LASER BEAMS

PACS numbers: 42.60.Da; 42.60.Jf
DOI: 10.1070/QE2011v041n01ABEH014422

Analytical description of a Gaussian beam in a ring resonator with a nonplanar axial contour and an odd number of mirrors

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***Abstract*. Stability conditions for a ring resonator with an odd number of mirrors and a nonplanar axial contour are studied analytically. New explicit expressions are derived to describe the transverse field distribution of the Gaussian mode with general astigmatism produced in this resonator. Field characteristics for a resonator with the specified parameters are calculated.**

Keywords: *ring cavity, nonplanar contour, Gaussian beam, general astigmatism.*

1. Ring resonators with a nonplanar axial contour (see, for example, [1 – 14]) produce the fundamental mode in the form of a Gaussian beam with general-type astigmatism. In this case, the expression for the function, which describes (in the scalar interpretation, without allowance for polarisation) the transverse field distribution of the fundamental mode in some resonator cross section in the zero approximation with respect to wavenumber k, has the form

$$u(r) = c\exp\left(\mathrm{i}k\,\frac{r^{\mathrm{t}}Hr}{2}\right),$$

where

$$r = \begin{pmatrix} x \\ y \end{pmatrix}, \quad r^{\mathrm{t}} = (x \quad y),$$

the elements of the quadratic matrix

$$H = \begin{pmatrix} 1/q_x & 1/q_{xy} \\ 1/q_{xy} & 1/q_y \end{pmatrix}$$

and the numerical factor c depend on the longitudinal coordinate z. The matrix H is symmetric and has the positive definite imaginary part for a beam concentrated in the vicinity of the resonator axis. In the case of an axially symmetric beam, the matrix H is proportional to a unit matrix ($q_x = q_y = q$, $1/q_{xy} = 0$). Otherwise, the beam is said to have astigmatism. Astigmatism is called simple if in some coordinate system, the matrix H has a diagonal shape: in this case, the directions of the eigenvectors of matrices $\mathrm{Re}H$ and $\mathrm{Im}H$ coincide, i.e., the major axes of the phase and intensity ellipses coincide and can be chosen as the coordinate axes. When such a beam propagates along the optical axis, the matrix H remains diagonal for all values of the longitudinal coordinate z. In the case of general astigmatism, the major axes of the intensity and phase ellipses are directed at some angle to each other, and $\mathrm{Re}H$ and $\mathrm{Im}H$ matrices cannot simultaneously have a diagonal form no matter what coordinate axes are chosen. Besides, these axes have different directions for different values of z (see, for example, [11 – 16]), which gave a reason to call such a beam rotating.

Let $w_{1,2}$ be the semiaxes of the intensity ellipse at the boundary of which the field amplitude decreases by e times compared to its value on the axis [7]. Then, the eigenvalues of the $\mathrm{Im}H$ matrix are equal to $2/(kw_{1,2}^2)$. {Note that if the intensity ellipse boundary is considered as in [11] to be the curve at the boundary of which the beam energy density rather than the amplitude decrease by e times (the beam energy density being proportional to the square of the amplitude), the semiaxes of such an ellipse turn $\sqrt{2}$ times smaller, and the eigenvalues of the $\mathrm{Im}H$ matrix are equal to $1/(kw_{1,2}^2)$.} The eigenvalues of $\mathrm{Re}H$ are the major curvatures of the beam wavefront and equal to $\rho_{1,2}^{-1}$, where $\rho_{1,2}$ are the major radii of curvature taking a sign 'plus' for a divergent beam or a sign 'minus' for a convergent beam; if the wavefront is hyperbolic, these radii have different signs. In what follows, when the directions of the eigenvectors of the matrices with the coordinate axes coincide, we will use both numerical and alphabetical notation x, y.

The quadratic matrix H satisfies the equation

$$HBH + HA = DH + C, \tag{1}$$

where A, B, C, and D are the real 2×2 matrices (for a passive resonator without losses). These matrices form the 4×4 ray matrix T of the round trip in the resonator (monodromy matrix [3])

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

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Received 22 September 2010
Kvantovaya Elektronika **41** (1) 43 – 54 (2011)
Translated by I.A. Ulitkin

Equation (1) follows from the relationship [9]

$$H_{\mathrm{out}} = (C + DH_{\mathrm{in}})(A + BH_{\mathrm{in}})^{-1},$$

which describes the transformation of a Gaussian beam when it travels across the system characterised by the matrix T, and from the condition for the beam recovery after the resonator round trip: $H_{\mathrm{out}} = H_{\mathrm{in}} = H$. The matrix T is symplectic [3, 9], which implies fulfilment of the condition

$$T^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}. \qquad (2)$$

A resonator is stable with respect to the first approximation if all the eigenvalues of the matrix T are modulo unity and the associated vectors are absent [1]. In this case, equation (1) has a symmetric solution with a positive imaginary part. This solution can be expressed by the components of the monodromy matrix eigenvectors [17, 1, 3–5, 9, 18–20]. Another approach is based on the analysis of the evolution of the Gaussian beam with general astigmatism and on the fact that such a beam is a mode of a linear resonator whose mirrors coincide with the beam fronts. Finally, the matrix equation (1) (or the resultant system of algebraic equations for the elements of the matrix H [6]) can be solved numerically, with subsequent selection of the solution that provides the concentration of the field near the optical axis. In the case of a resonator with a Gaussian aperture (Gaussian mirror), the solution can be found by simple iteration [12].

In this paper, for the problem of a resonator with an odd number of mirrors, one of which is nonplanar (e.g., spherical), we propose an alternative solution in which the field is described analytically, using explicit formulas; it is not required to search for the eigenvectors of the matrix T. It seems to us that such a description makes it possible to clearly trace the dependence of the characteristics of the light field on the resonator parameters. A similar problem for a resonator with an even number of mirrors was solved in paper [14], and a brief preliminary analysis of the differences between cases involving the even and odd numbers of mirrors was given in [13].

2. Consider a multimirror ring resonator with an odd number of mirrors and a nonplanar (in general) axial contour, whose length is denoted by L. The resonator contains a focusing element, which is, for example, a lens or one of the mirrors (spherical or elliptical); the remaining mirrors are considered flat.

Propagation of the field along the resonator is described by the matrix

$$T_L = \begin{pmatrix} E & LE \\ O & E \end{pmatrix},$$

where O and E are the zero and unit 2×2 matrices. With an appropriate choice of the coordinate systems, the matrix describing the reflection from a flat mirror represents a unit matrix; with each reflection, the orientation of the coordinate system changes and so does the direction of the angle readout (clockwise or counter-clockwise).

Passing through the focusing element (quadratic phase corrector [11]) is described by the matrix

$$T_\Psi = \begin{pmatrix} E & O \\ -\Psi & E \end{pmatrix},$$

where Ψ is a symmetric 2×2 matrix which is considered, without loss of generality, a diagonal matrix: $\Psi = \mathrm{diag}[\psi_x, \psi_y]$; otherwise, it can be reduced to a diagonal shape by turning the coordinate axes. For a focusing lens, $\psi_{x,y} = 1/f_{x,y}$, where $f_{x,y}$ are the focal lengths; for an astigmatism-free lens, $f_x = f_y = f$. For an elliptical mirror one axis of which lies in the plane of incidence (the xz plane) and the other is orthogonal to it (directed along the y axis), $\psi_x = 2(R_x \cos\alpha')^{-1}$, $\psi_y = 2R_y^{-1}\cos\alpha'$, where $\alpha' = \alpha/2$ is the angle of incidence; α is the angle between the incident and reflected axial rays; $R_{x,y}$ are the radii of curvature. For a spherical mirror, $R_x = R_y = R$; we will pay a special attention to this case below.

