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Lyapunov method and analysis of the emerging repetitively pulsed regime in semiconductor lasers with delayed feedback

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Abstract. **We study the nature of transition from stationary lasing to pulse-periodic oscillations when the phase of the delayed feedback and the diode laser pump current change. The appearance of oscillations can take place under the scenarios of soft or hard excitation of oscillations. We propose a semi-analytical approach to identify the nature of the transition and to determine the dynamic characteristics and stability of the arising spike regime with a change in the external parameters. Direct integration of the Lang–Kobayashi equations showed an acceptable accuracy of this approach.**

Keywords: *semiconductor laser, lasing dynamics, delayed feedback, bifurcation.*

1. Introduction

Semiconductor lasers with delayed feedback produced by reflecting a part of radiation from the external mirror attract attention due to a variety of lasing regimes [1]. The abundance of dynamic regimes in such a laser is due to the interference of the field reflected from the semiconductor face, which serves as an internal mirror, with a retarding field returned by the external mirror. The field established in the laser, in general, depends on the given initial field distributions and the population inversion in the time interval determined by the delay. Formally, this situation corresponds to the infinite-dimensional phase space. Even in this case, there are some asymptotic solutions to which stable laser generation approaches at almost any initial distributions. Several of these solutions can serve as a basis for the analysis of different regimes. The resultant complex dynamics is described by relatively simple equations [2] of the well-known Lang–Kobayashi (LK) model. Despite its apparent simplicity, the equations describe a number of phenomena occurring in the dynamics of diode lasers (DLs): a sequence of period-doubling bifurcations upon transition to chaos [3], as well as the effect of phase-locking of lasers through the exchange of radiation, including in the chaotic oscillation regime [4]. This effect, called the synchronisation of chaotic lasers, is used in the optical cryptography systems being developed [5]. The use of lasers with delayed feedback in cryptographic systems offers some advantages because the latter can generate hyperchaos, i.e., high-dimensional chaos [6]. There are several theoretical works [7, 8], devoted to the numerical study of nonlinear dynamics of lasers with delayed feedback, which investigate both the well-known phenomena (bistability) and unusual regimes, such as long-wavelength fluctuations. The latter are primarily associated with the development of instability at the moments when the radiation intensity is low. However, deep understanding of the processes causing the different dynamic regimes is still far away. The experiments illustrate the complex picture of dynamic regimes, but do not allow one to predict the behaviour of lasers with known parameters.

Lasers with delayed feedback can operate in a steady-state regime. The steady-state regime can be violated through bifurcation, accompanied by the appearance of oscillations. In the case of the classical Hopf bifurcation, there occurs soft excitation of harmonic oscillations whose amplitude increases away from the bifurcation point. In [9] we consider another type of bifurcation, when hard excitation of oscillations accompanied by a transition to the spike regime is realised, and show that the description of the behaviour of the system near this boundary may be reduced to solving an algebraic cubic equation whose coefficients depend on the known physical parameters of the laser and feedback. With the parameters changed, the number of real roots of the found cubic equation can vary from three to one, which in terms of the catastrophe theory corresponds to a cusp catastrophe [10]. The Hopf bifurcation corresponds to the variant, when there are three roots. One of the roots determines the frequency of small oscillations of the system. In [9] we show that in the case of one root the transition is hard. This means that at the bifurcation point on the phase plane there is an attracting limit cycle of finite radius.

A key feature of the DLs, resulting in a variety of dynamics, is the delayed feedback. The most complete description of the theory of difference-differential equations is given in [11–14].

In this paper, we study the regimes of established generation within the framework of the LK equations. Analysis of steady-state and regular dynamic solutions is performed in the linear approximation for perturbations. For the established oscillations, the perturbations are described by linear differential equations with periodic coefficients. Periodicity is a very strong property that is used in the Lyapunov theory of reducibility of the equations to simple ones. Therefore, the delayed feedback effect in the regime of established oscillations can be described in terms of an effective delayed feedback. Given the simplicity of arising oscillations, we derive a periodical solution near the bifurcation point from the steady-state solution. From the linear problem we can get a system of two transcendental characteristic equations for the repetition rate of oscillations. From the condition of their compatibility

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and the equation for the effective delayed feedback factor we can find the characteristics of the nonlinear solution.

