, 66] analysed equations (20) from the point of view of their solution with the help of the Riemann problem. They obtained the soliton solutions and noted that the single-soliton solutions are always described by an ordinary sech-shaped pulse. They showed that there exist solitons moving with a variable velocity and oscillating amplitudes.

There exist several versions of the GRMB equations describing the ultrashort pulse propagation in multilevel media. The simplest case is the models of atoms with three resonance energy levels, called V- and Λ-configurations (Fig. 4). It was found in [67, 68] that if the oscillator forces for each transition in V- and Λ-configuration of the energy levels are equal, there can exist a two-frequency pulse (characterised by two carrier waves with different frequencies), which propagates in a medium without the envelope shape distortion. The ultrashort pulse of this type was called a simulton [69]. The simultons are the single-soliton solutions of GRMB equations (20). The multisoliton solution corresponds to colliding simultons. Simultons with internal oscillations (as if colour breathers) are the two-level generalisation of the McCall–Hahn 0π-pulses. Note here that the simulton in the general case is unstable with respect to transformation into one single-frequency 2π-pulse and can stay a two-frequency pulse only if the medium is specially prepared, i.e., if the resonance levels are populated in a certain way [70].

The theory of propagation of two-frequency polarised ultrashort pulses in a three-level medium leads to the matrix variant of the GRMB system. This system was solved by generalising the IST to the case of the matrix Zakharov–Shabat–AKNS spectral problem (or matrix Manakov spectral problem). This case was considered in detail in papers [71–73].

The authors of paper [74] described coherent propagation of a three-frequency ultrashort pulse in a three-level medium where each harmonic component of the ultrashort pulse is in resonance with the corresponding transition, and in addition, the medium has a quadratic nonlinearity. The electric field is presented as a superposition of three fields,

$$E(z,t)=\sum_{j,k=1(j\neq k)}^{3}[E_{jk}\exp(-\mathrm{i}\omega_{jk}t+\mathrm{i}k_{jk}z)+\text{c.c.}],$$

where the frequencies ω_{jk} are close to the frequencies of the transitions between the jth and kth states of three-level atoms. The truncated Maxwell equations are written in the form

$$(E_{jk,t}+v_{jk}E_{jk,z})=\mathrm{i}a_{jk}\rho_{jk}+\mathrm{i}b_{jk}E_{jn}E_{nk}, \tag{22}$$

where v_{jk} is the group velocity for the wave with the envelope E_{jk};

$$a_{jk}=\frac{4\pi n_{\mathrm{a}}\omega_{jk}d_{jk}}{n^2(\omega_{jk})};\quad b_{jk}=\frac{2\pi\omega_{jk}\chi^{(2)}}{n^2(\omega_{jk})};$$

d_{jk} is the matrix element of the dipole transition operator between the jth and kth states; $\chi^{(2)}$ is the second-order nonlinear susceptibility; $n(\omega_{jk})$ is the refractive index at the frequency ω_{jk}. The elements of the density matrix ρ_{jk} satisfy the von Neumann equations

$$2\rho_{jk,t}=-\mathrm{i}\delta_{jk}\rho_{jk}+\mathrm{i}g_{jk}(\rho_{kk}-\rho_{jj})E_{jk}$$
$$+\mathrm{i}(g_{jn}E_{jn}\rho_{nk}-\rho_{jn}g_{nk}E_{nk}), \tag{23}$$
$$\rho_{kk,t}=\mathrm{Im}(g_{kj}E_{kj}\rho_{jk}+\text{c.c.}),$$

where $g_{jk}=(2a_{jk}/d_{jk})^{1/2}$.

We considered the cases when all the three transitions are allowed and one of the transitions is forbidden in the dipole approximation (for example, $d_{23}=0$). In both cases for equations (22) and (23) we obtained the Lax representation, which makes it possible to use the IST to find the soliton solutions.

2.2 Self-induced transparency in the case of two-photon resonance

Multiphoton processes can develop in the field of high-power optical pulses. The simplest situation appears when the resonant medium interacts with a couple of ultrashort pulses having different charier-wave frequencies ω_1 and ω_2 so that $\omega_1\pm\omega_2$ coincides with the resonance transition frequency ω_{21}. This situation refers to nondegenerate two-photon resonance, when $\omega_1=\omega_2$. The corresponding nonlinear processes are two-photon absorption ($\omega_1+\omega_2\approx\omega_{21}$) and Raman scattering ($\omega_1-\omega_2\approx\omega_{21}$) of the interacting waves. If the pulse duration is much smaller than the polarisation relaxation times and population differences their coherent propagation is possible. This SIT-like process, called two-photon self-induced transparency, is described by GRMB equations [75–77].

2.2.1 Nondegenerate two-photon resonance

Using the normalised variables, we can write the system of equations (see details in [74]) in the form:

for Raman scattering

$$\mathrm{i}e_{1,\zeta}=-k_{\mathrm{S}1}\langle n-n_0\rangle e_1-(1/2)\langle p\rangle e_2,$$
$$\mathrm{i}e_{2,\zeta}=-k_{\mathrm{S}2}\langle n-n_0\rangle e_2-(1/2)\langle p^*\rangle e_1,$$
$$p_{,\tau}=\mathrm{i}(\delta+\delta_{\mathrm{S}})p+\mathrm{i}ne_1e_2^*, \tag{24}$$
$$n_{,\tau}=(\mathrm{i}/2)(pe_1^*e_2-p^*e_1e_2^*);$$

for two-photon absorption

$$\mathrm{i}e_{1,\zeta}=-k_{\mathrm{S}1}\langle n-n_0\rangle e_1-(1/2)\langle p\rangle e_2^*,$$
$$\mathrm{i}e_{2,\zeta}=-k_{\mathrm{S}2}\langle n-n_0\rangle e_2-(1/2)\langle p\rangle e_1^*,$$
$$p_{,\tau}=\mathrm{i}(\delta+\delta_{\mathrm{S}})p+\mathrm{i}ne_1e_2, \tag{25}$$
$$n_{,\tau}=(\mathrm{i}/2)(pe_1^*e_2^*-p^*e_1e_2).$$

Here, $\delta_{\mathrm{S}}=2k_{\mathrm{S}1}|e_1|^2+2k_{\mathrm{S}2}|e_1|^2$ takes in into account the frequency shift of the transition due to the high-frequency Stark effect; p is the slowly varying polarisation; n is the population difference of resonance levels; n_0 is its equilibrium value. It is convenient to rewrite these equations in new variables, which are quadratic functions of normalised envelopes of ultrashort pulses $e_{1,2}$:

$$S_0=(|e_1|^2+m|e_2|^2),\ S_3=(|e_1|^2-m|e_2|^2),$$
$$p=R=R_1+iR_2,\quad n=R_3,$$
$$S=S_1+\mathrm{i}S_2=(1-m)e_1e_2+(1+m)e_1e_2^*,$$

where $m=1$ in the case of Raman scattering and $m=-1$ in the case of two-photon absorption. In these variables the both systems can be written in the form:

$$R_{,\tau}=\mathrm{i}[\delta+b^{(+)}S_0+b^{(-)}S_3]R+\mathrm{i}R_3S, \tag{26}$$
$$R_{3,\tau}=(\mathrm{i}/2)(RS^*-R^*S),$$

$$S_{,\zeta}=\mathrm{i}b^{(-)}\langle R_3\rangle S-\mathrm{i}m\langle R\rangle S_3, \tag{27}$$
$$S_{3,\zeta}=(\mathrm{i}/2)(\langle R\rangle S^*-\langle R^*\rangle S),\quad S_{0,\zeta}=0,$$

where $b^{(\pm)}=k_{\mathrm{S}1}\pm mk_{\mathrm{S}2}$.

Note that this implies the law of conservation of the 'modulus of the energy spin vector' $\boldsymbol{S}=(S_1,S_2,S_3)$:

$$(S_1^2+S_2^2+mS_3^2)_{,\zeta}=0,$$

i.e. $S_1^2+S_2^2+mS_3^2=S^2(\tau)$.

The authors of papers [78, 79] introduced the new variables:

$$r = R^* e^{i\Phi(\tau)}, \quad s = S^{-1} S^* e^{i\Phi(\tau)}, \quad r_3 = R_3, s_3 = S^{-1} S_3,$$

$$\eta = \int_0^\tau S(\tau_1) d\tau_1, \quad \Phi(\tau) = b^{(+)} \int_0^\tau S_0(\tau_1) d\tau_1.$$

With these variables, the system of equations (26) and (27) takes the form:

$$\begin{aligned} s_{,\zeta} &= imrs_3 + i\tilde{g}r_3 s, \quad s_{3,\zeta} = \frac{i}{2}(r^* s - rs^*), \\ r_{,\eta} &= i\tilde{g}rs_3 + ir_3 s, \quad r_{3,\eta} = \frac{i}{2}(rs^* - r^* s), \end{aligned} \tag{28}$$

where $\tilde{g} = b^{(-)}$. The laws of conservation of the modulus of the 'spin vectors' $\boldsymbol{s} = (s_1, s_2, s_3)$ and $\boldsymbol{r} = (r_1, r_2, r_3)$ are written in the form

$$m(s_1^2 + s_2^2) + s_3^2 = 1, \quad r_1^2 + r_2^2 + r_3^2 = 1.$$

As was shown by Steudel and Kaup [78, 79], the unified system of equations (28) derived in this way can be solved with the help of the IST if all the atoms are identical. The Bäcklund transform, which makes it possible to obtain soliton solutions of arbitrary multiplicity from single-soliton solutions of equations (28), is given in [80].

Using the Kaup–Steudel model, the authors of [81] studied, with the help of the IST, the propagation of ultrashort pulses under Raman resonance conditions. They considered the limit when the Raman scattering is initiated by polarisation fluctuations of a small-area Stokes pulse, and took into account the pump fluctuations. By using the obtained self-similar solution, the compression of the Stokes field fluctuations in a strongly nonlinear Raman scattering stage was explained.

The theory of soliton generation at the leading edge of a long pump pulse in the case of stimulated Raman scattering was developed by Kamchatnov [82]. Kamchatnov used the Kaup–Steudel model as the starting point and the MIST to formulate the Witham formalism and obtain periodic solutions with its help.

2.2.2 Degenerate two-photon resonance

In this case, two photons of one wave are absorbed upon a resonance transition between the states of a two-level atom. The normalised RMB equations have the form

$$\begin{aligned} e_{,\zeta} &= \frac{i}{2}(re^* - gr_3 e), \\ ir_{,\tau} &= 2(r_3 e^2 + ge^* e^* r), \\ r_{3,\tau} &= i(r^* e^2 - re^{*2}), \end{aligned} \tag{29}$$

where g is the coefficient proportional to the ratio of the nonlinear susceptibility responsible for the Stark shift to the susceptibility responsible for two-photon absorption. In the assumption that the population difference of the resonance states changes insignificantly, $r_3 \approx -1 + |r|^2/2$, these equations can be reduced to equations describing the SHG [83] in which, however, the field e plays the role of the pump wave and polarisation r is treated as a harmonic. Although the SHG equations

$$e_{1,\zeta} = -2e_2 e_1^*, \quad e_{2,\tau} = e_1^2$$

have the Lax representation and the IST makes it possible to find their solutions, these equations lack solitons.