Since the total number of mirrors in this problem is odd, the round trip in the resonator results in a change in the orientation of the coordinate system, so that the corresponding transformation is described by the ray matrix

$$T_V = \begin{pmatrix} V_\phi & O \\ O & V_\phi \end{pmatrix},$$

commuting with T_L, where

$$V_\phi = \begin{pmatrix} -\cos\phi & -\sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = -(\cos\phi)I - (\sin\phi)\sigma \qquad (3)$$

is the matrix of the operator of reflection from some straight line lying in the plane xy; ϕ is the doubled angle between the ordinate axis and this straight line (Fig. 1);

$$I = \mathrm{diag}[1, -1]; \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(Note that the values of ϕ in this paper and in [13] differ in sign.) In particular, when the contour is flat, this straight line is orthogonal to the plane of the contour, and if its direction coincides with the y axis, then $\phi = 0$, and transform (3) is reduced to a change in direction of the x axis. However, the y axis may have a different direction (recall that we tied the coordinate axes up with the directions of the eigenvectors of the Ψ matrix), then $\phi \neq 0$, and the problem in mathematical terms is no

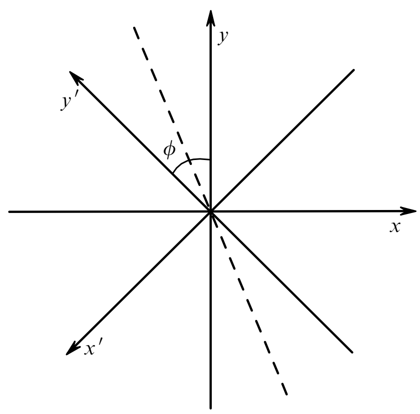

Figure 1. Transformation of the coordinate system during the round trip of light in the resonator.

different from the problem for the case of nonplanar contour. Similarly, in the general case, the angle ϕ is also determined not only by the geometrical characteristics of the contour, but also – through the direction of the coordinate axes – by the focusing properties of the element. This is an important qualitative difference of the problem under study from that considered in [14] for a resonator with an even number of mirrors.

Monodromy matrices T_- and T_+ in the beam cross sections, located immediately in front of and behind the focusing element, are calculated by the formulas

$$T_- = T_V T_L T_\Psi = \begin{pmatrix} V_\phi(E - L\Psi) & LV_\phi \\ -V_\phi \Psi & V_\phi \end{pmatrix},$$

$$T_+ = T_\Psi T_V T_L = \begin{pmatrix} V_\phi & LV_\phi \\ -\Psi V_\phi & (E - L\Psi)V_\phi \end{pmatrix}$$

(in both cases, use is made of the initial coordinate system). Of specially convenient form is the symmetrised matrix

$$T_0 = T_{\Psi/2} T_V T_L T_{\Psi/2} = \begin{pmatrix} V_\phi G & LV_\phi \\ (-V_\phi + GV_\phi G)/L & GV_\phi \end{pmatrix}, \quad (4)$$

where

$$G = E - L\Psi/2 = \mathrm{diag}[\gamma + \delta, \gamma - \delta] = \gamma E + \delta I$$

is the dimensionless symmetric diagonal matrix; $\gamma = 1 - (\psi_x + \psi_y)L/4$; $\delta = (\psi_y - \psi_x)L/4$. The matrix Ψ expressed by γ and δ has the form $\Psi = 2L^{-1}[(1-\gamma)E - \delta I]$.

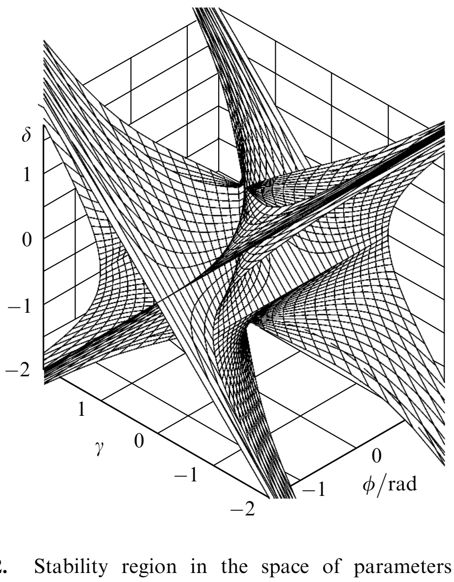

Figure 2. Stability region in the space of parameters γ, δ, ϕ at $\phi \in (-\pi/2, \pi/2)$. The figure is reproduced periodically long the ϕ axis.

The characteristic equations for λ – eigenvalues of the monodromy matrices – are reduced to the form

$$v^2 + (2\delta\cos\phi)v - (\gamma^2 - \delta^2) = 0,$$

where $v = (\lambda + \lambda^{-1})/2$ are the eigenvalues of the matrices of type $(T + T^{-1})/2$ as well as of the matrix $V_\phi G$ [the latter circumstance follows from expressions (2), (4)].

Sufficient stability conditions for the matrices $T_\pm$ [the values of λ lie on the unit circumference and different values of v lie in the interval $(-1,1)$ and also differ] have the form

$$-1 + 2|a| < \delta^2 - \gamma^2 < a^2 < 1, \quad (5)$$

where

$$a = \delta\cos\phi.$$

The necessary conditions ($|\lambda| = 1$) are obtained by substituting the strict inequality by the conditional inequality in (5).

In the space of three variables (ϕ, γ, δ), region (5) is symmetrical with respect to the planes $\phi = n\pi/2$ (n is the integer), $\gamma = 0$, $\delta = 0$, and with respect to the points $(n\pi/2, 0, 0)$ as well as periodic in ϕ with a period π. The general form of region (5) for $\phi \in (-\pi/2, \pi/2)$ is shown in Fig. 2 (branches tending to infinity are cut off). Region (5) is divided into subregions (connectivity components) – two components (for positive and negative values of γ) in each period $\phi \in (n\pi, (n+1)\pi)$. The cases of multiple λ values corresponding to points on the boundary surfaces

$$|\gamma| = |\delta\sin\phi|, \quad (6)$$

$$\gamma^2 = \delta^2 - 2|\delta\cos\phi| + 1 \quad (7)$$

require separate consideration because the associated vectors are possible to appear. The analysis shows that of all the boundary points the stability conditions are satisfied only at points on the straight line $\gamma = \delta = 0$ and on the straight sections $\phi = n\pi$, $\gamma = 0$, $\delta \in (-1, 1)$ connecting the subregions of set (5). Thus, taking into account these points, the set of parameter values for which the resonator is stable is connected.

Figure 3 presents the sections of region (5) in the coordinates γ, δ for different values of ϕ. In the case of $\phi = n\pi$, this area is a square $|\gamma| + |\delta| < 1$, or $0 < \psi_{1,2}L < 4$; the vertical ($\gamma = 0$) diagonal of this square lies on the boundary of a three-dimensional region (5). For $0 < |\cos\phi| < 1$, as shown above, the stability region splits into two subregions, bounded by segments of straight lines $|\gamma| = |\delta\sin\phi|$ and arcs of hyperbolas $\gamma^2 = (|\delta| - |\cos\phi|)^2 + \sin^2\phi$, that are tangents to the straight lines in their extreme

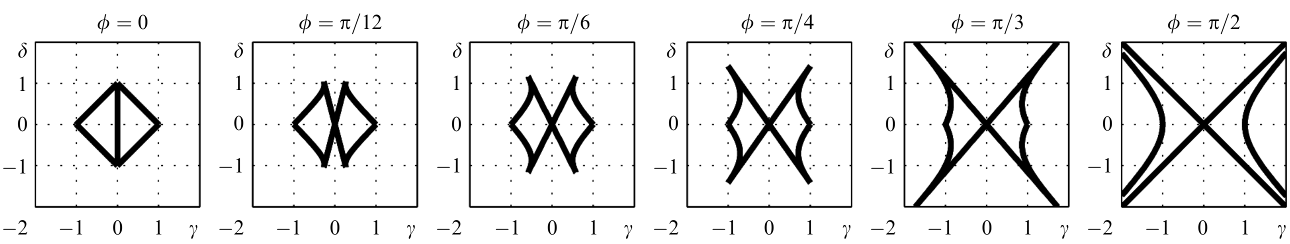

Figure 3. Stability regions in the $\gamma\delta$ plane for different ϕ.

points with coordinates $\delta = \pm\cos^{-1}\phi$, $\gamma = \pm\tan\phi$. If $\phi \in \pi/2 + n\pi$, these subregions are unbounded: the straight lines $\gamma = \pm\delta$ are not the tangents but the asymptotes of the hyperbolas $\gamma^2 - \delta^2 = 1$.