2. Basic equations of the model

The LK equations describe the lasing dynamics in a diode laser with external delayed feedback (Fig. 1). Lang and Kobayashi [2] took into account the delayed feedback effect by introducing the effective reflection coefficient $r_{\rm eff}$ at the laser crystal face end. The field $E_{\rm r}$ on this mirror (for a wave with a frequency Ω) is formed due to reflection of the wave E from the first boundary with an amplitude reflection coefficient r and (with some phase delay) from the second, external, boundary

$$E_{\rm r}{\rm e}^{{\rm i}\Omega t} = r_{\rm eff}E{\rm e}^{{\rm i}\Omega t} = [r + (1 - r^2)r_{\rm m}{\rm e}^{-{\rm i}\Omega\tau_{\rm d}}]E{\rm e}^{{\rm i}\Omega t}.$$

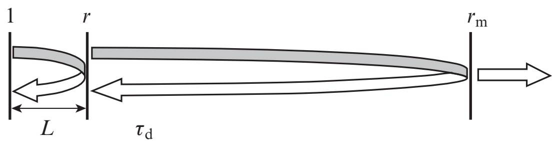

Figure 1. Diode laser with delayed feedback. The length of the laser cavity is L; the left mirror is highly reflecting; the right mirror has the reflection coefficient r; the external – $r_{\rm m}$. The signal delay time is $\tau_{\rm d}$.

The amplitude reflection coefficient $r_{\rm m}$ from the external mirror is assumed small. Then, the passive losses in the laser cavity of length L, taking into account the signal delay by time $\tau_{\rm d}$, is given by

$$\Pi = \frac{c}{2nL}\ln\frac{1}{r_{\rm eff}^2} = \frac{c}{nL}\ln\frac{1}{r[1 + (r_{\rm m}/r)(1 - r^2){\rm e}^{-{\rm i}\Omega\tau_{\rm d}}]}$$

$$= \frac{c}{nL}\ln\frac{1}{r} - 2M_{\rm d}{\rm e}^{-{\rm i}\Omega\tau_{\rm d}}.$$

Losses on a mirror in the LK model (first term) determine the photon lifetime and are considered below as distributed losses. The second term gives the contribution of the external mirror with allowance for the time delay of the reflected signal. For a laser with a cavity of length L and the amplitude reflection coefficient r from the crystal face, the photon lifetime in the medium with the speed of light c and the group refractive index n is defined as $\tau_{\rm ph}^{-1} = (c/n)(L^{-1}\ln r^{-1})$. Because the reflection from the external mirror is small and $\ln(1 + x) \approx x$, we have $M_{\rm d} = (c/2nL)(1 - r^2)(r_{\rm m}/r)$.

Let us position the origin of the coordinate system (z', t') on a highly reflecting mirror. Using the substitution $z = z'n/c\tau_{\rm ph}$, $t = t'/\tau_{\rm ph}$, we will pass to dimensionless coordinates. Then, the LK equations have the form ($z_L \equiv nL/c\tau_{\rm ph} = \ln r^{-1}$):

$$\frac{\partial E(t,z)}{\partial t} + \frac{\partial E(t,z)}{\partial z}$$

$$= (1 - {\rm i}R)NE + {\rm i}M{\rm e}^{{\rm i}\kappa}E(t - \tau_{\rm d}, z - \tau_{\rm d}), \tag{1}$$

$$T\frac{\partial N}{\partial t} = P - N - (2N + 1)I(t,z),$$

$$I(t,z) = |E(t,z)|^2{\rm e}^z + |E(t, 2z_L - z)|^2{\rm e}^{-z}, \quad 0 < z < z_L.$$

The first of the equations describes the behaviour of the slowly varying field amplitude envelope $E(t,z)$ in the form of a travelling wave with a round-trip in the cavity L [this imposes periodic boundary conditions $E(t, z + \tau_{2L}) = E(t,z)$, $\tau_{2L} = 2nL/c\tau_{\rm ph}$ is the a round-trip time of a laser beam in the cavity as a fraction of $\tau_{\rm ph}$], the second equation describes the time dynamics of population inversion N, caused by the evolution of the field intensity. Feedback in the equation is distributed; M is the dimensionless absolute value of the coupling constant module, which is determined by the amplitude coefficient of reflection from the external mirror $r_{\rm m}$: $M = (1 - r^2)(r_{\rm m}/r)/\tau_{2L}$; parameter κ is the coupling constant phase, which determines the phase incursion at a carrier frequency of the field in the loop with delayed feedback.

In the second equation the field intensity I is the total intensity of two counterpropagating waves, the two mirror points z and $\tau_{2L} - z$ coincide spatially (using the algebra modulo $\tau_{\rm 2L}$).