2.3 Coherent four-wave mixing

Counterpropagation of two pairs of waves in a cubic nonlinear medium, when the sum (or difference) of the carrier frequencies is close to the transition eigenfrequency in the medium, was considered in [84, 85]. If the entire field is represented as a superposition of the fields of four waves

$$E(z,t) = \sum_{j=1}^{2} [A_j \exp(-i\omega_{1j} t + ik_{1j} z)$$

$$+ B_j \exp(-i\omega_{2j} t + ik_{2j} z) + \text{c.c.}],$$

the truncated Maxwell equations are written in the form

$$\begin{aligned} (A_{1,z} + v_1^{-1} A_{1,z}) &= ia_{11} A_1 |B_1|^2 + ia_{12} A_2 (B_1 B_2^*) e^{i\delta}, \\ (A_{2,z} + v_2^{-1} A_{2,z}) &= ib_{11} A_2 |B_2|^2 + ib_{12} A_1 (B_2 B_1^*) e^{-i\delta}, \\ (B_{1,z} + u_1^{-1} B_{1,z}) &= ia_{22} B_1 |A_1|^2 + ia_{21} B_2 (A_1 A_2^*) e^{-i\delta}, \\ (B_{2,z} + u_2^{-1} B_{2,z}) &= ib_{22} B_2 |A_2|^2 + ib_{21} B_1 (A_2 A_1^*) e^{i\delta}, \end{aligned} \tag{30}$$

where v_1, v_2, u_1, u_2 are the group velocities; a_{jk} and b_{jk} are the coupling constants; $\delta = (k_{12} - k_{11} + k_{21} - k_{22})z$ is the phase (wave) detuning. The case of exact synchronism ($\delta = 0$) and the relations for the group velocities ($v_1 = v_2 = -u_1 = -u_2$, $v_1 = -v_2 = u_2 = -u_1$) were studied. The Lax representation was found for this system of equations and the IST was developed. Soliton and quasi-self-similar solutions describing the decay of the initial unstable state were constructed. Using the IST, the authors of [86] found the single-zone solution, which is a periodic wave*. The Witham equations were derived for slow envelopes.

The four wave interaction of counterpropagating waves under conditions of two-photon resonance was considered by Zabolotskii [87]. Using the IST previously developed for this problem, Zabolotskii found periodic (single-zone) solutions, which are periodic analogues of solitons and have the soliton properties.

2.4 Self-induced transparency in a medium with the Kerr nonlinearity

Nonlinear properties of optical fibres can be reinforced by introducing resonance impurities into them. At the end of the last century much attention was paid to the investigation of the erbium-doped fibres. Such fibres proved suitable for designing fibre amplifiers. The attention of the researchers was also drawn to coherent phenomena (photon echo, SIT) in these 'one-dimensional' media.

It is known that the NLSE, which is used to describe optical solitons in fibres, is as integrable [11, 23] as the RMB equations, which describe the coherent propagation regime in a resonant medium. The solution of the NLSE and RMB

*Solitons evolve from the initial conditions in the form of a solitary wave. If a periodic extended wave is the initial condition, instead of N-solitons there can appear N-zone solutions corresponding to the generalised cnoidal waves. The discrete spectrum of the IST (see Section 1.6) transforms in a set of allowed zones, which gave the name to periodic solutions.

equations is based on the same spectral problem of the IST. The system of equations, which describes the coherent propagation in a fibre with resonance impurities, combines both mentioned systems of equations, which indicates the possibility of its complete integrability under certain conditions.

The evolution of an ultrashort pulse propagating in a nonlinear single-mode fibre with resonance impurities is described by the system of equations [88–91]:

$$\begin{aligned} &\mathrm{i}e_{,\zeta} + se_{,\tau\tau} + \mu|e|^2 e + \kappa\langle p\rangle = 0, \\ &p_{,\tau} = \mathrm{i}\delta p + 2\mathrm{i}fer, \quad r_{,\tau} = \mathrm{i}f(e^* p - ep^*), \end{aligned} \tag{31}$$

where e is defined as a slowly varying complex ultrashort-pulse envelope (see Section 1.2). Interaction of radiation with resonance impurities is characterised by a dimensionless constant $f \sim \bar{d}A_0$, where $\bar{d}$ is the effective matrix element of the dipole moment operator of the transition between the resonance levels. The coefficient κ is expressed via the dispersion length L_{d} and the resonance absorption length L_{a} [30, 73]: $\kappa = L_{\mathrm{d}}(L_{\mathrm{a}}f)^{-1}$.

Propagation of polarised ultrashort pulses in a fibre with the Kerr nonlinearity, when the resonance levels of the impurities are degenerate with respect to the orientation of the total angular momenta j_1 and j_2, are described in some cases by completely integrable equations. For the electric field strength of ultrashort pulses in quasi-harmonic approximation we can write:

$$\begin{aligned} E_j(t,x,y,z) = {} & A_0 e_j(t,z)\Psi(x,y)\exp(-\mathrm{i}\omega_0 t + \mathrm{i}\beta_0 z) \\ & + A_0 e_j^*(t,z)\Psi^*(x,y)\exp(\mathrm{i}\omega_0 t - \mathrm{i}\beta_0 z). \end{aligned}$$

As in (20), the subscript is $j = \pm 1$; therefore, we can speak about the vector optical soliton. The same system of equations emerges if the nonlinear fibre contains three-level impurities and the ultrashort pulse is characterised by two carrier-wave frequencies: $\omega_0 \to \omega_{1,2}$ and $\Psi(x,y) \to \Psi(\omega_{1,2}; x, y)$.

The authors of papers [92, 93] studied theoretically propagation of polarised ultrashort pulses in a Kerr medium with resonance impurities. Apart from the two-level system, they considered the cases when the energy states of the two-level impurities can be degenerate. Only those systems of equations, which describe the transitions $j_1 = 0 \leftrightarrow j_2 = 1$ and $j_1 = 1 \to j_2 = 1$, proved to be integrable.

The equations for the normalised envelopes $\boldsymbol{e} = (e_1, e_2)$ and variables, which determine the state of the resonance impurities, can be written in the unified form:

$$\begin{aligned} &\mathrm{i}\boldsymbol{e}_{,\zeta} + s\boldsymbol{e}_{,\tau\tau} + \hat{\mu}(\boldsymbol{e}\cdot\boldsymbol{e}^*)\boldsymbol{e} = \kappa \sum_{a=1.2} \beta_{\mathrm{a}} \langle \boldsymbol{p}^{(a)} \rangle, \\ &\boldsymbol{p}^{(a)}_{,\tau} = \mathrm{i}\delta\boldsymbol{p}^{(a)} - \mathrm{i}f(\boldsymbol{e}\cdot\hat{m}^{(a)} - \boldsymbol{e}n^{(a)}), \\ &\hat{m}^{(a)}_{,\tau} = -\mathrm{i}f(\boldsymbol{e}^*\otimes\boldsymbol{p}^{(a)} - \boldsymbol{p}^{(a)*}\otimes\boldsymbol{e}), \\ &n^{(a)}_{,\tau} = -\mathrm{i}f(\boldsymbol{p}^{(a)*}\cdot\boldsymbol{e} - \boldsymbol{e}^*\cdot\boldsymbol{p}^{(a)}). \end{aligned} \tag{32}$$

The variables entering the Bloch equations in (32) were determined above (see Section 2.1).

The authors of papers [72, 92] found that both the system of equations (31) and system (32) are completely integrable but only when $L_{\mathrm{d}}L_{\mathrm{a}}^{-1} = 2f^2$. This condition means that the SIP soliton should be also a NLS soliton. In other words, the 2π-pulse amplitude and duration should be such that the dispersion broadening of the ultrashort pulse should be exactly compensated for by its compression due to self-phase modulation. The condition for the existence of SIT and NLS solitons was discussed in detail in [94] and then experimentally studied in an Er^{3+}-doped fibre [95]. At room temperature, the polarisation relaxation time T_2 is about of 0.1 ps and the dipole moment of the resonance (at a wavelength of 1.53 μm) transition is $|d_{12}| = 4.7\times10^{-3}$ D. To obtain a SIT soliton and a NLS soliton with approximately the same power, use was made of a fibre with a near-zero second-order group velocity dispersion. With the fibre cooled down to 4.2 K (in this case, $T_2 \approx 10$ ns), stable 2π- and 4π-pulses (SIT solitons) were observed.

Other examples of a completely integrable model describing propagation of an ultrashort pulse in fibre with impurities were presented in papers [96–98]. Instead of the NLSE, use was made of the Hirota-type equations to take into account the third-order group velocity dispersion, self-steepening of the pulse edge, and self-induced Raman scattering:

$$\begin{aligned} &\mathrm{i}e_{,\zeta} + (1/2)e_{,\tau\tau} + \hat{\mu}|e|^2 e \\ &\qquad + \mathrm{i}\epsilon(e_{,\tau\tau\tau} + 3|e|^2 e_{,\tau} + (3/2)e(|e|^2)_{,\tau}) + \kappa\langle p\rangle = 0, \\ &p_{,\tau} = \mathrm{i}\delta p + 2\mathrm{i}fer, \quad r_{,\tau} = \mathrm{i}f(e^*p - ep^*). \end{aligned} \tag{33}$$

It was shown that this system of equations has the Painlevé properties when some relation between the model parameters is fulfilled. The Lax representation was also found, which makes it possible to develop the IST and find its explicit soliton solution. Instead of the NLSE in (31), Makhan'kov et al. [99] considered the complex modified Korteweg–de Vries equation

$$\begin{aligned} &\mathrm{i}e_{,\zeta} - \mathrm{i}(e_{,\tau\tau\tau} + 6|e|^2 e_{,\tau} + 3e(|e|^2)_{,\tau}) + \kappa\langle p\rangle = 0, \\ &p_{,\tau} = \mathrm{i}\delta p + 2\mathrm{i}fer, \quad r_{,\tau} = \mathrm{i}f(e^*p - ep^*). \end{aligned} \tag{34}$$

They found the zero curvature representation and, using the Bäcklund transform, obtained the soliton solution of the system under study. In both cases the discussed systems of equations are completely integrable under condition that $L_{\mathrm{d}}L_{\mathrm{a}}^{-1} = 2f^2$.

Zabolotskii [100] considered propagation of an ultrashort pulse in a fibre with resonance impurities. Unlike other models, this model took into account the inertia of the Kerr nonlinearity in the fibre and correction to the refractive index, caused by a change in the populations of the resonance levels, the third-order group velocity dispersion being neglected. The model is described by the system of equations

$$\begin{aligned} &\mathrm{i}e_{,\zeta} + e_{,\tau\tau} + \hat{\mu}|e|^2 e + \mathrm{i}\epsilon(|e|^2 e)_{,\tau} - gre + \mathrm{i}\kappa p = 0, \\ &p_{,\tau} = \mathrm{i}\delta p + 2\mathrm{i}er, \quad r_{,\tau} = \mathrm{i}(e^*p - ep^*). \end{aligned} \tag{35}$$

With certain relations between the parameters in (35), this system of equations is completely integrable. For the given equations, the Lax pair and soliton solution were found.

3. Femtosecond optical solitons

Success in producing femtosecond electromagnetic-field pulses by compression or direct generation in laser systems has lead to the necessity to develop new models describing propagation of such pulses in which the slowly varying envelope approximation is not used. To this end, the model resulting in completely integrable equations and having soliton solutions would be rather attractive.

3.1 Self-induced transparency in a model of unidirectional waves

Probably, the first completely integrable model used to describe coherent propagation of ultrashort pulses in which the slowly varying envelope approximation was neglected was considered in [48, 50]. If the complete MB equations lack the soliton solutions, by assuming that the electromagnetic wave propagates only in one direction, we can obtain RMB equations (6), (7) for which there exists only one restriction on the pulse duration: it should be much shorter than the polarisation relaxation time. In the normalised form the RMB equations are expressed as

$$e_{,\zeta} = -\langle r_{1,\tau}\rangle, \quad r_{1,\tau} = -r_2,$$
$$r_{2,\tau} = r_1 + er_3, \quad r_{3,\tau} = -er_2. \tag{36}$$

Single-soliton solutions describe the half-cycle pulse – a unipolar spike of the electromagnetic field. Two-soliton (multisoliton) pulses correspond to the case of colliding single-soliton ultrashort pulses. Among the two-soliton solutions, there are breathers (corresponding to the coupled state of two solitons), which are similar to the few-cycle field-strength pulses.