Figure 4 shows the projection of the figure from Fig. 2 on the plane $\gamma\delta$ – a two-dimensional region bounded by hyperbolas $\gamma^2 - \delta^2 = \pm 1$ and representing a union of the stability regions for all values of ϕ. Let us formulate the conditions for stability with respect to $|\cos\phi|$ for various subsets of this region. For points on the segments $\gamma = 0$, $0 < |\delta| < 1$, the resonator is stable only at $\phi = n\pi$ ($|\cos\phi| = 1$). In the triangles $|\gamma| < |\delta| < 1 - |\gamma|$, $0 < |\gamma| < 1/2$, the stability condition has the form $|\sin\phi| < |\gamma/\delta|$ (i.e., $|\cos\phi| > [1 - (\gamma/\delta)^2]^{1/2}$). In the regions $1/2 + ||\delta| - 1/2| < |\gamma| < (1 + \delta^2)^{1/2}$, the resonator is stable if $|\cos\phi| < (1 + \delta^2 - \gamma^2)/(2|\delta|)$. In regions $1/2 + ||\gamma| - 1/2| < |\delta| < (1 + \gamma^2)^{1/2}$, two last conditions should be met: $[1 - (\gamma/\delta)^2]^{1/2} < |\cos\phi| < (1 + \delta^2 - \gamma^2)(2|\delta|)^{-1}$. Finally, for points inside the squares of $|\delta| < 1/2 + ||\gamma| - 1/2|$, the resonator is stable for all values of ϕ. We also present the stability conditions on the dividing lines and at the points of intersection: $\phi \neq (n + 1/2)\pi$ for $0 < |\delta| = |g| < 1/2$; $\phi \neq n\pi$ ($|\cos\phi| \neq 1$) for $0 < |\delta| = 1 - |\gamma| < 1/2$; $0 < |\cos\phi| < 1/(2|\delta|)$ for $|\delta| = |\gamma| > 1/2$; $(2|\delta| - 1)^{1/2}/|\delta| < |\cos\phi| < 1$ for $1/2 < |\delta| = 1 - |\gamma| < 1$; $\phi \neq n\pi/2$ ($|\cos\phi| \neq 0, 1$) for $|\delta| = |\gamma| = 1/2$; at $|\delta| = |\gamma| = 0$ the resonator is stable for any ϕ.

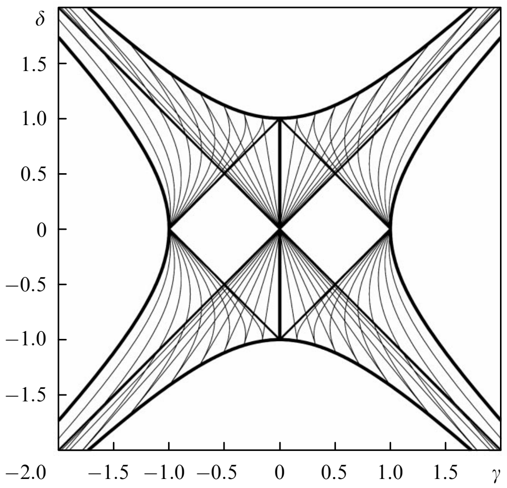

Figure 4. Projection of the stability region on the $\gamma\delta$ plane. Thin lines are the boundaries of stability regions for different ϕ spaced by 5°.

It is easy to verify that, except for sections $\phi = n\pi/2$, when $|\cos\phi|$ becomes zero or unity, the geometry of the stability region is markedly different from that in the case of a resonator with an even number of mirrors [13, 14].

3. Quite often, a spherical mirror is used as a focusing element. In this case, $\gamma = 1(\cos\alpha' + \cos^{-1}\alpha')L/(2R)$, $\delta = (\cos\alpha' - \cos^{-1}\alpha')L/(2R))$. One can see that the sector $\gamma < 1$, $\gamma - 1 < \delta < 0$ on the plane $\gamma\delta$ corresponds to the case under consideration; all other points in the stability region require the use of another mirror (or a lens).

The curves uniting all the points with the same R for all possible angles of incidence are the arcs of hyperbolas $\delta = -[(1-\gamma)^2 - (L/R)^2]^{1/2}$, while the lines corresponding to a fixed angle of incidence at various R are the rays

$$\delta = \frac{(\gamma - 1)\sin^2\alpha'}{1 + \cos^2\alpha'} \tag{8}$$

(for $\gamma < 1$). Such rays for any α and ϕ intersect the right stability subregion; to this there correspond the values

$$\frac{L}{R} < \frac{2}{(\cos\alpha' + \cos^{-1}\alpha') - |\sin\phi|(\cos\alpha' - \cos^{-1}\alpha')}.$$

In addition, if

$$\frac{\cos\alpha' - \cos^{-1}\alpha'}{\cos\alpha' + \cos^{-1}\alpha'} < \frac{1}{|\cos\phi| + |\sin\phi|}, \tag{9}$$

the ray also intersect the left subregion at

$$\frac{2}{(\cos\alpha' + \cos^{-1}\alpha') + |\sin\phi|(\cos\alpha' - \cos^{-1}\alpha')} < \frac{L}{R}$$

$$< (\cos\alpha' + \cos^{-1}\alpha') + |\cos\phi|(\cos\alpha' - \cos^{-1}\alpha').$$

Thus, for a resonator with an odd number of mirrors, depending on α and ϕ, there is one or two intervals of R values, in which the resonator is stable. The latter case is shown in Fig. 5; in Fig. 6 it corresponds to such α and ϕ for which the perpendicular recovered from a point on the abscissa (not shown in Fig. 6) intersects both stability subregions. In most cases, it holds true: the smallest value of the right-hand side of (9) is equal to $1/\sqrt{2}$, while the left-hand side of this inequality [the slope ratio for ray (8)] is smaller than $1/\sqrt{2}$, if $|\cos\alpha'| > \sqrt{2} - 1$, then the inequality holds for all ϕ. Beam (8) in this case lies above the hyperbola $\delta = (1 + \gamma^2)^{1/2}$ (the lower limit of the figure in Fig. 4), on which there lie the angular points of the stability regions for all possible ϕ, and for an arbitrary ϕ it intersects the left subregion of this region. The opposite situation – the only interval of R values, in which the ray intersects the hyperbola and for some ϕ is entirely below the specified subregion – occurs only in the case of sufficiently large (above 131°3′37″) values of α.

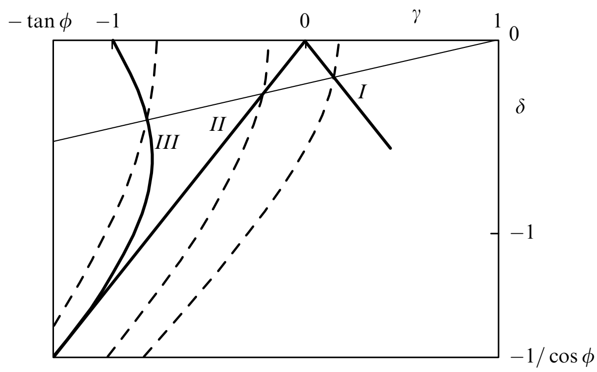

Figure 5. Stability region in the $\gamma\delta$ plane for a resonator with a spherical mirror. The straight line $\alpha = \text{const}$ (thin solid line) intersects both subregions, hyperbolae $R = \text{const}$ (dashed curves) go through the intersections of this straight line with the boundaries of the stability region $L/R = 2/[(\cos\alpha' + \cos^{-1}\alpha') - |\sin\phi|(\cos\alpha' - \cos^{-1}\alpha')]$ (*I*), $L/R = 2/[(\cos\alpha' + \cos^{-1}\alpha') + |\sin\phi|(\cos\alpha' - \cos^{-1}\alpha')]$ (*II*), and $L/R = 2/[(\cos\alpha' + \cos^{-1}\alpha') + |\cos\phi|(\cos\alpha' - \cos^{-1}\alpha')]$ (*III*). The point of the line break *I* corresponds to $\alpha = \pi$ (tangential incidence of the beam on the mirror).

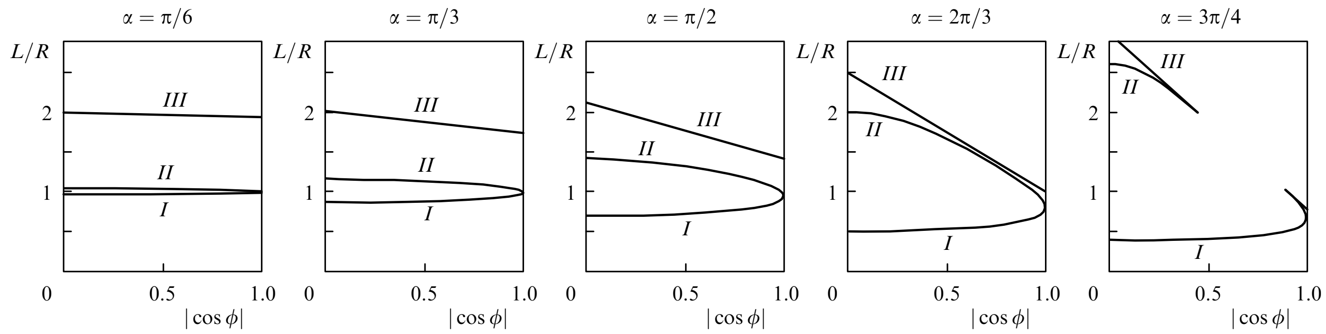

Figure 6. Regions of stability [below curve (I) and between curves (II) and (III)] and instability [between curves (I) and (II) and above curve (III)] for a resonator with a spherical mirror at different angles α. The equations for boundaries I, II, III are same as in Fig. 5.