The field gain G in the linear approximation near the threshold is proportional to the population inversion N. The dimensionless quantity N is expressed through the carrier concentration $N_{\rm c}$: $N = 0.5g\tau_{\rm ph}(N_{\rm c} - N_{\rm th})$, where g is the differential material gain $\{g = (c/n)\,\partial G/\partial N_{\rm c}$ [μm^3 ps^{-1}]$\}$. The threshold carrier density $N_{\rm th} = N_{\rm tr} + (g\tau_{\rm ph})^{-1}$ is determined by radiation losses at the facets of the crystal and by distributed losses. The latter can be described using the concept of effective transparency density $N_{\rm tr}$, which additionally includes passive losses in the volume. Next, we introduce $P = 0.5g\tau_{\rm ph}(j\tau_{\rm s} - N_{\rm th})$, the normalised pump intensity above the threshold, where $\tau_{\rm s}$ is the lifetime of carriers in the absence of stimulated transitions; j is the injection rate of carriers in a quantum well of thickness d. The dimensionless time is introduced through normalisation by the photon lifetime $\tau_{\rm ph}$. In particular, the dimensionless time of the inversion relaxation is defined as $T = \tau_{\rm s}/\tau_{\rm ph}$. A typical value is $T \approx 1000$.

In DLs the gain G is large, as a rule, so that the amplitude reflection coefficient of the output mirror is typically less than unity (~0.5). In the numerical integration of equations (1), there arise difficulties associated with the need to account for the discontinuity in the field amplitude on the right-hand mirror of the DL cavity. In view of the counterpropagating waves, the field intensity is $I = I_+{\rm e}^z + I_-{\rm e}^{-z}$, where the waves $I_\pm$ on the mirrors are transformed into each other with preserving the smoothness. The value of the field intensity discontinuity on the right-hand mirror is found (taking into account the smallness of the feedback) from the condition $J(z_L) = Pz_{\rm L}$, where $J = I_+{\rm e}^z - I_-{\rm e}^{-z}$.

For long delay times $\tau_{\rm d}$, compared to $\tau_{\rm 2L}$ the field distribution in the cavity changes in time as a whole, because the time of the field establishment is of the same order as the photon lifetime, and the oscillations of perturbations due to feedback have the scale associated with the delay time. Therefore, with an accuracy up to the shape of the field inside the cavity and the intensity normalisation, the dynamics of perturbations produced by the feedback can be studied by specifying the intensity with the help of a simple formula $I(t,z) = |E(t,z)|^2$. This formula is convenient for analytical consideration, and comparison with the calculations of the full model show the coincidence of the dynamics with the behaviour of the solutions obtained for a zero-dimensional LK model. Analysis provided in the Appendix gives the necessary conditions for the applicability of this model. Comparison of the frequencies of the increments of growth of perturbations without delayed feedback with the frequency ω for the perturbation oscillations due to feedback provides an additional condition $\omega T \gg 1$, under which the dynamics of solutions of equations (2) with distributed losses is close to the dynamics of the zero-dimensional model. Here, ω is the dimensionless frequency normalised to the inverse photon lifetime.

The steady state (SS) is determined by the conditions [1]: $\partial E(\tau,z)/\partial z = \mathrm{i}\beta$ and $\partial N/\partial\tau = 0$. Then, by substituting $E = E_{\mathrm{s}}\exp(\mathrm{i}\beta z + \psi(\tau,z))$, the first equation of system (1) in a moving coordinate system ($\tau = t - z$, $z = z$) is transformed to

$$\mathrm{i}\beta + \frac{\partial}{\partial z}\psi(\tau,z) = (1 - \mathrm{i}R)N$$

$$+\mathrm{i}M\mathrm{e}^{-\mathrm{i}\chi}\exp[\psi(\tau,z-\tau_{\mathrm{d}}) - \psi(\tau,z)],$$

where a new phase of the feedback is $\chi = \beta\tau_{\mathrm{d}} - \kappa$. Let the stationary value N be equal to N_0; then, it follows from the second equation of system (1) that $P - N_0 = (2N_0 + 1)E_{\mathrm{s}}^2$. The amplitude describing the slowly varying envelope of the wave field has the frequency detuning from the carrier frequency (it follows from the equality to zero of real components that $N_0 = -M\sin\chi$):

$$\mathrm{i}\beta = (1 - \mathrm{i}R)N_0 + \mathrm{i}M\mathrm{e}^{-\mathrm{i}\chi} = \mathrm{i}M(\cos\chi + R\sin\chi)$$

$$= \mathrm{i}M\sqrt{1+R^2}\sin[\chi + \arctan(1/R)].$$

The equations for ψ and $n = N - N_0$ (note that $n \ll 1$ because $M \ll 1$) take the form

$$\frac{\partial}{\partial z}\psi(\tau,z) = (1 - \mathrm{i}R)n$$

$$+iM\mathrm{e}^{-\mathrm{i}\chi}\{\exp[\psi(\tau,z-\tau_{\mathrm{d}}) - \psi(\tau,z)] - 1\}, \quad (2)$$

$$\frac{\partial n}{\partial\tau} = -\frac{n}{T_1} - \frac{\omega_{\mathrm{r}}^2}{2}[\exp(2\,\mathrm{Re}\psi) - 1].$$