3.2 High-order NLSE

To describe the nonlinear phenomena occurring with optical femtosecond pulses in nonlinear optics, the authors of [26, 101, 102] suggested using the high-order nonlinear Schrödinger equation. After passing to the normalised variables, it can be represented in the form of the NLSE with auxiliary terms:

$$\mathrm{i}e_{,\zeta} + se_{,\tau\tau} + \hat{\mu}|e|^2e + \mathrm{i}(\eta_3 e_{,\tau\tau\tau} + \mu_2|e|^2e_{,\tau}$$
$$+ \mu_3 e(|e|^2)_{,\tau}) = 0. \tag{37}$$

Here, the parameter η_3 corresponds to the third-order group velocity dispersion, while μ_2 and μ_3 correspond to two inertial contributions to the nonlinear polarisability, which are responsible for the formation of a shock wave and Raman self-scattering.

If $\eta_3 = 0$, $\mu_2 = \mu_3 = 1$ and $\hat{\mu} = 0$, the high-order NLSE (37) is reduced to the derivative nonlinear Schrödinger equation

$$\mathrm{i}e_{,\zeta} + se_{,\tau\tau} + \mathrm{i}(|e|^2e)_{,\tau} = 0. \tag{38}$$

This equation is completely integrable [103]. Using the IST, we can find both its soliton and multisoliton solutions. Note that the condition $\hat{\mu} = 0$ is insignificant for reducing the high-order NLSE to the completely integrable equation, and the resulant modified derivative NLSE

$$\mathrm{i}e_{,\zeta} + se_{,\tau\tau} + \hat{\mu}|e|^2e + \mathrm{i}(|e|^2e)_{,\tau} = 0 \tag{39}$$

remains a completely integrable equation [104].

If $\eta_3 = 1$, $\mu_2 = \pm 6$, $\mu_3 = 0$, equation (37) is reduced to the Hirota equation [105]

$$\mathrm{i}e_{,\zeta} + se_{,\tau\tau} + \hat{\mu}|e|^2e + ie_{,\tau\tau\tau} \pm 6\mathrm{i}|e|^2e_{,\tau} = 0, \tag{40}$$

which represents another example of a completely integrable equation.

In the general case, equation (37) is obviously not completely integrable. But if we assume that $\eta_3 = \epsilon_1$, $\mu_2 = \pm 6\epsilon_1$, $\mu_3 = 3\epsilon_1$, and $\hat{\mu} = 2s$, the high-order NLSE is reduced to a new integrable equation. In this case, the change of variables in (37) yields

$$u_{,\zeta} + \epsilon_1(u_{,\xi\xi\xi} + 6|u|^2u_{,\xi} + 3u|u|^2_{,\xi}) = 0. \tag{41}$$

Here, the electromagnetic field is related with the solution of equation (41) by the expression

$$e(\zeta,\tau) = u(\zeta,\xi)\exp\left(\frac{\mathrm{i}s\tau}{3\epsilon_1} + \frac{2\mathrm{i}s^3\zeta}{27\epsilon_1^2}\right).$$

This equation was derived by Sasa et al. [106], who showed that its solution can be obtained with the help of the IST. Unlike the NLS solitons, the solitons of the Sasa – Satsuma equation (41) can have two humps and change polarity, similarly to the SIT breathers. Mihalache et al. [107] found the one-parameter family of solitons, four-parameter family of breathers and the general N-soliton solution of the Sasa – Satsuma equation.

3.3 Alternative to the NLSE

Schäfer et al. [108] proposed a model describing propagation of few-cycle (up to one oscillation period) electromagnetic-field pulses in a nonresonance dielectric medium. The authors of [108] used a scalar wave equation describing propagation of a plane wave along the z axis:

$$E_{,zz} - c^{-2}E_{,tt} = 4\pi c^{-2}P_{1,tt}.$$

Polarisation P is divided into linear and nonlinear parts: $P = P_{\mathrm{lin}} + P_{\mathrm{nl}}$. In the general case, the linear part of polarisation is represented by an integral:

$$P_{\mathrm{lin}}(z,t) = \int_{-\infty}^{t} \chi^{(1)}(t-t')E(z,t')\mathrm{d}t'.$$

The integrand response function $\chi^{(1)}(t-t')$ – the linear susceptibility in the wavelength region between 1.6 and 3 μm – is approximated by the expression whose Fourier spectrum (in wavelengths) is

$$\chi^{(1)}(\lambda) = \chi_0^{(1)} + \chi_2^{(1)}\lambda^2,$$

where $\chi_0^{(1)} = 1.11$ μm^{-2}; $\chi_2^{(1)} = 0.0106$ μm^{-2}. As for the nonlinear part of polarisation, use is made of the expression $P_{\mathrm{nl}} = \chi^{(3)}E^3$, which means that only the cubic nonlinearity is taken into account, while the nonlinear susceptibility dispersion is absent (instantaneous response). These assumptions allow one to rewrite the initial wave equation in the form:

$$E_{,zz} - \upsilon^{-2}E_{,tt} - 4\pi^2\chi_2^{(1)}E = 4\pi\chi^{(3)}c^{-2}E^3_{,tt}, \tag{42}$$

where υ is the phase velocity (as in an ordinary dielectric).

The linear part of this equation coincides with the Klein–Gordon equation, which has solutions describing the waves travelling from left to right and from right to left. The next approximation is to be restricted to waves travelling only into one direction. The wave equation (42) can be reduced by using the multiscale perturbation method; the details of this procedure can be found in [108]. The resultant equation, called the Schäfer–Wayne short-pulse equation (SWSPE), has the form

$$E_{,zt} + 2\pi^2\chi_2^{(1)}E + 2\pi\chi^{(3)}c^{-2}E^3_{,tt} = 0. \tag{43}$$

If we appropriately substitute the variable and re-designate the independent variables, this equation can be written in a more elaborate form:

$$e_{\tau\xi} = e + (e^3/6)_{,\xi\xi}. \tag{44}$$

In this form the SWSPE was studied in papers [109, 110], which showed that it is completely integrable. The same result was achieved by the authors [111, 112], who additionally found the soliton solutions and Hamiltonian structure of the SWSPE. Using the IST, Victor et al. [110] derived an expression for the multisoliton solution of the SWSPE.

The numerical solution of the complete wave equation and comparison of its results with the results of the SWSPE solution showed that for few-cycle pulses the agreement between the obtained data and those that could be collected if the NLSE were solved instead of the SWSPE is much better.

The attempt to generalise this equation for the case of two-component (polarised) electromagnetic radiation [113] did not lead to equations having a physical sense.

3.4 Propagation of extremely short pulses

When we consider a resonant medium where all the relaxation processes are discarded as considerably slower (compared to the duration of an electromagnetic-field pulse), the resonance transition frequency ω_a yields the only time scale*. If the pulse duration t_p meets the condition $t_p\omega_a \gg 1$, its evolution can be described by using the quasi-harmonic wave representation (or, which is the same, considering slowly varying envelopes). Otherwise, if $t_p\omega_a \leqslant 1$, one should use complete Maxwell equations or – under certain conditions – the unidirectional wave approximation. The ratio $\varepsilon = \omega_R/\omega_a$, where ω_R is the instantaneous Rabi frequency [i.e., $(d/\hbar)\max|E|$], yields a new parameter. Let the amplitude of ultrashort pulses be such that the frequency ω_R is small compared to the minimal frequency of the resonance transition. Thus, we can use ε as a small parameter for solving Bloch equations (5) in the perturbation theory and substitute the polarisation medium obtained in this way into the wave equation, without using the approximation of slowly varying envelopes of ultrashort pulses [114]. In the unidirectional wave approximation the wave equation is written in the form

$$E_{,z} + c^{-1}E_{,t} = -(2\pi n_a d/c)\langle r_{1,t}\rangle, \tag{45}$$

where polarisation of the ensemble of two-level atoms with an accuracy to the third-order of smallness in ε is written in the form

$$\langle r_{1,t}\rangle = 2\langle d/\hbar\omega_a\rangle E - 2\langle d/\hbar\omega_a^3\rangle E_{,tt}$$
$$- 4\langle |d|^2 d/\hbar^3\omega_a^3\rangle E^3. \tag{46}$$

Substituting (46) into equation (45) and changing the variables (see details in [114]) lead to the modified Korteweg–de Vries equation known in the soliton theory [31]

$$e_{,\tau} + 6e^2 e_{,\xi} + e_{,\xi\xi\xi} = 0. \tag{47}$$

As is known [115], the Korteweg–de Vries equation is completely integrable and its soliton solutions are obtained by using the IST. The single-soliton solutions correspond to propagation of electromagnetic spikes (extremely short pulses, video pulses), and breathers – to few-cycle pulses (Fig. 8).

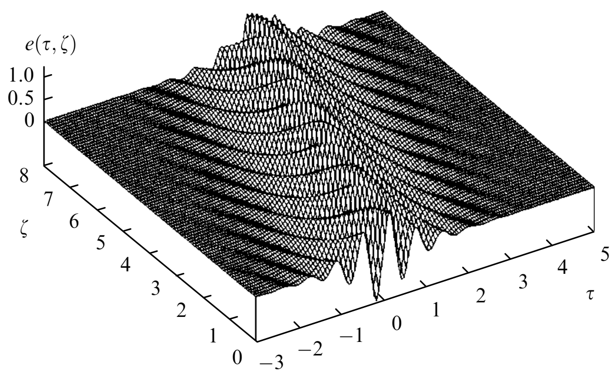

Figure 8. Modified Korteweg–de Vries breather – two-soliton solution of the modified Korteweg–de Vries equation (looks like an electromagentic few-cycle pulse).

Paper [116] considered the cases when the description can and should be performed without the slowly varying envelope approximation. As examples, Korteweg–de Vries (in a quadratic medium), modified Korteweg–de Vries (in a cubic medium) and sine-Gordon (in a resonant medium) equations were considered. In paper [117], comparing the results of description of few-cycle pulse propagation with the help of the modified Korteweg–de Vries, sine-Gordon equations, SWSPE and some model equations, the authors came to the conclusion that the most suitable is the combined modified Korteweg–de Vries–sine-Gordon equation [118, 119]:

$$e_{,\zeta} + c_1\sin\vartheta + 3c_2e^2e_{,\tau} + c_3e_{,\tau\tau\tau} = 0, \quad \vartheta_{,\tau} = e. \tag{48}$$

*Recall that in passing from microscopic electrodynamic equation to macroscopic ones, the fields were averaged over a physically small volume. This procedure determines the applicability limits of macroscopic Maxwell equations.

Under the condition $c_3 = 2c_2$, this equation is completely integrable [118] and its soliton (more precisely breather) solutions well reproduce the evolution of few-cycle pulses, obtained in numerical calculations.

The authors of [120–125] derived other equations to describe propagation of extremely short pulses. The review of some of them can be found in [126]. However, they are not completely integrable and have no soliton solutions. Thus, for example, propagation of a polarised ultrashort pulse in a resonant medium can be described by a two-component generalised modified Korteweg–de Vries equation, which is reduced to a completely integrable equation only in the case of fixed pulse polarisation. At the same time, numerical simulation of propagation and interaction of solitary waves with the help of the above equations shows their stability or weak attenuation. Collision of such pulses is accompanied by weak emission of linear dispersive waves. All this makes them similar to solitons.