4. Let us consider now the general case and write the expressions for the matrices $H_{\mp}$, corresponding the cross sections located directly in front of and behind the focusing element, in the form

$$H_{\mp} = H_0 \pm \frac{\Psi}{2}, \tag{10}$$

where H_0 is a purely imaginary [for internal points of region (5)] symmetric matrix of the form

$$H_0 = \frac{\mathrm{i}[2aG + (1-\gamma^2+\delta^2-d)V_{\phi}]}{Lt}\,\mathrm{sign}(\gamma a). \tag{11}$$

Here, the quantities d and t coinciding in modulus with the determinant and the trace of the matrix $LV_{\phi}\mathrm{Im}H_0$, are defined by the equalities

$$d = \sqrt{\left(1-\gamma^2+\delta^2\right)^2 - 4a^2}, \tag{12}$$

$$t = \sqrt{2\left(1-\gamma^2+\delta^2-2a^2-d\right)}. \tag{13}$$

The value of d vanishes on the surface (7), and the value of t – on the surface (6), on the plane $\delta = 0$, as well as at $\cos\phi = 0$ [$\phi = (n+1/2)\pi$]. On the surface (7), $t(d=0) = 2[|a|(1-|a|)]^{1/2}$, while on the surface (6), $d(t=0) = 1-|a|^2$. The highest value of d in the closed region (5) equals unity and is achieved when $|\gamma| = |\delta|$ and $a = 0$, i.e., at $\cos\phi = 0$ on the straight lines $|\gamma| = |\delta|$ or at the origin of the coordinates of the plane $\gamma\delta$. The highest value of t is also equal to unity and is achieved on the surface (7) when $|a| = 1/2$.

The function d can be represented as a product

$$d = u_+u_-,$$

where

$$u_{\pm} = \sqrt{1-\gamma^2+\delta^2 \pm 2a}. \tag{14}$$

The function t is also expressed through $u_{\pm}$:

$$t = |u_+ - u_-|\sqrt{1 - \frac{(u_+ + u_-)^2}{4}}.$$

At each segment of the boundary (7), the cofactor vanishes

$$u_< = \min\{u_+, u_-\} = \sqrt{1-\gamma^2+\delta^2-2|a|},$$

while the other cofactor

$$u_> = \max\{u_+, u_-\} = \sqrt{1-\gamma^2+\delta^2+2|a|}$$

takes the value $2\sqrt{|a|}$.

The matrices H_+ and H_- satisfy (1), where A, B, C, D are the blocks of the matrices T_+ and T_-, respectively. Matrix (11) describes the transverse field distribution in the equivalent resonator [7], whose curved mirror is replaced by a flat mirror with an astigmatic lens adjacent to it in the section corresponding to the flat mirror. It also meets (1); the corresponding monodromy matrix has the form (4). Here we do not present the derivation of (11)–(13): the way to explicitly get the solution of equations (1) is described in detail in [21] (see also [22, 23]). It is simpler to consider the equation with respect to H_0; the fact that the block B of the matrix T_0 is symmetric and the block $D = A^{\mathrm{t}}$ appreciably facilitates the solution of equation (1) that breaks down in this case into the system of matrix equations

$$H_0LV_{\phi}H_0 = (-V_{\phi} + GV_{\phi}G)/L, \tag{15}$$

$$H_0V_{\phi}G = GV_{\phi}H_0, \tag{16}$$

which makes it possible to use a simplified technique [24].

The matrix $\mathrm{Im}H_0$ is positive and its eigenvalues are positive in the region (5) because its trace

$$\mathrm{tr}\,\mathrm{Im}H_0 = \frac{4|\gamma a|}{Lt} \tag{17}$$

and the determinant

$$\det\mathrm{Im}H_0 = \frac{d}{L^2} \tag{18}$$

are positive (from whence it follows, taking into account the condition $d \leqslant 1$, the restriction on the area of the ellipse intensity: $\pi w_1 w_2 \geqslant 2\pi L/k$). The angles between the eigenvectors of this matrix, which determine the direction of the axes of the ellipse intensity, and the coordinate axes are

$$\theta_0 = \frac{1}{2}\arctan\left(\frac{1-\gamma^2+\delta^2-d}{1-\gamma^2-\delta^2-d}\tan\phi\right) \tag{19}$$

(with an accuracy to the term multiple of $\pi/2$).

On the boundary (7), where $d = 0$, the area of this ellipse, according to (18), becomes infinite, and the resonator becomes unstable. On the boundary (6), where $t = 0$, the quantity in (17) tends to infinity (with a nonzero numerator) and the value of (18) is limited, which means that one of the eigenvalues of $\operatorname{Im} H_0$ is infinite, and the other vanishes, and the resonator also becomes unstable. Below, we will specifically consider the behaviour of the matrix H_0 in the vicinity of the boundary of the stability region, as well as some special values of the parameters [in particular, those for which t vanishes (13)].

As for the matrices $H_\pm$, their imaginary parts coinciding with $\operatorname{Im} H_0$, are positive defined, and the real parts, equal to $\mp\Psi/2$, are diagonal, in contrast to the resonator with an even number of mirrors in which the imaginary parts of the required matrices are diagonal.

5. Expressions (11), (17) are not defined for the parameters, at which the denominator vanishes. To describe the behaviour of the matrix H_0 for the values of δ and $\cos\phi$ close to zero, we transform these equations, by multiplying their numerators and denominators by the conjugate expressions. Then, we arrive at the equalities

$$H_0 = \frac{\mathrm{i}t' \operatorname{sign}\gamma}{2L\sqrt{\eta}}(G + \xi V_\phi), \tag{20}$$

$$\operatorname{tr}\operatorname{Im} H_0 = \frac{|\gamma| t'}{L\sqrt{\eta}}, \tag{21}$$

where

$$\xi = \frac{2a}{1 - \gamma^2 + \delta^2 + d}$$

does not exceed unity in modulus, the value of

$$\eta = \gamma^2 - (\delta \sin\theta)^2 \tag{22}$$

lies in the interval between zero and $(1 - |a|)^2$, and

$$t' = \sqrt{2(1 - \gamma^2 + \delta^2 - 2a^2 + d)}\,. \tag{23}$$

It follows from the equality $\gamma^2 - \delta^2 = \eta - a^2$ that

$$d = \sqrt{(1 + a^2 - \eta)^2 - 4a^2},$$

$$t' = \sqrt{2(1 - a^2 - \eta + d)}.$$

The quantity t' vanishes, where the equalities $t = d = 0$ are met simultaneously. This occurs at points $\delta = 0$, $|\gamma| = 1$ of the plane $\gamma\delta$ (for arbitrary ϕ), as well as on the curves $\gamma^2 = \delta^2 + 1$ in sections $\phi = (n + 1/2)\pi$ ($\cos\phi = 0$). In this case, H_0 becomes a zero matrix, and the intensity ellipse extends indefinitely in all directions. All these points lie on the boundary (7) of the stability region (5). In addition, $t' = 0$ on the space curves

$$\delta^2 = \gamma^2 + 1 = \cos^{-2}\phi \tag{24}$$

[the 'cusps' of the boundary of the stability region, common for expressions (6) and (7)], on which, however, the quantity η (22) – the radical expression in the denominator of (20), (21) – also vanishes. Therefore, as seen from (11), (17), the matrix H_0 on these curves does not vanish, and, moreover, its elements and trace grow without bound. On the surface (6), $t'(t = 0) = [2(1 - a^2)]^{1/2}$, while on the surface (7), $t'(d = 0) = t'(t = 0) = 2[|a|(1 - |a|)]^{1/2}$. The highest value of t' in the closed region (5) is equal to two, and reached at the same values of parameters for which $d = 1$.