We introduced here

$$\omega_{\mathrm{r}}^2 = 2\frac{P - N_0}{T} \text{ and } \frac{1}{T_1} = \frac{1}{T}\frac{1+2P}{1+2N_0} \approx \frac{1}{T}.$$

Note that the gain is concentrated inside the diode cavity, the cavity round-trip length is $2L \sim c\tau_{\mathrm{ph}}$ and much smaller than the external cavity length $c\tau_{\mathrm{d}}$. Account for the circular boundary conditions (in the laboratory coordinate system) for the field inside the crystal makes it possible to consider the solution to be periodic along the spatial coordinate z (with a period $2z_L$) and arbitrary changing with time. Formally, the solution is sought for on the surface of the cylinder. The sweep of this integration region is a band of width $2z_L$ with an edge along the line $t - z = \mathrm{const}$ in the laboratory coordinate system. Therefore, in numerical calculations of the equations it is necessary to store the data acquired during an interval of the delay time $\tau_{\mathrm{d}}/\tau_{\mathrm{ph}}$ in the computer memory.

3. Perturbation analysis taking into account the delay

The properties of nonlinear equations (2) can be studied by switching to the analysis of linear equations for the perturbations. We deal here with the replacement $\psi \to \psi + \delta\psi$, $n \to n + \delta n$, when the initial exact solutions of the nonlinear system are varied to obtain a linear system of equations for small perturbations. As is known [15], in the absence of delayed feedback, equations (2) have a stable stationary solution $\psi = 0$. Initial perturbations decay with time. The shape of oscillations (in the case of a low decay rate) can be obtained analytically. Feedback changes the behaviour of the system and its phase portrait. In particular, there may emerge new dynamic solutions, which under certain conditions become stable, and their orbits in phase space are attracting. Such transitions speak of restructuring of the entire solution of the system. Different types of solutions are separated by bifurcation points.

Since the feedback leads to the restructuring of the solution even in the case of a small feedback factor, the account for its influence cannot be correct within the finite-order perturbation theory. To take into account the perturbations in all orders, use is made of Green's function. Because we investigate the oscillations near the frequency of relaxation oscillations ω_{r}, which are much larger than the inverse times of inversion relaxation T^{-1}, then under these assumptions the zero-dimensional model is valid ($\partial\psi/\partial z \to \partial\psi/\partial\tau$):

$$\frac{\partial}{\partial\tau}\delta\psi = (1 - \mathrm{i}R)\delta n + \mathrm{i}M\exp[-\mathrm{i}\chi + \psi(\tau - \tau_{\mathrm{d}}) - \psi]$$

$$\times[\delta\psi(\tau - \tau_{\mathrm{d}}) - \delta\psi], \quad (3)$$

$$\frac{\partial\delta n}{\partial\tau} = -\frac{\delta n}{T} - \omega_{\mathrm{r}}^2\exp(2\,\mathrm{Re}\psi)\delta\mathrm{Re}\psi.$$

The homogeneous part of this system of equations does not contain delayed feedback and, therefore, is simple enough. Its solution can be obtained in an explicit form, and the particular solution of (3) can be derived by the method of variation of constants. The complete solution of the system is an integral equation whose kernel (Green's function) depends on the trajectory ψ. The properties of the solutions depend strongly on the spectral properties of Green's function [16]. It follows directly from (3) that the relaxation term leads to a temporal decay of the homogeneous solutions; therefore, only a partial solution of the inhomogeneous system determines the dynamics on large time intervals. The particular solution is an integral between two instants of time (the upper limit is the current time). The upper limit specifies the behaviour of the solution, while the bottom limit is the solution of the homogeneous system and disappears in the asymptotic limit.

Since one of the equations is the complex, instead of two equations we obtain three real equations. For the real and imaginary parts $\delta\psi$ we introduce the notations $\delta x = \delta\,\mathrm{Re}\psi$, $\delta y = \delta\,\mathrm{Im}\psi$, and for the inversion perturbations we keep the notation δn. In equation (3) the terms in the homogeneous part of the system determine the form of the matrix A; the matrix B takes into account only the contribution of delayed feedback:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -R \\ -\omega_{\mathrm{r}}^2\mathrm{e}^{2\mathrm{Re}\psi} & 0 & -T^{-1} \end{pmatrix},$$

$$B(\tau) = M\mathrm{e}^{x(\tau-\tau_{\mathrm{d}})-x}\begin{pmatrix} \sin\zeta & -\cos\zeta & 0 \\ \cos\zeta & \sin\zeta & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\zeta = \chi + y - y(\tau - \tau_d)$. Then, the solution of the linearised system is written in the form of an integral equation, which is derived from the solution of the homogeneous system by using the method of variation of constants [17] (for the established periodic solutions the external integral is indefinite):