3.5 Self-induced transparency in a two-level medium with a permanent dipole moment

Usually, the models of resonance interaction of electromagnetic pulses with matter took into account only those state of atoms and molecules for which the dipole moment operator does not have diagonal elements. However, there exist media (polar molecules, asymmetric quantum dots, quantum wells) for which the dipole moment operator has diagonal elements [127–129]. Propagation of short and extremely short electromagnetic pulses in such media was studied in [130–140]. The authors of these papers considered a plane electromagnetic wave propagating in a medium from atoms and molecules whose transition from the ground state to the excited is characterised by the dipole moment operator with both nonzero nondiagonal (d_{12}, d_{21}) and diagonal (d_{11}, d_{22}) matrix elements. Because of the diagonal matrix elements of the dipole moment operator, transitions of atoms and molecules have constant polarisability. It is believed that the resonance system has a permanent dipole moment. If the dipole–dipole interaction is smaller than the thermal fluctuation energy and the external constant electric field is absent, the macroscopic polarisation of the medium is zero. All the papers mentioned in this Section considered only paraelectrics. In the case of extremely short pulses, all the relaxation processes in the system of atoms can be neglected.

The assumption that the waves propagate only in one direction allows one to reduce the complete MB equations to a simpler system of equations, i.e., RMB equations (6), (7), which take into account diagonal matrix elements of the dipole transition moment operator. For the case under study the RMB equations have the form [126]:

$$E_{,z} + c^{-1}E_{,t} = -\frac{\pi n_{\rm a}}{c}\langle (d_{22} - d_{11})r_{3,t} + 2d_{12}r_{1,t}\rangle, \tag{49}$$

$$\begin{aligned} r_{1,t} &= -\left[\omega_{\rm a} + \frac{(d_{11} - d_{22})E}{\hbar}\right]r_2, \\ r_{2,t} &= \left[\omega_{\rm a} + \frac{(d_{11} - d_{22})E}{\hbar}\right]r_1 + \frac{2d}{\hbar}Er_3, \\ r_{3,t} &= -\frac{2d}{\hbar}Er_2. \end{aligned} \tag{50}$$

These equations differ from (6), (7) by auxiliary terms proportional to $d_{11} - d_{22}$.

Equations (49), (50) can be rewritten in the normalised form as was done previously for (9) and (10) so that to obtain a normalised system of equations

$$e_{,\zeta} = -\langle r_{1,\tau} - \mu r_{3,\tau}\rangle, \tag{51}$$

$$\begin{aligned} &r_{1,\tau} = -(1 + \mu e)r_2, \\ &r_{2,\tau} = (1 + \mu e)r_1 + er_3, \quad r_{3,\tau} = -er_2, \end{aligned} \tag{52}$$

where $\mu = (d_{11} - d_{22})/2d_{12}$ is the measure of influence of the permanent dipole moment on the resonance system response to an electromagnetic field. Equation (51) can be expressed as $e_{,\zeta} = \langle r_2\rangle$.

Agrotis et al. [131] found in the absence of inhomogeneous broadening of the resonance line that the system of equations (51), (52) admits the Lax representation. Caputo et al. [132] considered another (gauge equivalent) Lax representation. The Hamiltonian structure of equations (51) and (52) was determined in [135]. Account for the inhomogeneous broadening of the resonance line preserves the property of complete integrability of this system of equations [136].

Using the method of Darboux transforms, the authors of [137–139] obtained soliton solutions of the system of equations (51), (52), in particular, the breather solution whose appearance and evolution was studied numerically in [140]. Analytic expressions for two-soliton solutions of equations (51) and (52), including solutions for breathers, were previously derived in [131].

If we set the parameter μ equal to zero in equations (51) and (52), we can obtain a system of RMB equations, having soliton solutions which can describe both the few-cycle pulses (breathers) and unipolar pulses – electromagnetic field spike (extremely short pulse) [39, 42]. The solitons of the RMB equations have an 'area'

$$\vartheta(\zeta) = \int_{-\infty}^{\infty} e(\zeta, \tau)\mathrm{d}\tau,$$

multiple of 2π; this means that under the action of such a pulse, atoms undergo transition from the ground state to the excited state and vice versa such that the soliton propagation turned the state of the resonant medium into initial. The multiplicity of the pulse 'area' to 2π depends on the number of changes in the populations of ground and excited states. The breather has an 'area' equal to zero; therefore, it is called a 0π-pulse. When μ is nonzero, the breather area is also nonzero and its shape is asymmetric, which was observed in calculations [140]. In addition, its group velocity can be much smaller than the velocity of light in a medium.

4. Multicomponent solitons

4.1 Vector optical solitons

In the general case, the vector soliton corresponds to the soliton solution of the system of nonlinear equations, which can be represented as a row matrix. Thus, for example, the vector NLS soliton is the solution of the multicomponent (vector) NLSE

$$ie_{,\zeta} + se_{,\tau\tau} + \hat{\mu}(e \cdot e^*)e = 0, \tag{53}$$

where $e = (e_1, e_2, \ldots, e_n)$. The subscripts of the vector components can have here different physical sense, for example, the frequency (for a polychromatic pulse) or projection of the polarisation vector of the polarised electromagnetic pulse.

Equation (53) is an example of a completely integrable equation that is used in nonlinear optics. It belongs to the hierarchy of AKNS equations which can be solved by using the IST together with the Manakov spectral problem [15]. The multisoliton solution for the n-component NLSE was obtained in [99, 141].

As a simple vector generalisation of the derivative NLSE, we can use the equation

$$\mathrm{i}e_{,\zeta} + e_{,\tau\tau} - \mathrm{i}[(e \cdot e^*)e]_{,\tau} = 0. \tag{54}$$

In [142] the IST was applied to the vector (two-component) derivative NLSE. The results were used to study the modulation instability of circularly polarised long Alfvén waves.

The authors of [143] found that the completely integrable equation

$$\mathrm{i}e_{,\zeta} + e_{,\tau\tau} \pm \mathrm{i}(e \cdot e^*)e_{,\tau} = 0, \tag{55}$$

taking into account a nonlinear change in the group velocity, has soliton solutions.

Another example of the vector nonlinear waves emerges while considering propagation of a femtosecond optical pulse in an optical fibre, taking into account birefringence, highest orders of the group velocity dispersion, and nonlinear susceptibility. Quite integrable in this case is the two-component Sasa – Satsuma equation [144], which generalises equation (41):

$$u_{,\zeta} + \epsilon[u_{,\xi\xi\xi} + 6(u \cdot u^*)u_{,\xi} + 3u(u \cdot u^*)_{,\xi}] = 0. \tag{56}$$

The authors of [144] considered also the three-component Sasa – Satsuma equation. They found that this equation has the Lax representation and can be solved using the IST. However, the exact soliton solution can be simply obtained by using the method of Darboux – Bäcklund transforms.

4.2 Self-induced transparency in a three-level medium for polarised pulses

A particular case of interaction of polarised radiation with a resonant medium was considered in [145]. It was assumed that a two-frequency ultrashort pulse propagates in a three-level medium in the case when one of the V- or Λ-configuration levels is not degenerate, while other two levels are triply degenerate with respect to projections of the angular momentum. Or, vice verse, two levels are degenerate (triplets), while the adjacent level is not degenerate (singlet).

4.2.1 The $j_1 = 0 \to j_2 = 1 \to j_3 = 0$ transition

The electric-field components of an ultrashort pulse can produce a 2×2 matrix $\hat{e} = (e_k^j)$, where the subscript denotes the 'colour' – the carrier-wave frequency ω_1 or ω_2, and the superscript – circular polarisation. For left-polarised radiation we have $j = -1$, while for right-polarised radiation – $j = +1$. The elements of the transition matrix are collected in the matrices $\hat{p}$, $\hat{n}$, and $\hat{m}$ according to the following rules:

for V-transition $(j, l = \pm 1)$

$$p_1^j = \langle j_1, 0|\hat{\rho}|j_2, j\rangle,\ p_2^j = \langle j_3, 0|\hat{\rho}|j_2, j\rangle,\ n_1^1 = \langle j_1, 0|\hat{\rho}|j_2, 0\rangle,$$

$$n_2^2 = \langle j_3, 0|\hat{\rho}|j_3, 0\rangle,\ m_{jl} = \langle j_2, j|\hat{\rho}|j_2, l\rangle,\ n_1^2 = \langle j_1, 0|\hat{\rho}|j_3, 0\rangle = n_1^{1*};$$

for Λ-transition $(k, l = \pm 1)$

$$p_1^k = \langle j_2, -k|\hat{\rho}|j_1, 0\rangle,\ p_2^k = \langle j_3, -k|\hat{\rho}|j_2, 0\rangle,$$

$$n_1^1 = -\langle j_1, 0|\hat{\rho}|j_2, 0\rangle,\ n_2^2 = -\langle j_3, 0|\hat{\rho}|j_3, 0\rangle,$$

$$m_{kl} = -\langle j_2, -l|\hat{\rho}|j_2, -k\rangle,\ n_1^2 = -\langle j_1, 0|\hat{\rho}|j_3, 0\rangle = n_2^{1*}.$$

The RMB equations (in the slowly varying envelope approximation) corresponding to this case of transition in a three-level medium are written as a system of matrix equations:

$$\begin{aligned}
&\hat{e}_{,\zeta} = -\mathrm{i}\langle\hat{p}\rangle,\\
&\hat{p}_{,\tau} = \mathrm{i}\delta\hat{p} + \mathrm{i}(\hat{n}\hat{e} - \hat{e}\hat{m}),\\
&\hat{m}_{,\tau} = \mathrm{i}(\hat{p}^+\hat{e} - \hat{e}^+\hat{p}),\\
&\hat{n}_{,\tau} = \mathrm{i}(\hat{p}\hat{e}^+ - \hat{e}\hat{p}^+).
\end{aligned} \tag{57}$$

This system of equations has the Lax representation and is solved with the help of the IST. The single-soliton solutions, the Bäcklund transform (which made it possible to construct soliton solutions of an arbitrary multiplicity), and an infinite sequence of the conservation laws were found for this system. As in the case of a polarised 2π-pulse, the normalised envelope of a two-frequency polarised soliton – solution of the system of equations (57) – is found from the expression

$$\hat{e}(\tau, \zeta) = l^{(\mathrm{p})} \otimes l^{(\mathrm{c})} 2\eta\,\mathrm{sech}[2\eta(\tau - \tau_0) - b\zeta]$$

$$\times \exp(-2\alpha\tau + \mathrm{i}\kappa\zeta),$$

where $l^{(\mathrm{p})}$ is the unit vector in the space of polarisations and $l^{(\mathrm{c})}$ is the unit vector in the space of frequencies (colour). Collision of two such solitons leads to the rotation of these vectors. The expressions determining the rotation of the polarisation and colour vectors are similar to (21). The vectors $l^{(\mathrm{p})}$ rotate independently of the vectors $l^{(\mathrm{c})}$. Thus, when the solitons collide, polarisation changes and the energy between the components of two-frequency polarised solitons is redistributed.