Let us present formulas relating the quantities used in (20) with the functions $u_\pm$ (14):

$$\xi = \frac{u_+ - u_-}{u_+ + u_-},$$

$$\eta = \left[1 - \frac{(u_+ - u_-)^2}{4}\right]\left[1 - \frac{(u_+ + u_-)^2}{4}\right],$$

$$t' = (u_+ + u_-)\sqrt{1 - \frac{(u_+ - u_-)^2}{4}},$$

and then

$$H_0 = \mathrm{i}\operatorname{sign}\gamma\left[2L\sqrt{1 - \frac{(u_+ - u_-)^2}{4}}\right]^{-1} \times [(u_+ + u_-)G + (u_+ - u_-)V_\phi],$$

$$\operatorname{tr}\operatorname{Im} H_0 = |\gamma|(u_+ + u_-)\left[L\sqrt{1 - \frac{(u_+ + u_-)^2}{4}}\right]^{-1}.$$

We will transform expression (11) for the angle θ_0 so that to eliminate the uncertainty at $\delta = 0$ and $\cos\phi = 0$:

$$\theta_0 = -\frac{1}{2}\arctan\frac{2\sin\phi\cos\phi}{1 - \gamma^2 + \delta^2 - 2\cos^2\phi + d}. \tag{25}$$

At $a = 0$, when $d = 1 - \eta$ and $t' = 2\sqrt{1 - \eta}$, matrix (20) transforms into the matrix

$$H_0(a = 0) = \frac{\mathrm{i}\operatorname{sign}\gamma}{L}\sqrt{\frac{1 - \eta}{\eta}}\,G.$$

In particular, if $\cos\phi = 0$, we have $\eta = \gamma^2 - \delta^2$,

$$H_0(a = 0) = \frac{\mathrm{i}}{L}\sqrt{\frac{1 - \gamma^2 + \delta^2}{\gamma^2 - \delta^2}}\,G\operatorname{sign}\gamma,$$

the matrices H_0 and $H_\pm$ are diagonal, $\theta_0(\cos\phi = 0) = 0$, and the resonator produces a beam with simple astigmatism (at $\delta \neq 0$).

In the case $\delta = 0$, when $\eta = \gamma^2$, at an arbitrary ϕ we obtain

$$H_0(\delta = 0) = \mathrm{i}L^{-1}\sqrt{1 - \gamma^2}\,E \tag{26}$$

[taking into account that $d(\delta = 0) = 1 - \gamma^2$, $t'(\delta = 0) = 2(1 - \gamma^2)^{1/2}$]; in this case, $\Psi = 2L^{-1}(1 - \gamma)E$, the astigmatism is absent, and the beam is axially symmetric.

However, there is a special case $\gamma = \delta = 0$, when the solution $\mathrm{i}L^{-1}E$ following from (26) is not unique: in this case, the desired solution is any matrix of the form

$$H_0(\delta - \gamma = 0) = \frac{\mathrm{i}}{L}\,\frac{(1 + \zeta^2)E + 2\zeta V_{\phi+\pi/2}}{1 - \zeta^2}, \tag{27}$$

where ζ is a complex parameter ($|\zeta| < 1$). This is due to the fact that the points of the straight line $\gamma = \delta = 0$ lie on the boundary of region (5), and the matrix T_0 at these points has multiple eigenvalues. Beams, described by matrix (27), with $\zeta \neq 0$ have simple astigmatism; in this case, $\theta_0 = \phi/2 + \pi/4$.

The matrices of family (27) are closely related to the asymptotic behaviour of the solution H_0 (20) in the vicinity of this straight line: if γ and δ are both close to zero, then (20) in the first approximation coincides with that matrix of family (27), for which

$$\zeta = \left(\gamma - \sqrt{\eta}\,\mathrm{sign}\,\gamma\right)(\delta \sin\phi)^{-1}. \qquad (28)$$

Varying the values of γ and δ, lying in the stability region, we, however, do not obtain the whole family (27), but only a subfamily of the real ζ.

6. Now let $\phi = n\pi$, then $\sqrt{\eta} = |\gamma|$,

$$d(\sin\phi = 0) = d_0 = \sqrt{\left[1 - (\gamma+\delta)^2\right]\left[1 - (\gamma-\delta)^2\right]},$$

$$t'(\sin\phi = 0) = t'_0 = \sqrt{2\left(1 - \gamma^2 - \delta^2 + d_0\right)}$$

$$= \sqrt{1 - (\gamma+\delta)^2} + \sqrt{1 - (\gamma-\delta)^2}, \qquad (29)$$

$$H_0(\sin\phi = 0) = \frac{\mathrm{i}}{L}\,\mathrm{diag}\left[\sqrt{1 - (\gamma+\delta)^2}, \sqrt{1 - (\gamma-\delta)^2}\right],$$

$$\mathrm{tr}\,\mathrm{Im}\,H_0(\sin\phi = 0) = \frac{t'_0}{L}.$$

This is a case of separating variables; the fundamental mode is the product of solutions of two two-dimensional problems.

When $\gamma = 0$, we are again confronted with an ambiguity: the solution is not only the matrix $\mathrm{i}E(1-\delta^2)^{1/2}/L$, which gives formula (29), but also any matrix of the family

$$H_0(\gamma = \sin\phi = 0) = \frac{\mathrm{i}\sqrt{1-\delta^2}}{L}\,\frac{\left(1+\zeta^2\right)E - 2(-1)^n\zeta\sigma}{1-\zeta^2}. \qquad (30)$$

In particular, this ambiguity takes place in a three-mirror resonator, considered in [25]. As before, this is due to the fact that the considered points with coordinates $\gamma = 0$, $|\delta| < 1$, $\phi = n\pi$ lie on the boundary (5), and the matrix T_0 at these points has multiple eigenvalues. If $\delta \neq 0$, then the beams described by solutions of (30), have simple (at $\zeta = 0$) or general (at $\zeta \neq 0$) astigmatism, and in the latter case, $\theta_0 = \pi/4$. Family (30) is associated with the asymptotic behaviour of the matrix H_0 in the vicinity of straight lines $\gamma = 0$, $\phi = n\pi$: at near-zero values of γ and $\sin\phi$, matrix (20) in the first approximation coincides with matrix (30), where ζ is again determined by formula (28). As before, we obtain in this way subfamily (30), corresponding to real values of ζ.

If $\gamma = \delta = \sin\phi = 0$, the family of the solutions is described by the expression

$$H_0(\gamma = \delta = \sin\phi = 0) = \frac{\mathrm{i}}{L}\,\frac{\left(1+\zeta^2\right)E - 2(-1)^n\zeta\sigma}{1-\zeta^2}, \qquad (31)$$

$$|\zeta| < 1,$$

following from (27) or (30). The beams corresponding to these solutions at $\zeta \neq 0$ have simple astigmatism; in this case, $\theta_0 = \pi/4$. The matrices of family (31) for real ζ (28) describe in the first approximation the behaviour of the solutions (20) at the values of γ, δ, and $\sin\phi$ that are simultaneously close to zero.

All the considered cases of ambiguous solutions (27), (30), (31) are of a common origin: the ambiguity arises if $G = \delta V_\phi$, and equation (16) holds identically. In this case, an arbitrary symmetric matrix

$$H_0(G = \delta V_\phi) = \frac{\mathrm{i}\sqrt{1-\delta^2}}{L}\,\frac{\left(1+\zeta^2\right)E + 2\zeta V_{\phi+\pi/2}}{1-\zeta^2}; \qquad (32)$$

meets equation (15); the imaginary part of (32) is positive at $|\zeta| < 1$. It is easy to see that (27), (30), and (31) are particular cases of expression (32).

Note that if we extend analytically our solution to the case of nonreal γ and δ (which corresponds to a resonator with imperfect mirrors, whose reflection coefficient depends on transverse coordinates) and turn the imaginary parts to zero, the matrices of family (32) with nonreal ζ can be obtained as a result of this transformation under specific relationship between the parameters of the resonator.

7. We will formulate without proof some results of the analysis of the behaviour of the matrix $H_0 = \mathrm{i}\,\mathrm{Im}\,H_0$ and its eigenvalues in the vicinity of other boundary points of the stability region (5). [This analysis is based mainly on the relations (17), (18), (21) and (23).]