$$\begin{pmatrix}\delta x(\tau)\\ \delta y(\tau)\\ \delta n(\tau)\end{pmatrix} = \int^{\tau} \mathrm{d}t\,\widehat{T}\left(\exp\left[\int_t^{\tau}\mathrm{d}t' A\right]\right)B(t)\begin{pmatrix}\delta x(\tau-\tau_d)-\delta x(t)\\ \delta y(\tau-\tau_d)-\delta y(t)\\ 0\end{pmatrix}, \quad (4)$$

here $\widehat{T}(\exp)$ is the chronological exponent [18]. In the case of the stationary solution*, both matrices

$$A_0 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & -R\\ -\omega_r^2 & 0 & -T^{-1}\end{pmatrix} \text{ и } B_0 = M\begin{pmatrix}\sin\chi & -\cos\chi & 0\\ \cos\chi & \sin\chi & 0\\ 0 & 0 & 0\end{pmatrix}$$

are constant; therefore,

$$\int_t^{\tau}\mathrm{d}t' A = (\tau - t)A_0.$$

The behaviour on the periodic attractor is determined by the integration region near the upper limit, when $t \to \tau$. On the time interval where transient processes play the role, the oscillation frequency varies smoothly, starting from the frequency of relaxation oscillations. In the asymptotic limit the phase trajectory goes to the periodic attractor. For the established periodic oscillations, the Lyapunov theory for the reducibility (solvability) of equations are valid. The initial system of differential equations belongs to the class of equations with periodic coefficients. Any system with periodic coefficients is reducible (see Ref. [19]), i.e., it can be converted to a system of equations with constant coefficients using some (nonlinear) Lyapunov transformation. Derivation of the statement is based on the most common assumptions associated with the periodicity; the Lyapunov transformation itself is not defined. For the integral equation (4) with constant matrices A_0 and B_0, the Lyapunov transformation corresponds to Green's function $\exp[A_0(\tau - t)]$. Thus, the problem is solvable and reduced to algebraic.

Analytic properties of the operator kernel in an infinitely small vicinity of the point at the upper integration limit of (4) form the dynamic characteristics of the system oscillations directly on the periodic attractor. Before taking the limit $t \to \tau$, it is necessary to expand the solution vector in the eigenvectors A and integrate it with the corresponding eigenfunction:

$$\begin{pmatrix}\delta x\\ \delta y\\ \delta n\end{pmatrix} = \int^{\tau}\mathrm{d}t\exp[A_0(\tau - t)]B_0\begin{pmatrix}\delta x(t-\tau_d)-\delta x\\ \delta y(t-\tau_d)-\delta y\\ 0\end{pmatrix}. \quad (5)$$

Of the three eigenvectors of the matrix A_0, the two form a two dimensional complex subspace, the third – a one-dimensional real subspace with a zero eigenvalue. The half-sum of the first two vectors, along with the third vector create a real two-dimensional subspace. The third row of these vectors is zero in the limit $t \to \tau$, which is a prerequisite, because the bottom row of the matrix B_0 is zero. Thus, for $\exp[A_0(\tau - t)]$ the eigenvectors and eigennumbers have the form:

$$\frac{1}{2}\begin{pmatrix}1\\ -R\\ \pm\mathrm{i}\omega_r\end{pmatrix}\exp[\pm\mathrm{i}\omega_r(\tau - t)], \begin{pmatrix}0\\1\\0\end{pmatrix} \to \left(\begin{pmatrix}1\\ -R\end{pmatrix}\cos[\omega_r(\tau-t)] \atop (-\omega_r)\sin[\omega_r(\tau - t)]\right), \begin{pmatrix}0\\1\\0\end{pmatrix}$$

(because $\omega_r \gg 1/T$, we assume $1/T \to 0$).

Of interest are the bifurcation points in which there appear oscillation perturbations not decaying in time. For oscillations at the frequency ω, the general form of $\delta\psi$ is proposed in [20]. Without loss of generality, the real part of the solution is sought for in the form $\delta x = \cos(\omega t)$. In this case, the general form of the imaginary component has the form $\delta y = \alpha\cos(\omega t) + \beta\cos(\omega t + \vartheta)$, where $\vartheta = -\omega\tau_d/2$. The solution splits on the plane into two vectors: one is a mode $\propto \cos(\omega t)$, the other is a phase-shifted mode, $\cos(\omega t + \vartheta)$. For each of the modes we derive a separate equation. To solve these equations, we expand the first vector in the eigenvectors of the matrix A, transform the formulas of the product of trigonometric functions, and after integration and substitution of the variable at the upper limit of the indefinite integral, we obtain

$$\begin{pmatrix}1\\ \alpha\end{pmatrix} = 2\frac{\beta}{\omega}M\sin\vartheta\left\{\begin{pmatrix}\cos\chi\\ -R\cos\chi\end{pmatrix}\frac{\omega^2}{\omega^2-\omega_r^2} + \begin{pmatrix}0\\ R\cos\chi - \sin\chi\end{pmatrix}\right\}. \quad (6)$$