4.2.2 The $j_1 = 1 \to j_2 = 1 \to j_3 = 0$ transition

In this case, the elements of the transition matrix are collected in the scalar n and in the matrices $\hat{p}$, $\hat{r}$, $\hat{k}$, $\hat{m}$ according to the following rules:

for the V-transition

$$p_1^k = \langle j_1, -k|\hat{\rho}|j_2, j\rangle, \quad p_2^k = \langle j_3, -k|\hat{\rho}|j_2, 0\rangle,$$

$$n = -\langle j_2, 0|\hat{\rho}|j_2, 0\rangle, \quad m_{kl} = -\langle j_1, -l|\hat{\rho}|j_1, -k\rangle,$$

$$k_{kl} = -\langle j_3, -l|\hat{\rho}|j_3, -k\rangle, \quad r_{kl} = -\langle j_1, -l|\hat{\rho}|j_3, -k\rangle;$$

for the Λ-transition

$$p_1^k = \langle j_2, 0|\hat{\rho}|j_1, k\rangle, \; p_2^k = \langle j_2, 0|\hat{\rho}|j_3, k\rangle, \; n = \langle j_2, 0|\hat{\rho}|j_2, 0\rangle,$$

$$m_{kl} = \langle j_1, k|\hat{\rho}|j_1, l\rangle, \; k_{kl} = \langle j_3, k|\hat{\rho}|j_3, l\rangle, \; r_{kl} = -\langle j_3, l|\hat{\rho}|j_3, k\rangle.$$

It is convenient to form here a four-component electric-field vector of the ultrashort pulse $\boldsymbol{e} = (e_1^{-1}, e_1^{+1}, e_2^{-1}, e_2^{+1})$, polarisation matrix $\boldsymbol{p} = (p_1^{-1}, p_1^{+1}, p_2^{-1}, p_2^{+1})$, and 4×4 matrix $\hat{M}$ constructed from the blocks:

$$\hat{M} = \begin{pmatrix} \hat{m} & \hat{r} \\ \hat{r}^{+} & \hat{k} \end{pmatrix}.$$

Using these notations, the RMB equation can be written in the form

$$\begin{aligned} &\boldsymbol{e}_{,\zeta} = -\mathrm{i}\langle \boldsymbol{p}\rangle, \\ &\boldsymbol{p}_{,\tau} = \mathrm{i}\delta\boldsymbol{p} - \mathrm{i}\boldsymbol{e}\cdot M + \mathrm{i}\boldsymbol{e}n, \\ &\hat{M}_{,\tau} = -\mathrm{i}(\boldsymbol{e}^{*}\otimes\boldsymbol{p} - \boldsymbol{p}^{*}\otimes\boldsymbol{e}), \\ &n_{,\tau} = -\mathrm{i}(\boldsymbol{p}^{*}\cdot\boldsymbol{e} - \boldsymbol{e}^{*}\cdot\boldsymbol{p}). \end{aligned} \tag{58}$$

The derived system of equations resembles system (20). Thus its solution can be easily obtained by using the IST. As in the previous case, solitons were found and their interaction character was determined.

4.3 Self-induced transparency in an anisotropic/isotropic two-level medium with a permanent dipole moment

The anisotropy of the dipole moment operator (it can be related to the crystal structure of the matrix containing resonance atoms) and the effect of the permanent dipole moment are taken into account in the Hamiltonian of the system of two-level atoms by introducing the following term – interaction Hamiltonian:

$$\hat{H}_{\mathrm{int}} = -\hat{\sigma}_{+}(d_{zx}^{(1)}E_x + d_{zy}^{(1)}E_y) + \hat{\sigma}_{-}(d_{zx}^{(2)}E_x + d_{zy}^{(2)}E_y)$$
$$+\hat{\sigma}_1(d_{xx}E_x + d_{xy}E_y) + \hat{\sigma}_2(d_{yx}E_x + d_{yy}E_y),$$

where use is made of matrices $\hat{\sigma}_{\pm} = (1 \pm \hat{\sigma}_3)$ and ordinary Pauli matrix $\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3$. The transverse components of the electric-field vector are designated as E_x and E_y; $d_{zx}^{(a)}, d_{zy}^{(a)}$ are the diagonal matrix elements of the dipole moment operator in the ground ($a = 1$) and excited ($a = 2$) states; d_{jk} are the real tensor components of the dipole moment of the transition between the states $a = 1$ and $a = 2$ [146 – 148].

In normalised variables the system of generalised RMB equations has the form:

$$e_{x,\zeta} = (\kappa_x r_1 + \kappa_y r_2 - r_3)e_y - r_2,$$

$$e_{y,\zeta} = -(\kappa_x r_1 + \kappa_y r_2 - r_3)e_x + \rho r_1,$$

$$r_{1,\tau} = (1 - \kappa_x e_x - \kappa_y e_y)r_2 + e_y r_3, \tag{59}$$

$$r_{2,\tau} = -(1 - \kappa_x e_x - \kappa_y e_y)r_1 - e_x r_3,$$

$$r_{3,\tau} = e_x r_2 - e_y r_2.$$

Here, e_x and e_y are the normalised projections to the corresponding electric-field axes of the pulse; κ_x, κ_y, and ρ are the parameters, expressed via the matrix elements of the permanent dipole moment and real tensor components of the dipole transition moment; r_1, r_2, r_3 are the components of the Bloch vector.

Using the IST, the author of [146] found the solution for the isotropic limit ($\rho = 1$) of the general integrable model of interaction between a two-component electric field and two-level atoms. A more complicated problem corresponding to the anisotropic case ($\rho \neq 1$) was solved in [147, 148]. In this case, the IST is based on the spectral problem of a new type, which was previously not encountered in the soliton theory. Recent paper [149] presents a detailed review of the IST, which is used to solve exactly the system of the RMB equations, taking into account the field polarisation and the permanent dipole moment, and gives explicit expressions for their soliton solutions.

4.4 Coherent propagation of polarised extremely short pulses

Propagation of few-cycle pulses in a resonant medium of two-level atoms was described in the unidirectional wave approximation by a completely integrable model based on scalar RMB equations (6) and (7). Zabolotskii [150] took into account the electromagnetic-field polarisation, which preserves the integrable properties. He considered the interaction of extremely short pulses with a two-level medium consisting of atoms with the energy σ-transition ($j = 1/2, m = -1/2 \leftrightarrow j = 1/2, m = 1/2$, where j and m are the total angular momentum and its projection to the quantization axis, respectively). In this case, the matrix element of the dipole moment is written in the form $\boldsymbol{d} = d_x\hat{\boldsymbol{x}} - d_y\hat{\boldsymbol{y}}$, where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ are the unit vectors of the Cartesian coordinate system, which determine the axes x and y, orthogonal to the z axis of the wave propagation. The equations for the electromagnetic pulse components have the form

$$E_{x,zz} - c^{-2}E_{x,tt} = 4\pi c^{-2} n_{\mathrm{a}} d_x r_{1,tt},$$

$$E_{y,zz} - c^{-2}E_{y,tt} = 4\pi c^{-2} n_{\mathrm{a}} d_y r_{2,tt},$$

and the Bloch equations have the form

$$r_{1,t} = -\omega_0 r_2 + (d_y/\hbar)E_y r_3,$$

$$r_{2,t} = \omega_0 r_1 - (d_y/\hbar)E_x r_3,$$

$$r_{3,t} = (d_x/\hbar)E_x r_2 - (d_y/\hbar)E_y r_1.$$

Using the unidirectional wave approximation and introducing the normalised dependent (e_1, e_2) and independent (ζ and τ) variables, the system of the vector RMB equations can be written in the form [150 – 152]

$$e_{1,\zeta} = r_{1,\tau}, \quad e_{2,\zeta} = \rho^2 r_{2,\tau},$$
$$r_{1,\tau} = -r_2 + e_2 r_3, \quad r_{2,\tau} = r_1 - e_1 r_3, \tag{60}$$
$$r_{3,\tau} = e_1 r_2 - e_2 r_1,$$

where $\rho = d_y/d_x$. If we make a substitution $(r_1, r_2, r_3) \to (-r_1, -r_2, r_3)$ and set $e_2 = 0$, system (60) transforms into the RMB equations (6) and (7).

The isotropic case ($d_y = d_x$) considered in [152] yields a system of the vector RMB equations, which can be integrated with the help of the IST based on the Kaup–Newell spectral problem [103]. This problem was used to solve the derivative NLSE (38). The soliton solutions can be found in [103]. Steudel et al. [152] obtained the multisoliton solutions of the system of the vector RMB equations by using the Darboux and Bäcklund transforms. These solutions have breathers, which describe few-cycle electromagnetic-field pulses:

$$e_{\mathrm{br}}(\zeta,\tau) = -2\mathrm{i}\kappa \sin 2\phi \, \frac{\cosh[\tau \sin 2\phi - (\zeta/2)\cot\phi + \mathrm{i}\phi]}{\cosh^2[\tau\cos 2\phi + (\zeta/2) + \mathrm{i}\phi]}$$
$$\times \exp(\mathrm{i}t\cos 2\phi + \mathrm{i}\zeta/2), \tag{61}$$

where κ and ϕ are two constants depending on the initial condition.

If we treat the ratio d_y/d_x as arbitrary, the system of equations (60) can be still solved using the IST, but the spectral problem should be different. This new spectral problem and the zero curvature representation for system (60) were found in [150]. The author of [150] obtained the soliton solution but because of its awkwardness we do not present it here. The multisoliton solutions, including breathers, were later obtained by Steudel et al. [153], who, using the results of [150] constructed the Darboux and Bäcklund transforms for (60) in the general case. Nontrivial symmetry groups of the corresponding zero curvature representation were studied in [146, 154]. The same papers showed that the solutions of the vector RMB equations are expressed by the solutions of the matrix Riemann–Hilbert problem. Many results associated with the description of propagation of extremely short electromagnetic pulses in a two-level medium are collected in review [149].

4.5 Optical domain walls

Investigation of polarisation state dynamics of two optical pulses or beams counterpropagating in a nonlinear Kerr medium resulted in formulation of the models predicting formation of domain walls separating the regions with different polarisations.

Thus, Tratnik et al. considered counterpropagation of optical pulses in an isotropic cubic nonlinear medium [155]. The waves propagate in the directions $\hat{z}$ and $-\hat{z}$ along the z axis and are characterised by slowly varying envelopes $\boldsymbol{e}^{(\pm)} = e_x^{\pm}\hat{\boldsymbol{x}} + e_y^{\pm}\hat{\boldsymbol{y}}$. The system of truncated equations for the envelopes has the form:

$$e^{(\pm)}_{j,\tau} \pm e^{(\pm)}_{j,\zeta} = \mathrm{i}g_{\mathrm{int}}\{[(\boldsymbol{e}^{(+)*}\cdot\boldsymbol{e}^{(+)}) + (\boldsymbol{e}^{(-)*}\cdot\boldsymbol{e}^{(-)})]e^{(\pm)}_j$$
$$+ (\boldsymbol{e}^{(\mp)*}\cdot\boldsymbol{e}^{(\pm)})e^{(\mp)}_j\}, \tag{62}$$

where $\zeta = k_0 z$, $\tau = k_0 v_{g_{\mathrm{int}}} t$ are the normalised coordinate and time; k_0 is the wavenumber of the carrier wave; $v_{g_{\mathrm{int}}}$ is the group velocity (the group velocity dispersion was neglected). The interaction constant g_{int} is determined by the linear $[\chi^{(1)}]$ and nonlinear $[\chi^{(3)}]$ susceptibilities: $g_{\mathrm{int}} = 2\pi\chi^{(3)}/(1+4\pi\chi^{(1)})$.

The polarisation state is often described by the Stokes vector $\boldsymbol{S}$. In the present problem two such vectors are used:

$$\boldsymbol{S}^{(\pm)} = (e_x^{(\pm)*}e_y^{(\pm)} + \text{c.c.},$$
$$\mathrm{i}e_x^{(\pm)*}e_y^{(\pm)} + \text{c.c.}, \ |e_x^{(\pm)}|^2 - |e_y^{(\pm)}|^2).$$

The moduli of these vectors, $|\boldsymbol{S}^{(\pm)}|^2 = (\boldsymbol{e}^{(\pm)}\cdot\boldsymbol{e}^{(\pm)*})$, remain constant so that the polarisation state is determined only by the orientation of the Stokes vectors.