On the surface (7), the quantity d (12) is zero, one of the eigenvalues $\mathrm{Im}\,H_0$ also vanishes, and the other takes the value

$$\frac{2|\gamma|}{L}\sqrt{\frac{|a|}{1-|a|}}. \qquad (33)$$

In this case

$$\theta_0(d = 0) = \frac{1}{2}\arctan\left(\frac{\sin\phi}{|\cos\phi| - |\delta|}\right)\mathrm{sign}\cos\phi.$$

In the vicinity (7), the value of d is small when $|a|$ is not close to zero,

$$d \sim 2^{3/2}\sqrt{|\gamma\delta\Delta\gamma\cos\phi|}, \qquad (34)$$

where $\Delta\gamma$ is the distance to the boundary of the variable γ. The presence of square-root singularity leads to the fact that the d, θ_0, and eigenvalues $\mathrm{Im}\,H_0$ dramatically change in the small neighbourhood of the boundary. In this case, unlike the resonator with an even number of mirrors, both eigenvalues decrease. Accordingly, there is a quick turn of the intensity ellipse accompanied by an unlimited increase in the length of one of its semiaxes and a less pronounced increase in the length of the other. The values of $|a|$ decrease with decreasing simultaneously the limiting value (33) of the second eigenvalue on the boundary and the size of the near-boundary region. The width of this region, where relation (34) is still fulfilled, is determined by the inequality $\Delta\gamma \ll 2|a/\gamma|$. When the opposite inequality $1 \gg \Delta\gamma \gg 2|a/\gamma|$ is fulfilled (in particular, at $\cos\phi = 0$), the dependence of d on $\Delta\gamma$ is close to linear:

$$d \sim 2|\gamma\Delta\gamma|.$$

If the value of δ is close to zero, and that of γ – to unity, the eigenvalues of the matrix $\mathrm{Im}\,H_0$ are small and coincide with values of $u_\pm/L$ in the first approximation. The value of θ_0 is close in this case to $\phi/2$, if $|\cos\phi|$ is not small. If $1-|\gamma|$, δ, and $\cos\phi$ are close to zero simultaneously, the dependence of the ratio of these parameters may exhibit different asymptotic behaviours of the angle

$$\theta_0 = -\frac{1}{2}\arctan\frac{2\sin\phi\cos\phi}{d+\sqrt{d^2+4a^2-2\cos^2\phi}}.$$

Consider now the vicinity of the surface (6) on which

$$\eta \sim 2|\gamma|(|\gamma| - |\delta|\sin\phi|)$$

vanishes; in this case, $\theta_0(\eta=0) = \phi/2 - \pi/4$ and

$$H_0(\eta\to 0) \sim \frac{\mathrm{i}\,\mathrm{sign}\,\gamma}{L}\sqrt{\frac{1-a^2}{\eta}}(G + aV_\phi).$$

When approaching the surface (6) one of the eigenvalues of the matrix $\mathrm{Im}\,H_0$ tends to zero, while the other – to infinity:

$$\frac{2}{kw_1^2} \sim \sqrt{\frac{(1-a^2)\eta}{2L|\gamma|}},$$

$$\frac{2}{kw_2^2} \sim \frac{2|\gamma|}{L}\sqrt{\frac{1-a^2}{\eta}},$$

the intensity ellipse in one direction extends indefinitely, and in the other narrows into a line, while its area remains limited and tends to $2\pi L/[k(1-a^2)^{1/2}]$. Therein lies a qualitative difference between this case and the above-discussed case of the vicinity of the surface (7), where the intensity ellipse turned not into a straight line but into a strip. Common in these situations is the presence of square-root singularity (this time in the denominator), because of which a drastic change occurs in the shape of the beam in a narrow boundary layer and outside it the beam parameters vary fairly smoothly. In this case, when approaching the stability boundary a quick change in the direction of ellipse semiaxes does not occur, in contrast to the above case of the vicinity of the surface (7). Note that this analysis requires refinement in the region of small $|\gamma|$, where the behaviour of the eigenvalues is determined by the ratio $\sqrt{\eta}/|\gamma|$; this case was considered above.

Another special case is the vicinity of the 'cusps' of the stability region, lying on space curves (24). On these curves, $|a|=1$, so in their vicinity the difference $1-|a|$ is small as well as the value of η, which does not exceed $(1-|a|)^2$. Then, putting

$$p = \frac{\eta}{(1-|a|)^2}$$

[the value of p lies in the interval (0, 1)], we obtain

$$H_0 \sim \frac{\mathrm{i}\,\mathrm{sign}\,\gamma}{L\sqrt{p(1-|a|)}}\sqrt{1+\sqrt{1-p}}\,(G+V_\phi),$$

$$\theta_0(|a|=1) = \frac{\phi}{2} - \frac{\pi}{4},$$

$$\frac{2}{kw_1^2} \sim \frac{(1-|a|)^{3/2}}{L|\gamma|}\sqrt{\frac{p(1-p)}{1+\sqrt{1-p}}},$$

$$\frac{2}{kw_2^2} \sim \frac{2|\gamma|}{L}\sqrt{\frac{1+\sqrt{1-p}}{p(1-|a|)}}.$$

Under the assumption that the value of $|\gamma|$ is bounded away from zero, the second eigenvalue for $a\to 1$ tends to infinity, and the first – to zero (note that it contains an additional factor $[p(1-p)]^{1/2}$, vanishing at the boundaries). In the case of small values of $|\sin\phi|$, the factor $|\gamma|$ is also small, and therefore the uncertainty arises in the expressions for the eigenvalues, so that at certain ratios of parameters it might be that the first eigenvalue is large, and the second is small, or that they are small simultaneously. In any case, however, their product for $a\to 1$ tends to zero, and the area of the intensity ellipse – to infinity.

Finally, the last case – the large absolute values of γ, δ, belonging to the stability region, when the value of $\cos\phi$ is close to zero, and lying (see Fig. 4) in small neighbourhoods of the bisectors of the coordinate angles, as well as of hyperbolae $\gamma^2-\delta^2=\pm 1$ for which the bisector are asymptotes. Of course, these points are close also to the stability boundaries (6) and (7) lying between these hyperbolae. Between the boundaries the value of η rapidly changes from zero to $(1-|a|)^2$, and with the matrix itself H_0 quickly changes. The eigenvalue

$$\frac{2}{kw_1^2} \sim \frac{d\sqrt{\eta}}{L|\gamma|t'}$$

in the region under study is small and vanishes on both boundaries, and the eigenvalue

$$\frac{2}{kw_2^2} \sim \frac{|\gamma|t'}{L\sqrt{\eta}}$$

changes from the value (33) on the surface (7) to infinity on the surface (6). Although expression (33) contains a large cofactor $|\gamma|$, the value of this expression in the case under study is not obligatory large (in particular, at $\cos\phi=0$ it vanishes). Nevertheless, in any case in the considered region the eccentricity of the intensity ellipse is always large, the angle θ_0 is close to zero, and the matrix $\mathrm{Im}\,H_0$ is close to diagonal (more precisely, one of the diagonal elements is much larger than other elements of this matrix).

8. Let us square the monodromy matrix T_0 (4):

$$T_0^2 = \begin{pmatrix} 2G'G - E & 2LG' \\ 2(GG'G - G/L & 2GG' - E \end{pmatrix}, \tag{35}$$

where $G' = V_\phi G V_\phi$.

It is easy to verify that matrix (35) coincides with the monodromy matrix of a linear resonator (of the length L) bounded by two identical elliptical mirrors, whose principal directions of curvature have the angle ϕ with respect to each other, and the values of the principal curvatures coincide with the eigenvalues of the matrix $\Psi/2$ [26, 27]. This means that the Gaussian beam traversing a linear resonator is converted in the same way as during two round trips in the ring resonator, so that any solution for the ring resonator simultaneously describes the field in the corresponding linear resonator. The converse is true if all the eigenvalues of matrix (4) are different from $\pm\mathrm{i}$: in this case, all the eigenvectors of matrix (35), through which the solutions of equation (1) can be expressed are also eigenvectors of matrix (4), because the corresponding eigenvalues of matrices (4) and (35) have the same multiplicity. If the eigenvalues of matrix (4) (simple or double) are equal to $\pm$i, matrix (35) has the eigenspace with the dimension 2 or 4, respectively, corresponding to the eigenvalue -1, then equation (1) for matrix (35) has solutions that do not satisfy a similar equation for matrix (4). These solutions describe the beams, which are reproduced not after a single but after a double trip in the considered ring resonator, and then the mode is a superposition of two Gaussian beams that switch back and forth between each other while traversing the resonator.

9. Consider the case when the reflecting mirror is spherical. Figure 7 shows the dependence of L/R on the eigenvalues of the dimensionless matrix $L\mathrm{Im}H_0$ and the angles of inclination $\theta_{1,2}$ to the coordinate axis x of the corresponding eigenvectors (axes of the intensity ellipse) at constant ϕ and α for the above case, when the straight line $\alpha = \mathrm{const}$ in Fig. 5 intersects both stability subregions. Figure 7 illustrates the above-noted qualitative differences between the resonators with an even and odd number of mirrors: in the latter case, the number of stability subregions does not exceed two (not three); at the boundaries of the subregions when one of the eigenvalues vanishes, the other tends to infinity (for an even number of mirrors, it increases sharply, but remains finite); and finally, with the highest value of L/R only one eigenvalue vanishes (with an even number of mirrors – both).