Similarly from the equation for the second vector:

$$\begin{pmatrix}0\\ \beta\end{pmatrix} = \frac{2}{\omega}M\sin\vartheta\left\{\begin{pmatrix}\sin\chi\\ -R\sin\chi\end{pmatrix}\frac{\omega^2}{\omega^2-\omega_r^2} + \begin{pmatrix}0\\ \cos\chi + R\sin\chi\end{pmatrix}\right\}$$
$$+\frac{2}{\omega}M\sin\vartheta(\alpha + 2\beta\cos\vartheta)$$
$$\times\left\{\begin{pmatrix}-\cos\chi\\ R\cos\chi\end{pmatrix}\frac{\omega^2}{\omega^2-\omega_r^2} + \begin{pmatrix}0\\ \sin\chi - R\cos\chi\end{pmatrix}\right\}. \quad (7)$$

It follows from the first row of these conditions that $\tan\chi = \alpha + 2\beta\cos(\omega\tau_d/2)$ and $\omega^2 - \omega_r^2 = -2\beta\omega M\sin(\omega\tau_d/2)\cos\chi$. Analysis of the second row together with the first formula gives the expression for the coefficient $\beta = [-2/(\omega\cos\chi)]M\sin(\omega\tau_d/2)$. Removing β from the second formula, we have the first characteristic equation (obtained previously in [20]):

$$\omega^2 - \omega_r^2 = 4M^2\sin^2(\omega\tau_d/2). \quad (8)$$

From the second row for the first harmonic we derive the second characteristic equation:

$$2\omega^2\sin\chi + 2M\omega\sin(\omega\tau_d) = \omega_r^2(\sin\chi - R\cos\chi). \quad (9)$$

Thus, at the bifurcation point there appear some transcendental conditions for the frequency of sought-for oscillations. The unknown parameter ω is one; however, there are two conditions. Overdetermination of the number of equations is apparently due to the requirement of real frequencies. The mathematical reason for this peculiarity lies in the presence of the function $\mathrm{Re}\psi$ in the original system of LK equations (2). From general considerations we can assert that the joint solution of equations (8), (9) is not always possible. Note that the well-known Hopf bifurcation is characterised by the compatibility of these two equations.

* In the general case, the commutators that appear because of the chronological operator before the exponential, lead to corrections of higher order of smallness in $(\tau - t)$, their contribution is not significant when $t \to \tau$.

To find the position of bifurcation points, we will use (5) by specifying the perturbations in the form of exponentials: $\delta x = \exp(\gamma t)$ and $\delta y = \alpha \exp(\gamma t)$. Then, it follows from (5) that

$$\gamma\begin{pmatrix}1\\ \alpha\end{pmatrix} = M(\mathrm{e}^{-\gamma\tau_\mathrm{d}} - 1)\left[\frac{\gamma^2}{\omega_\mathrm{r}^2 + \gamma^2}(\sin\chi - \alpha\cos\chi)\begin{pmatrix}1\\ -R\end{pmatrix}\right.$$

$$\left. + \begin{pmatrix}0\\ \cos\chi + R\sin\chi + \alpha(\sin\chi - R\cos\chi)\end{pmatrix}\right].$$

Thus, for the two unknowns γ and α there are exactly two equations.

For a more correct description of the stability criterion associated with the position of the roots near the imaginary axis of the complex number γ, we should take into account the processes of inversion relaxation in the active medium ($1/T$ is nonzero). This leads to the fact that in the previous formula we deal with a formal replacement:

$$\frac{\gamma^2}{\omega_\mathrm{r}^2 + \gamma^2} \to \frac{\gamma(\gamma + T^{-1})}{\omega_\mathrm{r}^2 + \gamma(\gamma + T^{-1})}.$$

Then,

$$\gamma = M(\mathrm{e}^{-\gamma\tau_\mathrm{d}} - 1)\frac{\gamma(\gamma + T^{-1})}{\omega_\mathrm{r}^2 + \gamma(\gamma + T^{-1})}(\sin\chi - \alpha\cos\chi),$$

$$R\gamma = +M(\mathrm{e}^{-\gamma\tau_\mathrm{d}} - 1)(\cos\chi + R\sin\chi)$$

$$+ \alpha[M(\mathrm{e}^{-\gamma\tau_\mathrm{d}} - 1)(\sin\chi - R\cos\chi) - \gamma].$$