By using the new independent variables $2\eta = \zeta - \tau$ and $2\xi = -(\zeta + \tau)$ and the system of equations (62), the authors of [155] obtained two coupled Bloch equations describing the rotation of the introduced Stokes vectors around each other:

$$\partial_\xi \boldsymbol{S}^{(+)} = g_{\mathrm{int}}\boldsymbol{S}^{(-)} \times \boldsymbol{S}^{(+)},$$
$$\partial_\eta \boldsymbol{S}^{(-)} = -g_{\mathrm{int}}\boldsymbol{S}^{(-)} \times \boldsymbol{S}^{(+)}. \tag{63}$$

The authors of papers [156, 157] considered the birefringent medium with a cubic nonlinearity. The equations for the Stokes vectors are more cumbersome but, by neglecting the Kerr self-action compared to their cross action, these equations can be simplified and reduced to completely integrable equations of motion of an anisotropic chiral field in the O(3) group [158, 159]:

$$\partial_\xi \boldsymbol{S}^{(+)} = \boldsymbol{S}^{(+)} \times \hat{J}\boldsymbol{S}^{(-)},$$
$$\partial_\eta \boldsymbol{S}^{(-)} = \boldsymbol{S}^{(-)} \times \hat{J}\boldsymbol{S}^{(+)}, \tag{64}$$

where $\hat{J}$ is the diagonal matrix determined by the nonlinear medium susceptibilities. When the matrix elements $\hat{J}$ are different, solutions (64) determine stable polarisation states in the case of parallel or orthogonal vectors of counterpropagating waves.

The models under study resemble the model of an isotropic (63) or anisotropic (64) ferromagnetic. It is known that ferromagnetic have domain walls describing the transition from one stable state of magnetic moments to another. In optics the domain wall corresponds to the region where the polarisation state of the interacting waves changes, for example, the left-polarised waves turn into the right-polarised ones.

5. Deformed solitons

In the conventional IST, the spectral parameter corresponding to the linear problem on eigenvalues is assumed constant. The potentials entering the spectral IST problem change (deform) according to the system of integrable equations and because the spectral parameter is constant the given integrable equations prescribes isospectral deformation. If the spectral parameter is not constant, we can derive equations which can be solved by the inverse scattering transform; however, the soliton solutions in these equations do not preserve their shape during propagation.

Some problems of nonlinear fibre optics make use of the models based on the NLSE with variable coefficients. For example, propagation of optical solitons in a dispersion- and loss-compensated fibre. In the general case, use is made of the equations, which are not completely integrable and do not have soliton solutions. However, there are cases when a change in the variables allows one to reduce a nonautonomous NLSE to completely integrable equation [160, 161].

5.1 Deformed Maxwell – Bloch equations

Burtsev et al. [162] suggested generalising the IST in which the spectral parameter depends on the coordinate and time. The nonlinear evolution equations solved by this method are sometimes called deformed (nonisospectral) integrable equations. We will call the corresponding soliton solutions the deformed solitons.

In nonlinear optics an example of such a system of equations was found. This system represents generalised MB equations describing the ultrashort-pulse evolution in a two-level medium where the excited atomic state is steadily populated, while the ground states is depleted so that the difference between the populations changes at a constant velocity. The deformed RMB equations in this case have the form [162]:

$$e_{,\zeta} = p, \quad p_{,\tau} = er, \quad r_{,\tau} = -\frac{1}{2}(e^* p + ep^*) - 4\bar{c}. \tag{65}$$

The parameter $\bar{c}$ describes pumping in a two-level medium.

Another example of deformed RMB equations presented in [163] is given by the equations

$$e_{,\zeta} = p, \quad p_{,\tau} + \bar{c}p = er, \quad r_{,\tau} + \bar{c}r = -\frac{1}{2}(\mathrm{e}^* p + ep^*). \tag{66}$$

This system assumes that the population difference and polarisation in a two-level medium relax at the save rate.

Paper [163] also considers a system of equations

$$\begin{aligned} &\mathrm{i}e_{,\zeta} + \frac{1}{2}e_{,\tau\tau} + \frac{1}{4}|e|^2 e = \mathrm{i}p, \\ &p_{,\tau} = er, \quad r_{,\tau} + \frac{1}{2}(e^* p + ep^*) = -4\bar{c}, \end{aligned} \tag{67}$$

describing propagation of ultrashort pulses in an optical fibre with resonance impurities in the case of exact resonance and cw pump. Porsezian et al. [164] determined the condition under which (65) and (67) have the Painlevé property. The two-soliton solution of equations (65) and (67) was given in [165], which made it possible to find the Bäcklund transform allowing the multisoliton solutions to be constructed from single-soliton ones.

Zabolotskii [166] considered the evolution of extremely short pulses within the framework of two integrable systems of generalised RMB equations. The first model describes interaction of the field with a two-level nondegenerate medium, taking into account the permanent dipole moment and external cw pump. The system of the equations generalising (51) and (52) has the form:

$$\begin{aligned} &e_{,\zeta} = r_2, \\ &r_{1,\tau} = -(1 + \mu e)r_2 + \bar{b}(\zeta), \\ &r_{2,\tau} = (1 + \mu e)r_1 + er_3, \\ &r_{3,\tau} = -er_2 + \bar{c}(\zeta), \end{aligned} \tag{68}$$

where the parameter $\bar{b}$ described induced polarisation of the medium. Zabolotskii showed that this system is integrated with the help of the IST, and presented a general expression for the N-soliton solution. By the example of particular solutions he studied the effect of the permanent dipole moment and pump on the soliton dynamics.

The second model describes interaction of pulses of a two-component electric field with a two-level degenerate medium (the ground state is a triple and the excited state is a singlet) under cw pumping of the upper level and depletion of lower levels. The system of equations of this model is expressed as

$$\begin{aligned} &e_{1,\zeta} = -p_1, \quad e_{2,\zeta} = -p_2, \\ &p_{1,\tau} = -(1 + \mu_1 e_1)s_1 - 2n_1 e_1 + q_1 e_2, \\ &p_{2,\tau} = -(1 + \mu_2 e_2)s_2 - 2n_2 e_2 + q_1 e_1, \\ &s_{1,\tau} = (1 + \mu_1 e_1)p_1 + q_2 e_2, \ s_{2,\tau} = (1 + \mu_2 e_2)p_2 - q_2 e_1, \\ &q_{1,\tau} = -(e_1 p_2 + e_2 p_1), \quad q_{2,\tau} = (e_1 s_2 - e_2 s_1), \\ &n_{1,\tau} = 2e_1 p_1 + e_2 p_2 + (\bar{c}_1 + \bar{c}_2), \\ &n_{2,\tau} = e_1 p_1 + 2e_2 p_2 + (\bar{c}_1 + \bar{c}_2). \end{aligned} \tag{69}$$

Here, e_1, e_2 are the normalised circular right- and left-polarised electric field components; $\mu_1 = (d_{22} - d_{11})/d_{12}$ and $\mu_2 = (d_{33} - d_{11})/d_{12}$ are the coefficients taking into account the Stark shifts of the energy levels; $\bar{c}_1$ and $\bar{c}_2$ are the population and depletion rates of the upper and lower levels, respectively; the density matrix elements are related to the variables in equations (69) by the expressions:

$$\begin{aligned} &\mathrm{i}p_1 = \varrho_{12} - \varrho_{21}, \quad \mathrm{i}p_2 = \varrho_{13} - \varrho_{31}, \quad s_1 = \varrho_{12} + \varrho_{21}, \\ &q_1 = \varrho_{23} + \varrho_{32}, \quad \mathrm{i}q_2 = \varrho_{23} - \varrho_{32}, \quad n_1 = \varrho_{11} - \varrho_{22}, \\ &n_2 = \varrho_{11} - \varrho_{33}. \end{aligned}$$

The example of soliton solutions for different initial populations of the sublevels showed that the pump leads to a change in the polarisation dynamics. The two-soliton solution is used to analyse the interaction of solitons in a two-level medium in the presence of an external pump.

In some problems the model equations differ from completely integrable by the terms with a small parameter. In this case, use can be made of the perturbation theory for solitons [167, 168]. Zabolotskii [169] presented the perturbation theory for the systems of evolution equations, which are close to the systems integrable by the IST with a spectral parameter depending on spatiotemporal variables. This theory is used to study the evolution of soliton light pulses in a two-level medium with an upper pump level, taking into account linear and nonlinear losses and dispersion. Zabolotskii also studied the contribution of the radiation part of the solution with allowance for perturbations and showed that there exists a parameter domain where this contribution can be neglected.

5.2 Deformed equations in the case of two-photon absorption

Burstev et al. [163] proposed a system of equations which correspond to deformations of Kaup–Steudel equations (26) and (27) describing the SIT in the case of two-photon resonance:

$$\begin{aligned}
r_{3,\zeta} &= \frac{\mathrm{i}}{2}(s^*r - sr^*) - \frac{\bar{c}}{4}r_3, \\
r_{,\zeta} &= \mathrm{i}sr_3 + \mathrm{i}gs_3r - \frac{\bar{c}}{4}r, \\
s_{3,\tau} &= -\frac{\mathrm{i}}{2}(s^*r - sr^*) + \epsilon\frac{\bar{c}}{4}r_3, \\
s_{,\tau} &= \mathrm{i}gr_3s + \mathrm{i}\epsilon rs_3 + \epsilon\bar{c}\left(\mathrm{i}f - \frac{g}{2}\right)r.
\end{aligned} \tag{70}$$

Here, $\epsilon = 1$ corresponds to two-photon absorption and $\epsilon = -1$ corresponds to stimulated Raman scattering. The parameter f is defined as $f^2 = (\epsilon - g^2)/4$.

Another example of deformation of Kaup–Steudel equations has the form

$$\begin{aligned}
r_{3,\zeta} &= \frac{\mathrm{i}}{2}(s^*r - sr^*) + 4\bar{h}s_3, \\
r_{,\zeta} &= \mathrm{i}sr_3 + \mathrm{i}gs_3r - 8\bar{h}\left(\mathrm{i}f + \frac{g}{2}\right)s, \\
s_{3,\tau} &= -\frac{\mathrm{i}}{2}(s^*r - sr^*) - \bar{c}s_3, \\
s_{,\tau} &= \mathrm{i}gr_3s + \mathrm{i}\epsilon rs_3 - \bar{c}s,
\end{aligned} \tag{71}$$

where $\bar{h} = \mathrm{i}f\bar{c}/[4\epsilon(\mathrm{i}f + g/2) - g/2]$. However, these systems did not attract much attention.

5.3 Deformed NLSEs

The authors of paper [170] developed the Riemann problem method to solve the deformed NLSE

$$\mathrm{i}e_{,\tau} + e_{,\zeta\zeta} \pm 2|e|^2e + \frac{\mathrm{i}}{2\tau}e = 0, \tag{72}$$

which contains an auxiliary term, linear in the field and inversely proportional to time (for spatial solitons) or to coordinate (for temporal solitons). The equation itself was presented in [162] as an example of the NLSE deformation, its solution being described by cylindrically diverging waves in a cubic nonlinear medium. This deformation is trivial because with a proper change in the variable, equation (72) can be reduced to an ordinary NLSE.