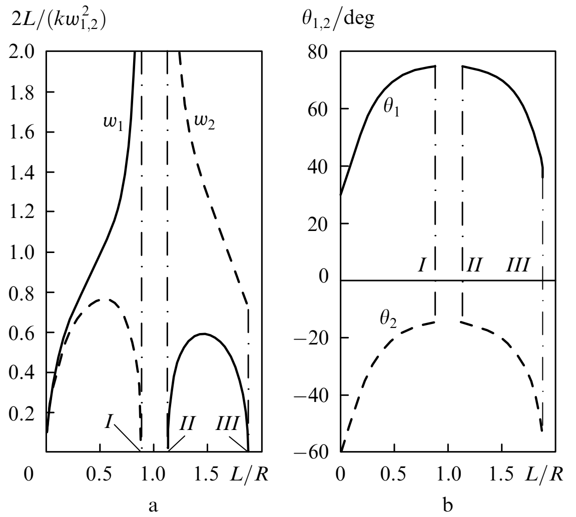

Figure 7. Dependences of eigenvalues of the dimensionless matrix $L\mathrm{Im}H_0$ (a) and angles of incidence of the corresponding eigenvectors to the x axis (b) on L/R ($\alpha = \pi/3$, $\phi = \pi/3$). Roman numerals indicate the values of L/R corresponding to the formulas presented in the caption to Fig. 5.

Figure 8 shows the dependences of similar characteristics of the dimensional matrix $\mathrm{Im}H_0$ on $|\cos\phi|$ for a resonator with an odd number of mirrors, the parameters of which coincide with those of the four-mirror resonator considered in [9, 14]. As seen from Fig. 6, such a resonator is stable for all values of ϕ, which, in the case of an even number of mirrors, is impossible for any values of the parameters.

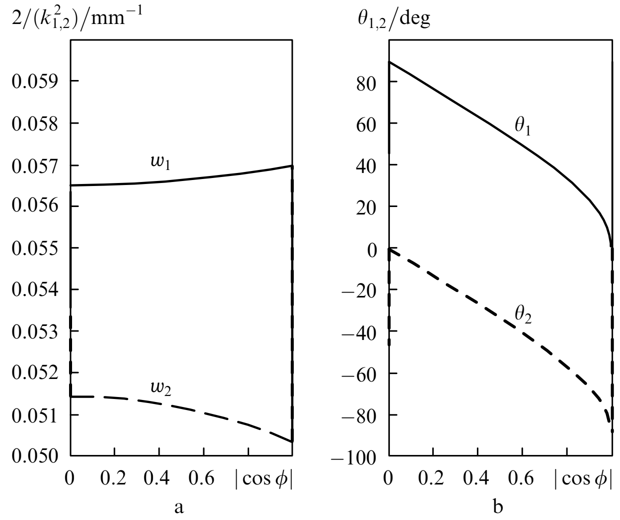

Figure 8. Dependences of eigenvalues of the matrix $\mathrm{Im}H_0$ (a) and angles of incidence of the corresponding eigenvectors to the x axis (b) on $|\cos\phi|$ at $R = 50$ mm, $L \approx 12.22$ mm, $\alpha = \pi/3$ ($\gamma \approx 0.7531$, $\delta \approx -0.0353$, $\phi = \pi/3$). For $\tan\phi < 0$, the values of the angles differ in sign from those presented in this figure.

10. Consider now the evolution of the matrix H when the beam propagates along the resonator contour. The analysis will be based on expression [28]

$$H(z) = \frac{H_+ + (z\det H_+)E}{1 + z\,\mathrm{tr}\,H_+ + z^2\det H_+}, \quad (36)$$

where the matrix $H_+ = H(0)$ is found from (10),

$$\mathrm{tr}\,H_+ = \frac{1}{L}\left[2(\gamma - 1) + \frac{\mathrm{i}|\gamma|t'}{\sqrt{\eta}}\right],$$

$$\det H_+ = \frac{(1-\gamma)^2 - \delta^2 - d}{L^2}\left(1 + \frac{\mathrm{i}t'}{2\sqrt{\eta}}\right),$$

in this case the matrices $H(L)$ and H_- are related by a similarity relation

$$H_- = V_\phi H(L) V_\phi.$$

The beam has a symmetry centre at point $z = L/2$, $x = y = 0$; in this case,

$$H(L/2) = \frac{1}{L[(1+\gamma)^2 - \delta^2 - d]}$$

$$\times\left[4\delta V_{\phi+\pi/2}\sin\phi + \frac{\mathrm{i}t'(1+\gamma^2-\delta^2-d)}{\sqrt{\eta}}(E + \xi V_\phi)\right],$$

$$\mathrm{tr}\,H(L/2) = \mathrm{i}|\mathrm{tr}\,H(L/2)| = \frac{2\mathrm{i}t'(1+\gamma^2-\delta^2-d)}{L\sqrt{\eta}\left[(1+\gamma)^2 - \delta^2 - d\right]}, \quad (37)$$

$$\det H(L/2) = -\frac{4(\gamma d + \delta^2\sin^2\phi)}{L^2\left\{\gamma\left[(1+\gamma)^2 - \delta^2\right] - \delta^2\sin^2\phi\right\}}.$$

It follows from (37) that for $z = L/2$ the major semiaxes of the intensity ellipse are directed at angles $(\phi + n\pi)/2$ to the x axis, and the directions of principal curvatures of the wavefront coincide with the bisectors of the angles between these axes and form angles $\phi/2 + (2n+1)\pi/4$ with the x

axis. The wavefront in this section has the shape of a saddle, the principal radii of curvature of which are equal modulo $L[(1+\gamma)^2 - \delta^2 - d]/(4\delta \sin\phi)$.

The formula similar to (36), allows one to express $H(z)$ through the matrix $H(L/2)$, its trace, and determinant. Multiplying and dividing (36) by the expression that is complex conjugate with the denominator, we can distinguish the real and imaginary parts of the matrix $H(z)$. We do not present these formulas because of their awkwardness; we will write only the expressions for the angles of incidence of the intensity and phase ellipses, $\theta_{\rm Im}$ and $\theta_{\rm Re}$, to the x axis [i.e., the eigenvectors of the matrices $\operatorname{Im} H(z)$ and $\operatorname{Re} H(z)$]:

$$\theta_{\rm Im} = \frac{1}{2}\left\{\phi - \arctan\frac{(2\kappa-1)\left[(1+\gamma)\left(1-\gamma^2+\delta^2+d\right) - 2\delta^2\cos^2\phi\right]\tan\phi}{\gamma\left[(1+\gamma)^2-\delta^2-(2\kappa-1)^2 d\right] - 2[1-2\kappa(1-\kappa)]\delta^2\sin^2\phi} + n\pi\right\}, \tag{38}$$

$$\theta_{\rm Re} = \frac{1}{2}\left\{\phi - \operatorname{arccot}\frac{(2\kappa-1)\left[\gamma\left[(1+\gamma)^2-\delta^2-d\right] - 2\delta^2\sin^2\phi\right]\cot\phi}{\gamma\left[(1+\gamma)^2-\delta^2-(2\kappa-1)^2 d\right] - 2[1-2\kappa(1-\kappa)]\delta^2\sin^2\phi} + n\pi\right\}, \tag{39}$$

where $\kappa = z/L$ [the formulas are given for the internal points of region (5) when all the expressions entering into it are defined].

Angles (38) and (39) are determined up to a term $n\pi/2$ (semiaxes of the ellipses are orthogonal) and are measured from the initial direction of the x axis or the direction in which it goes after one or more reflections from flat mirrors; the direction is determined by the orientation of the coordinate system, i.e., by the number of reflections. Transition to other coordinate systems leads to the addition of a constant term and (with changing the orientation) to a change in the sign in expressions (38), (39).

It is easy to see that the angle $\theta_{\rm Re}(0)$ is multiple of $\pi/2$: the directions of the principal curvatures of the wave front coincide with the coordinate axes. In particular, when $-\pi < \phi < 0$ or $0 < \phi < \pi$, the mentioned angle is equal to $n\pi/2$ or $(n+1)\pi/2$, respectively. In this case, the values of $\theta_{\rm Re}(\kappa = 1)$ are equal to $\phi + (n+1)\pi/2$ and $\phi + n\pi/2$, and $\theta_{\rm Re}(1/2) = \phi/2 + (n+1)\pi/4$ (at any ϕ). It is somewhat more difficult to verify that $\theta_{\rm Im}(0)$ for some n coincides with θ_0 (19), (25); for this, in particular, it is sufficient that $\tan[2\theta_{\rm Im}(0)] = \tan(2\theta_0)$. In the middle of the contour, $\theta_{\rm Im}(1/2) = (\phi + n\pi)/2$. Figure 9 shows the dependences of $\theta_{\rm Im}$ and $\theta_{\rm Re}$ on κ for a resonator with the selected parameters.