Excluding α, we obtain the equation to find the eigenvalue γ (see [21]):

$$f \equiv [\gamma(\gamma + T^{-1}) + \omega_\mathrm{r}^2] + (M/\gamma)^2(\mathrm{e}^{-\gamma\tau_\mathrm{d}} - 1)^2\gamma(\gamma + T^{-1})$$

$$- (M/\gamma)(\mathrm{e}^{-\gamma\tau_\mathrm{d}} - 1)\{\gamma(\gamma + T^{-1})(\sin\chi + R\cos\chi)$$

$$+ [\omega_\mathrm{r}^2 + \gamma(\gamma + T^{-1})](\sin\chi - R\cos\chi)\} = 0. \qquad (10)$$

The roots of this transcendental equation determine the bifurcation points of the system, in this case, for stationary solutions of the system. If the roots with positive real part are absent, then the solution of the system is stable; otherwise, the solution is unstable. The points, when the right half-plane has the roots, correspond to bifurcation from the stable state to the regime with periodic pulsations. The presence of the roots in this part of the complex plane can be determined by the methods of complex analysis by setting into one-to-one correspondence the roots of expression (10) to the poles on the complex plane for the logarithmic derivative f. The integral in a closed contour enclosing the right half-plane calculated with the residues theory gives the number of the zeros of equation (10) with $\mathrm{Re}\gamma > 0$ (see Fig. 2).

$$n_\mathrm{rt} = 1 + \frac{1}{2\pi\mathrm{i}}\int_{+\mathrm{i}\infty}^{-\mathrm{i}\infty}\mathrm{d}\gamma\,\frac{f'(\gamma)}{f(\gamma)}. \qquad (11)$$

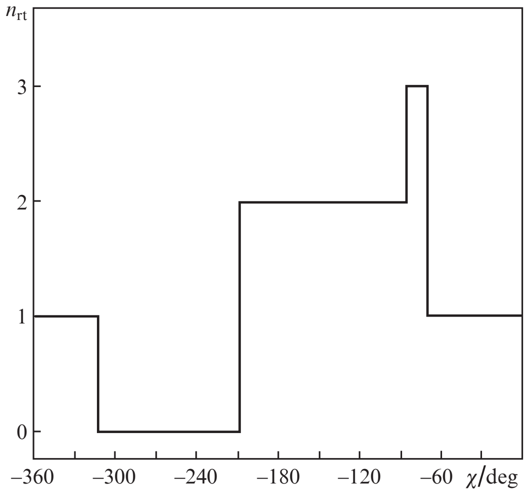

Figure 2. Number of roots of equation (10) in the right half-plane, obtained in calculating the loop integral (11), at the given values $P = 0.8$, $M = 0.02$, $T = 1000$, $\tau_\mathrm{d} = 40$. The bifurcation $n_\mathrm{rt} = 0-2$ investigated below is located in this figure at $\chi = -209°$.

4. Lyapunov principle and single-frequency approximation

It follows from the analysis based on the solutions for the steady states that the loss of stability leads to oscillations. However, the number of conditions per frequency of the latter is overdetermined [see Eqns (8), (9)]. This suggests that data on a nonlinear periodic solution can be derived by studying the linear problem (4). Moreover, we assume that the nonlinear periodic solution $\psi(\tau, z) = x + \mathrm{i}y$ can be represented in the first approximation as a harmonic with the frequency ω against the background of a constant term. The nonlinear solution in general is a spectrum of multiple frequencies ω, which makes, however, the analysis much more intricate. Even in the case of a single frequency function $\psi(\tau, z)$, in equation (4) there appear the harmonics of the fundamental frequency, because this function is contained in the equation in the exponential. Equation (4) was obtained from the linear system of equations (3) for small variations $\delta\psi, \delta n$ relative to the initial nonlinear periodic solutions. This linear system contains periodic coefficients and, according to the Lyapunov principle, is reducible to a simpler system, namely, to a system with constant coefficients. Note that in equation (3) one of the key factors leading to the emergence of periodic coefficients is given by $M\mathrm{e}^{-\mathrm{i}\chi}\exp[\psi(\tau - \tau_\mathrm{d}) - \psi]$. According to the Lyapunov principle of reducibility, there is a couple of the effective feedback constants $M_1\mathrm{e}^{-\mathrm{i}\chi_1}$, which provide mapping into a problem with constant coefficients. For the eigenvalues with a zero real part, the solution has a characteristic frequency ω with all other frequencies being multiple of it. Therefore, it seems natural in the first approximation to calculate the effective feedback factors by performing averaging in time over the period of oscillations. In this approximation we will search for a single-frequency probe signal having a zero growth rate. Similarly, one can calculate the Lyapunov exponent by averaging the value of variations on a large time interval.