A more general example of the NLSE deformation is presented in [171, 172]:

$$\mathrm{i}e_{,\tau} + (f(\zeta)e)_{,\zeta\zeta} + 2f(\zeta)|e|^2e + 2e\int_{-\infty}^{\zeta} f_{,\zeta_1}|e|^2\mathrm{d}\zeta_1 = 0. \tag{73}$$

Here, the spatial inhomogeneity is given by an arbitrary function $f(\zeta)$. At $f(\zeta) = 1$, we obtain an ordinary NLSE. The equation is completely integrable and has solutions in the form of deformed solitons, which move at an accelerating velocity in particular cases of the inhomogeneity selection.

5.4 NLS solitons in an inhomogeneous medium

To describe an electromagnetic wave in an inhomogeneous medium, Chen et al. [14] proposed the equation

$$\mathrm{i}e_{,\tau} + e_{,\zeta\zeta} + 2(|e|^2 - \alpha\zeta)e = 0, \tag{74}$$

which, as was shown, can be solved with the help of the IST under the condition that the spectral parameter changes linearly with τ. The soliton solution has the form:

$$e_{\mathrm{s}}(\tau,\zeta) = \frac{2\eta\exp\{2\mathrm{i}(\xi - \alpha\tau)\zeta - 4\mathrm{i}[\alpha^2\tau^3/3 - \alpha\zeta\tau^2 + (\xi^2 - \eta^2)\tau]\}}{\cosh[2\eta(\zeta + 2\alpha\tau^2 - 4\xi\tau - \zeta_0)]}. \tag{75}$$

Here, the parameters η, ξ, are found from the initial conditions and ζ_0 is the position of the centre of gravity of a soliton at the initial instant of time. If in a standard soliton the position of the centre of gravity changes linearly with time ($\zeta = \zeta_0 + 4\xi\tau$) so that ξ is the group velocity, the trajectory of the centre of gravity for a deformed soliton is given by the expression

$$\zeta = \zeta_0 + 4\xi\tau - 2\alpha\tau^2.$$

Thus, the soliton as a whole moves with uniform acceleration. At $\alpha > 0, \xi > 0$ the trajectory of the soliton motion resembles the trajectory of a body thrown vertically (Fig. 9). Chen et al. [14] found also the N-soliton solution and derived a recurrent expression to calculate the integrals of motion.

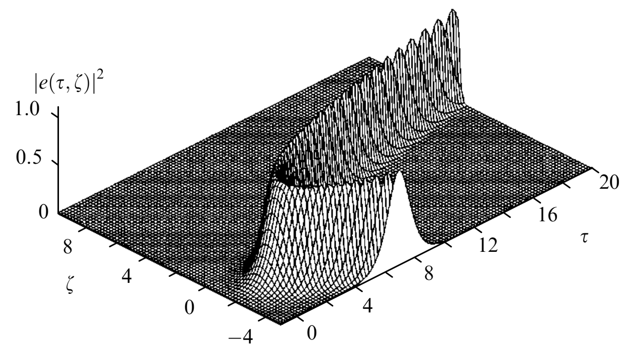

Figure 9. Turn of a soliton in an inhomogeneous medium – uniform acceleration.

The solitary waves in a medium, whose inhomogeneity is described by a quadratic function, were studied by Balakrishnan [173] using the equations:

$$\mathrm{i}e_{,\tau} + e_{,\zeta\zeta} + 2(|e|^2 + \alpha\zeta^2)e = 0. \tag{76}$$

Using the generalised MIST, soliton solutions corresponding to nonstationary pulses and accelerated pulses were found. Interest in this equation is caused by the studies of the Bose condensate of ultracold atoms. If we consider the condensate which is in a cigar-like magnetic trap and is confined by an optical parabolic trap in the longitudinal direction, the Gross–Pitaevskii equation [174] can be written in the form of (76).

Cupta et al. [175] considered the NLSE with a variable coefficient in the linear term. The dependence of this coefficient on the coordinate is given by the parabolic function with a maximum for the positive value of the coordinate:

$$\mathrm{i}e_{,\tau} + \frac{1}{2}e_{,\zeta\zeta} + \mathrm{i}e|^2 e + \mathrm{i}\gamma e - \left(\alpha - \frac{1}{2}\beta\zeta^2\right)e = 0. \quad (77)$$

Cupta et al. found the solution in the form of a solitary wave. At some parameters, reflection of a solitary wave from an inhomogeneity is possible.

5.5 Solitons of a nonautonomous NLSE

The equation

$$\mathrm{i}e_{,\zeta} + \frac{1}{2}D(\zeta)e_{,\tau\tau} + R(\zeta)|e|^2 e + \mathrm{i}\gamma(\zeta)e = 0 \quad (78)$$

is called the nonautonomous nonlinear Schrödinger equation. In some cases (for example, in describing the spatial deformed solitons), the variables ζ and τ change places. At a certain ratio between the coefficients, this equation (after changing the variables) can be transformed to an ordinary NLSE.

Under the condition that the coefficients responsible for the group velocity dispersion D, nonlinearity R, and dissipation γ in (78) are related as

$$2\gamma DR = RD_{,\zeta} - R_{,\zeta}D,$$

it is possible to find the zero curvature representation for (78) [176] and thus the multisoliton solutions of the nonautonomous NLSE. The authors of [176] used the Darboux transform following directly from the IST equation to find the soliton and multisoliton solutions. For a single-solitons solution the trajectory of the centre of gravity of a soliton in the plane (ζ, τ) is given by the equation

$$\tau = \tau_0 + a_1 \int_0^{\zeta} D(\xi)d\xi,$$

where a_1 is a constant found from the initial conditions. If the dispersion parameter D is specified by the periodic function, the soliton will periodically change its direction of motion (Fig. 10). The soliton amplitude can increase, decrease, or remain constant, which results from the choice of the coefficients in (78). It is pertinent to note that the collision of solitons in this case does not lead to their destruction. Such deformed solitons are genuine solitons, i.e., the solutions of a completely integrable equation, as is shown in [177].

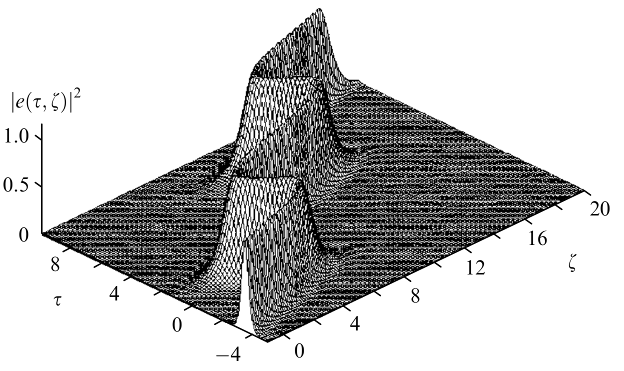

Figure 10. Variable acceleration of a soliton in a medium with periodically varying parameters.

Apart from bright solitons with a zero asymptotic, the NLSE has the solutions, which are called grey and dark solitons and represent a moving dip of the wave intensity against the constant nonzero background. The authors of [178] considered grey solitons in the case of a nonautonomous NLSE. They found the soliton solutions at a certain ratio between the parameters in equation (78). They also showed that if the initial soliton has a phase modulation (chirp), the grey deformed soliton can experience self-compression.

Tian et al. [179] considered the generalised two-component nonautonomous NLSE

$$\mathrm{i}e_{1,\zeta} + \frac{1}{2}D(\zeta)e_{1,\tau\tau} + R(\zeta)(|e_1|^2 + |e_2|^2)e_1 + \mathrm{i}\gamma(\zeta)e_1 = 0,$$
$$(79)$$
$$\mathrm{i}e_{2,\zeta} + \frac{1}{2}D(\zeta)e_{2,\tau\tau} + R(\zeta)(|e_1|^2 + |e_2|^2)e_2 + \mathrm{i}\gamma(\zeta)e_2 = 0.$$

If the relation

$$2\gamma\frac{R}{D} = \frac{\partial}{\partial\zeta}\left(\frac{R}{D}\right)$$

between the coefficients describing the group velocity dispersion, nonlinear response, and dissipation is met, it is possible to find the zero curvature representation for system (79) and using the IST to find the soliton solutions, including the two-soliton solutions, which describe the collision of two solitons.

Recently, Serkin et al. [180] considered the nonautonomous NLSE

$$\mathrm{i}e_{,\zeta} + \frac{1}{2}D(\zeta)e_{,\tau\tau} + R(\zeta)|e|^2 e - (2\alpha(\zeta)\tau + \Omega^2(\zeta)\tau^2)e = 0, \quad (80)$$

which has soliton solutions with the parameters (phase and group velocities, amplitude, and phase) varying with ζ. Serkin et al. called these solitons nonautonomous [180]. Like ordinary (with fixed parameters) solitons, the nonautonomous solitons interact elastically. Moreover, Kundu [181] showed that equation (80) transfers into a standard NLSE after some transformations.

6. Application of integrable models

Nonlinear optics has some phenomena, which can be described with the help of integrable systems of equations such as NLSE, RMB, three-wave equations, etc., or models which are analysed by using the IST formalism.

6.1 Propagation of an ultrashort pulse through a thin film

Propagation of the ultrashort pulses through an interface between two dielectric media on which a thin (with a thickness l shorter than the wavelength of a carrier wave) film of resonantly absorbing atoms is located, was considered in some papers, including [182]. Let a thin film be at point $z = 0$ on the axis along which an electromagnetic wave propagates. The wave equation has the from

$$E_{,zz} - n^2 c^{-2} E_{,tt} = 4\pi c^{-2} P_{tt} l \delta(z).$$

Polarisation is $P = n_a d \langle r_1 \rangle$, where r_1 is found from the solution of Bloch equations (7). The refractive index n of dielectric media is defined as $n(z) = 1$ at $z < 0$ and $n(z) = n > 1$ at $z > 0$. Outside the film, the solution of the wave equation for the harmonic wave has the form:

$$E^-(z,t) = E_{\rm in} \exp(\mathrm{i}k_1 z - \mathrm{i}\omega t)$$

$$+ E_{\rm ref} \exp(-\mathrm{i}k_1 z - \mathrm{i}\omega t) + \text{c.c.}, \quad z < 0,$$

$$E^+(z,\omega) = E_{\rm tr} \exp(\mathrm{i}k_2 z - \mathrm{i}\omega t) + \text{c.c.}, \quad z > 0.$$

Here, $k_1 = \omega/c$, $k_2 = \omega n/c$ are the wavenumbers of incident and refracted waves. The amplitudes of the reflected and propagated waves can be expressed by the incident wave amplitude and the thin film polarisation. For a quasi-harmonic ultrashort-pulse wave, this can be rather cumbersome. The authors of paper [182] suggested solving a system of equations for an auxiliary field $\tilde{E}(z,t)$, which is determined by the equation

$$\tilde{E}_{,z} = \mathrm{i}(4\pi\omega_0 n_a dl/n)\langle r_1(t)\rangle \delta(z).$$

with the boundary conditions

$$\lim_{z\to-\infty} \tilde{E}(z,t) = E_{\rm in}, \quad \lim_{z\to+\infty} \tilde{E}(z,t) = E_{\rm tr}.$$

The equation for $\tilde{E}(z,t)$ and Bloch equations (7), as was shown in [182], can be solved with the help of the IST, thus finding the refracted ultrashort pulse. If we assume, for example, that the incident wave envelope $E_{\rm in}(t) = 2\eta^{-1}\operatorname{sech}(\eta t)$, the refracted wave envelope will have the form:

$$E_{\rm tr}(t) = 2\eta^{-1}\{\operatorname{sech}(\eta t) + \mathrm{e}^{\mathrm{i}\varphi_0}\operatorname{sech}[\eta(t - t_0)]\},$$

where the phase shift φ_0 and t_0 are determined in solving the linear IST equations. Thus, when ultrashort pulses coherently interact with thin film atoms there emerges a delayed pulse.