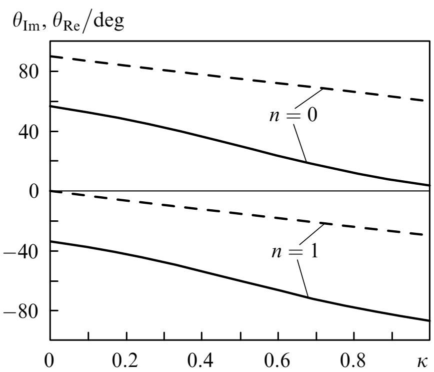

Figure 9. Dependences of the angles of incidence of the semiaxes of the ellipse of intensity $\theta_{\rm Im}$ (solid curves) and phase $\theta_{\rm Re}$ (dashed curves) on $\kappa = z/L$ for $\gamma \approx 0.7531$, $\delta \approx -0.0353$, and $\phi = \pi/3$.

Figure 10 shows the κ dependence of the eigenvalues $2/(kw_{1,2}^2)$ of the matrix $\operatorname{Im} H$ ($w_{1,2}$ are the semiaxes of the intensity ellipse) and the eigenvalues $\rho_{1,2}^{-1}$ of the matrix $\operatorname{Re} H$ ($\rho_{1,2}$ are the principal radii of the wavefront curvature) for the specified resonator. In the middle of the contour, the eigenvalues of the matrix $\operatorname{Im} H$ take maximum values and the values of $w_{1,2}$ are, therefore, minimal. The eigenvalues of $\operatorname{Re} H$ at this point coincide in moulus and differ in sign, and the wavefront has the shape of a saddle. At points located symmetrically with respect to the middle of the contour, one of the eigenvalues vanishes and the shape of the wavefront is cylindrical. Note that the form of these curves is very close to the same curves for a resonator with an even number of mirrors [14]. We do not present explicit analytical expressions for the eigenvalues and for the matrices $\operatorname{Im} H$, $\operatorname{Re} H$ themselves because of their awkwardness.

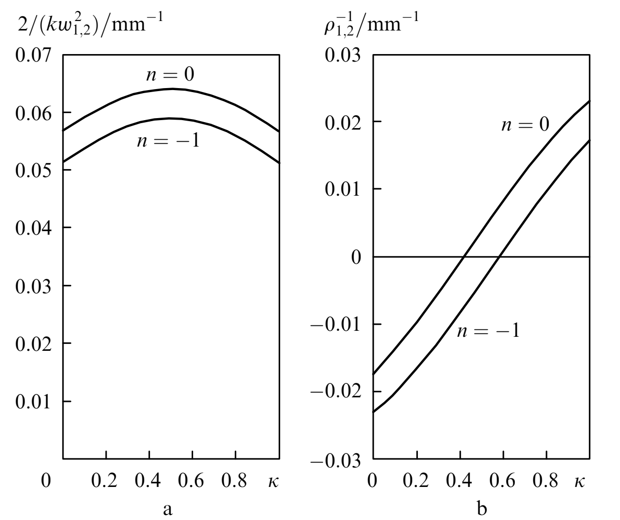

Figure 10. Dependences of the eigenvalues of the matrices $\operatorname{Im} H_0$ (a) and phase $\operatorname{Re} H_0$ (b) on $\kappa = z/L$ for $\gamma \approx 0.7531$, $\delta \approx -0.0353$, and $\phi = \pi/3$

11. Here are a few words about the possible generalisations of the results.

(i) In this study we dealt only with formulas for the fundamental mode. To construct the formulas for the higher modes, it is necessary to write the creation operators, analogous to quantum mechanical ones, which requires the eigenvectors of the monodromy matrix to be determined [1, 3, 20]. It turns out [29] that the expressions for these vectors in the problem under study can be found in explicit form, with substantial assistance from the found matrix H_0: in fact, here use is made of the inverted traditional method in which this matrix is expressed through the components of these vectors. Note that the definition of the eigenvectors of the monodromy matrix for the case of an even number of mirrors is a significantly more complex problem.

(ii) We still have focused our attention on describing the shape of the transverse distribution and have not considered the natural frequencies (more precisely, the eigenvalues of the wavenumber k), since this issue was studied in [1, 3]. Nevertheless, the technique used allows us to somewhat modify the corresponding formulas.

(iii) In this paper, we have restricted our consideration to the principal term of asymptotic expansion of the field in the resonator. Of interest here can be the issues related to the accuracy of the derived expressions, the limits of their applicability, the introduction of correction terms or even a complete asymptotic series – more specifically, related to possible simplifications (with regard to the specific character of the problem) of the procedure of its construction described in [1, 3].

(iv) Our results can be generalised to the case of complex γ and δ, when the reflection coefficient of the astigmatic mirror depends on the transverse coordinates according to the Gaussian law. In need of modification are the formulas containing moduli and signs of the quantities, which are now complex. In this case, it is necessary to reconsider the issues related to the choice of the signs in radicals, which provides a property of having a fixed sign in the matrix $\mathrm{Im}\,H$, issues related to the possibility and uniqueness of this choice (i.e., stability of the resonator with a complex monodromy matrix [18, 19]), the issues about the losses of the fundamental and higher modes in such resonators, the field nonreciprocity, etc. A special analysis is required when the real parts of γ and δ lie outside region (5), as well as near its boundaries (6), (7). Some technical difficulties can also arise in the case of the Gaussian mirror, described by a symmetric complex matrix Ψ of general form, which is not diagonalized by rotation axes, so that the matrix G will not be diagonal.

(v) The analysis shows that the mathematical structure of the problem does not change if an axially symmetric lens is mounted in the middle of the resonator contour. In this case, the monodromy matrix will continue to be of form (4), but with a somewhat different matrix G and a certain effective value L, different from the length of the resonator contour. Note that if the transmittance of the lens is assumed dependent on the radius according to the Gaussian law, or if the lens is replaced by a Gaussian aperture, not only γ, δ but also L turn to be the complex values.

(vi) The following natural generalisation is the problem, in which the specified lens is astigmatic. In the particular case when the matrix describing the lens commutes with the matrix V_ϕ, this problem is very close to that considered in [26, 27] (the problem of a two-mirror resonator with different directions of principal curvatures) and is not too superior to the latter in terms of complexity.

(vii) We have considered the problem in a scalar formulation; however, no difficulties arise during its reformulation for Maxwell's equations. In the main approximation the resulting electromagnetic wave is transverse and a plane-polarised (unlike the case of an even number of mirrors, where the polarisation is circular [10]), since the polarisation vector should obviously be an eigenvector of the matrix V_ϕ. Each of these two vectors generates its own series of natural frequencies of the resonator.

12. Let us sum up the main results of this study.

In the present work, we studied the ring resonators with a nonplanar axial contour and an odd number of mirrors, one of which is nonplanar. In such resonators, as a result of the round trip of light in the resonator contour the orientation of the coordinate system changes, i.e., this system is reflected with respect to some direction, not related, in general, with the principal directions of the curvature of nonplanar mirror. The resultant fundamental mode in this case has the form of a Gaussian beam with astigmatism, of general type. We investigated in detail the geometry of the stability region of the resonator in the space of dimensionless parameters that determine the resonator properties. We presented the explicit expressions for the quadratic matrix describing the transverse field distribution of the fundamental mode for all admissible parameters. We analysed in detail the behaviour of the matrix and the character of the field in the vicinity of the boundaries of the stability region and specific values of the parameters. We studied the singular points and surfaces in the stability region for which the quadratic matrix is not uniquely determined, as well as the behaviour of the matrix in the vicinity of these points and surfaces. We investigated the dependence of the transverse field distribution on the longitudinal coordinate and presented explicit expressions for the angles of incidence of the semiaxes of the intensity and phase ellipse as a function of this coordinate.

Results are specified for the case when the nonplanar mirror is spherical. For such resonators, we investigated the dependences of the beam characteristics on the radius of curvature, angle of incidence, and direction with respect to which reflection of the coordinate system takes place during the trip of light in the resonator.

The results obtained showed sharp qualitative differences between the resonators with an odd and even number of mirrors (the latter case was investigated earlier [14]). These differences relate to both forms of the stability region (in particular, the number of subregions into which this area splits does not coincide) and the transverse field distribution. In particular, the semiaxes of the intensity ellipse and phase ellipse are directed along the principal directions of the curvature of nonplanar mirrors in the resonator with an even number of mirrors and with an odd number of mirrors, respectively. Common is only that in both cases the fundamental mode is a Gaussian beam with general astigmatism, symmetrical with respect to the middle of the contour.

As an example, we used a resonator whose parameters coincide with those of the resonator with an even number of mirrors (considered in [9, 14]). In this case, we found an unexpected similarity, which is not only qualitative but also quantitative, in the shape of beams formed in the resonators with an odd and even number of mirrors.

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