The authors of [183] presented generalisation to the case of polarised radiation or three-level atoms. They also considered the case of the ultrashort pulse incident on a thin film at an arbitrary angle and showed that the number of solitons in the reflected and refracted waves depends on the angle of incidence.

6.2 Raman soliton

Druhl et al. [184] observed anomalous reversible pump depletion in the case of Raman scattering and formation of a short spike of Stokes radiation. This phenomenon was interpreted as soliton formation under the action of a random phase shift in the Stokes pulse and the short spike itself has been known since then as a Raman soliton. Numerical simulation [185] showed that the phase shift mechanism can lead to amplification of the Raman soliton even in the presence of coherence damping and of detuning from resonance, and the possibility of propagation of a stable Raman soliton was found in an optical fibre. The experimental results, confirming the prediction that the quantum fluctuations lead to the formation of a spontaneous Raman soliton, are presented in [186].

The authors of [187, 188] considered a model describing the three-wave interaction:

$$e_{1,\zeta} = -qe_2, \quad e_{2,\zeta} = q^* e_1, \quad q_\tau = -\vartheta q + e_1 e_2. \tag{81}$$

In these equations e_1, e_2 are the pump-pulse and scattered-wave envelopes; q is the medium polarisation envelope; ϑ is the polarisation (coherence) damping coefficient. Using the IST, Kaup [188] solved the problem with initial conditions for this system of equations. He obtained the condition necessary for the appearance of the Raman solitons and determined their parameters.

To interpret the experiment from paper [184], Claude et al. [189] used another model, which does not take into account the coherence damping:

$$e_{,\tau} = \gamma a_1 a_2^*, \quad a_{1,\zeta} = ea_2, \quad a_{2,\zeta} = e^* a_1. \tag{82}$$

Here, a_1, a_2 are the amplitudes of the excited and ground states of the medium. The boundary conditions

$$\lim_{\zeta\to-\infty} a_1 = 1, \quad \lim_{\zeta\to-\infty} a_2 = 0, \quad \lim_{\zeta\to\pm\infty} e = 0$$

correspond to the assumptions that before the soliton is formed, the medium was in an excited state. Thus, this model described the decay of the excited state, accompanied by generation of a radiation pulse. The equations were solved with the help of the IST. Because the spectrum of the spectral IST problem does not contain a discrete part, the Raman soliton observed in the experiment does not in fact correspond to the solution of equation (82). In the case of a long pulse, the equations were modified to take into account the coherence (phonon wave) damping losses; however, the system derived is not completely integrable.

Later, the Raman solitons were studied in [190 – 192] using the IST, which allows one to solve boundary problems on a semi-axis. The main attention in studies of stimulated Raman scattering was concentrated on the analysis of the nonsoliton part of the spectrum of the IST equation.

6.3 Model of a laser amplifier and superfluorescence

The authors of [193] suggested using SIT equations to describe amplification of electromagnetic pulses during their propagation through an inversely populated medium. The model of a two-level lossless amplifier was studied by Lamb [194]. He found the SIT self-similar solution, which follows from the equation

$$\upsilon\phi_{,zz} + \phi_{,z} - \sin\phi = 0.$$

The numerical solution shows that the stationary pulse is absent but if nonlinear losses are introduced in the wave equation, it is possible to obtain a stationary solution [195] called a π-pulse. The inverse scattering transform in analysing the laser amplifier model was first used by Manakov [196]. He found that the solitons in this model are absent. The problem on the propagation of an ultrashort pulse in an extended two-level amplifier was reformulated

[197, 198] into the boundary problem on a semi-axis. The authors of these papers showed that the amplified pulse always has a quasi-self-similar form. Near the leading edge the ultrashort pulse is described by the Painlevé equation while at the trailing edge the solution passes to the regime of self-similar high-frequency oscillations.

The SIT equations were used in [199] to describe the superfluorescence phenomenon (i.e., formation of short light pulses from polarisation fluctuations in a two-level inversely populated medium). Unlike SIT, the superfluorescence pulses are produced from the unstable state of the resonant medium and the IST should be reformulated to solve the boundary problem on the time semi-axis. The solution of this problem was discussed in [191, 199, 200].

6.4 Development of the IST and its application in nonlinear optics

It is pertinent to note that the use of the IST in boundary problems on a finite interval or semi-axis allows one to describe rigorously transient processes during the stimulated Raman scattering [189, 190] and some other processes in nonlinear optics [201, 202].

Steudel et al. [200] developed the effective S-matrix method to solve the equations on the finite interval. As an example of the developed formalism, they studied SHG and stimulated Raman scattering. The Riemann–Hilbert problem method for integrating the equations specified on the semi-axis was presented by Doctorov et al. [202]. They considered application of this approach to develop the perturbation theory, which makes it possible to analyse the systems close to completely integrable, as well as to the case of SIT and stimulated Raman scattering on a limited interval. They showed that because of the atom dephasing, the Raman soliton (Raman spike of the electromagnetic field) can be stabilised.

6.4.1 Cnoidal waves

The periodic solutions of the Korteweg–de Vries equations were called cnoidal waves because thay are expressed by the function *cn*, known as the Jacobi cosine. It would be well to call ordinary harmonic waves the cosine waves. Apart from the solutions corresponding to solitary waves, the completely integrable systems also have the solutions, which describe periodic waves, more general than the cnoidal ones (see the footnote in Section 2.3). The IST was used to construct periodic solutions of some nonlinear optics equations. Using the IST-based Witham method, a theory of soliton generation at the leading edge of a long pump pulse was developed for the two-photon absorption and stimulated Raman scattering in the Kaup–Steudel model (28) [82, 203]. The periodic solutions describing the four-wave interaction of counterpropagating waves under the two-photon absorption were first found in [87]. Note, however, that these solutions lack any physical sense because one should not neglect polarisation relaxation and population difference for periodic waves infinitely extended in time.

6.4.2 Relations with the field theory models

Of interest is the fact the both the McCall–Hahn SIT equations and the generalised RMB equations in the case when inhomogeneous resonance-line broadening is absent and exact resonance is fulfilled can be transformed into the equations of motion of the principal chiral field model [204]. The same model involves the Kaup–Steudel equations describing the SIT in the case of two-photon resonance [205]. The authors of paper [206] have recently studied different generalisations of the generalised RMB equations for multilevel systems and have found that these systems have the hidden non-Abelian symmetry.

Similarly to that how the unitary transform couples the Hamiltonians of different theories of optical resonance and quantum optics, the authors of [207] found that this transform allows one to obtain from a Lax representation of one exactly integrable system a Lax representation of another system. As an example of unitary transforms in the theory of integrable systems, the authors of [208, 209] considered the propagation model of differently polarised pulses under the Raman scattering and two-photon resonance conditions on the $j_1 = 0 \to j_2 = 0$ and $j_1 = 1 \to j_2 = 1$ transitions. The models were obtained by applying the unitary transform to the Lax representation of the equations of the polarisation double-resonance theory in the three-level system with the Λ-configuration of the energy levels.

6.4.3 Method of Stark switching for optical solitons

One of the coherent spectroscopy methods is the Stark switching technique. The resonance atomic or molecular medium is placed in the field of cw laser radiation whose carrier-wave frequency differs from the transition frequency. Then, with the pulse of the dc electric field, the transition frequency because of the Stark effect changes such that this frequency is in resoance. Similarly, we can 'switch on' and 'switch off' the resonance interaction. This technique helped to study coherent transition processes: optical nutations, free induction decay, photon echo, coherent Raman beats.

The same method can be used to excite optical solitons in a resonant medium. Using the IST theory, Basharov et al. [210] developed a theory describing propagation of ultrashort pulses under the condition that additional Stark pulses 'switch on' the resonance interaction. The basis of the IST consists of the nontraditional spectral problem. The author of [211] presented the generalisation to the case of polarised ultrashort pulses.

6.4.4 Solitons and the electromagnetically induced transparency

The model of three-level atoms is the simplest model which is used to study the electromagnetically induced transparency (EIT). Nazarkin et al. [212] showed that two-photon self-induced transparency can emerge due to relaxation suppression in the medium such that the electromagnetic pulse propagation regime becomes coherent.

The authors of papers [213, 214] considered the interaction of solitons under the EIT conditions by using the system of equations (21). Rybin et al. [215, 216] discussed the electromagnetic pulse deceleration under the EIT condition by using the model of three-level atoms with the Λ-configuration of the energy levels. They showed that the group velocity of a soliton monotonically decreases with increasing intensity of the control laser field.

6.4.5 Solitons in artificial media

A great deal of interest has arisen in the last few years in artificial materials (metamaterials) with extraordinary electrodynamic properties. Negative refraction is the most discussed property of such materials [217]. To date, such metamaterials – transparent in the optical spectrum – are absent, but their possible nonlinear properties are being actively studied. Transparent metamaterials can have NLS solitons or solitons in a two-component NLSE [218, 219]. Interesting phenomena can be found in metamaterials due to the simultaneous existence of forward and backward waves* in them [220]. Recent paper [221] indicated the SIT possibility for a backward wave.

Basharov [222] described an artificial medium consisting of microcavities filled with atoms and photons. He showed that interaction of an external field under Raman resonance conditions with an optically forbidden atomic transition involving photons of the microcavity is described in some cases by ordinary MB equations for single-photon resonance. In this case, atoms and photons in microcavities are treated as a new elementary emitter – an atom – photon cluster on the states of which the irreducible representations of dynamic symmetry algebra are implemented, i.e., third-order polynomial algebra.

7. Conclusions

The sequential use of the IST has lead to the discovery many equations that produce soliton solutions. The soliton solutions, as a rule, are stable with respect to collisions and small perturbations (although this stability is not asymptotic – small perturbations remain small, while the solitons themselves can change their parameters). This property makes the search for such equations attractive for the development of the nonlinear wave theory. Physics has few models based on completely integrable systems; therefore, these systems can be considered as the first order of the perturbation theory. The authors of papers [167, 168] developed a special perturbation theory for equations close to integrable, this theory being widely used in different fields of physics including nonlinear optics.

There exists a field of nonlinear optics where the term 'soliton' is used to designate stable (robust soliton) electromagnetic pulses. The systems of equations describing their propagation and interaction are not integrable and even the perturbation theory is not efficient here. The most known example is the gap (Bragg) solitons which can propagate in periodic nonlinear media. Apart from these solitons, vortex (topological) and parametric (in particular, quadratic) solitons, solitons in photonic crystals and in fibre gratings, dissipative and noncoherent solitons are being studied. Recent books [223 – 226] are devoted to all these so-called solitons.

Acknowledgements. The author is grateful to his colleagues A.M. Basharov, S.O. Elyutin, I.R. Gabitov, N.N. Rozanov, E.V. Kazantseva, J.-G. Caputo, A.A. Zabolotskii, B.I. Mantsyzov, S.V. Sazonov, and S.A. Kozlov for many discussions of different problems of the theory of solitons, their place in nonlinear optics and in the problems on propagation of ultrashort pulses in resonant and nonresonant media. The author also would like to thank Laboratoire de Mathematiques, INSA de Rouen for support and hospitability during the preparation of the manuscript. The work was partially supported by the Russian Foundation for Basic Research (Grant No. 09-09-00701-a) and the Federal Target Program 'Scientific and Scientific-Pedagogical Manpower of the Innovation Russia'.

*In an isotropic medium the phase and group velocities in the backward wave have opposite directions, while in the forward wave the directions of these velocities are identical.

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