Expansion of the exponential of the complex amplitude harmonic in the Fourier series is expressed through the generalised Thompson (Kelvin) functions ber_0, bei_0, which are the analytic continuation of the classical Bessel functions:

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}t \exp(z\mathrm{e}^{\mathrm{i}\varphi}\sin t) = I_0(z\mathrm{e}^{\mathrm{i}\varphi}) = J_0(\mathrm{i}z\mathrm{e}^{\mathrm{i}\varphi})$$

$$= \mathrm{ber}_0(z,\varphi) + \mathrm{i}\,\mathrm{bei}_0(z,\varphi).$$

Further generalisation of the Thompson (Kelvin) functions yields the functions that arise when the exponential functions for two phase-shifted sines are integrated over a period. Performing separately a series expansion for each sine and multiplying the two series, we obtain a new alternating series, where even indices correspond to the expansion of the real part of, and the odd indices – to the imaginary part. Using Graf's addition theorem [22], the series reduces to I_0:

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}t \exp[x\sin t + \mathrm{i}y\sin(t+\xi)]$$

$$= \sum_{k=-\infty}^{+\infty} \mathrm{i}^k I_k(x) J_k(y)\cos(k\xi) = I_0\left(\sqrt{x^2 - y^2 + \mathrm{i}2xy\cos\xi}\right).$$

Therefore, the constants of feedback modulus and phase for oscillations of type $x_1[\sin t + \mathrm{i}\alpha\sin t + \mathrm{i}\beta\sin(t+\vartheta)]$ are renormalised according to the rule

$$M_1\exp(-\mathrm{i}\chi_1) = M\exp(-\mathrm{i}\chi)$$
$$\times I_0\left(2x_1\sin\vartheta\sqrt{(1+\mathrm{i}\alpha+\mathrm{i}\beta\cos\vartheta)^2 + (\mathrm{i}\beta\sin\vartheta)^2}\right), \quad (12)$$

where $\vartheta = -\omega\tau_{\mathrm{d}}/2$.

For the time-dependent matrix A, the two-dimensional subspace of eigenvectors retains the form with changing the eigenvalues. These two eigenvectors have the form

$$\begin{pmatrix}1\\-R\\0\end{pmatrix}\cos\left(\omega_{\mathrm{r}}\int_t^{\tau}\mathrm{d}t'\exp x\right), \begin{pmatrix}0\\1\\0\end{pmatrix}$$

(the third row is zero, where $1/T$ again tends to zero).

In the first-order accuracy $[x = x_0 + x_1\cos(\omega t)]$, we have

$$\int_t^{\tau}\mathrm{d}t'\exp(x - x_0) = \int_t^{\tau}\mathrm{d}t'[I_0(x_1) + 2I_1(x_1)\cos(\omega t')]$$

$$= \{I_0(x_1)(\tau - t) + (2/\omega)I_1(x_1)[\sin(\omega\tau) - \sin(\omega t)]\},$$

$$\exp(2x_0)\overline{\exp[2(x - x_0)]} \approx \exp(2x_0)I_0^2(x_1).$$

According to the second equation of system (2), the time average of the function $\overline{\exp(2x)} \to 1$ for any stable steady-state and periodic solutions. This eliminates the parameter $\exp(x_0)$, and, consequently, the time dependence of the eigenvector

$$\exp\left(\mathrm{i}\omega_{\mathrm{r}}\int_t^{\tau}\mathrm{d}t'\exp x\right)$$

$$= \exp[\mathrm{i}\omega_{\mathrm{r}}(\tau - t)]\exp\{\mathrm{i}p[\sin(\omega\tau) - \sin(\omega t)]\}$$

$$\approx \sum_n J_n^2(p)\exp[\mathrm{i}(\tau - t)(\omega_{\mathrm{r}} + \omega n)].$$

Here, $p = 2[\omega_{\mathrm{r}}I_1(x_1)/\omega I_0(x_1)]$. Therefore, the first eigenvector contains some harmonics providing resonances at frequencies $\omega_{\mathrm{r}} + \omega n$:

$$\begin{pmatrix}1\\-R\\0\end{pmatrix}\sum_n J_n^2(p)\cos[(\tau - t)(\omega_{\mathrm{r}} + \omega n)].$$

Taking into account these corrections, original formula (6), (7) are transformed. The changes concern, first, the feedback factors M, χ, which as a result of renormalisation (12) are transformed into M_1, χ_1. Secondly, the quantity $(\omega^2 - \omega_{\mathrm{r}}^2)^{-1}$ is replaced by another quantity denoted by S:

$$S = \sum_{n=1}^{\infty} n\frac{J_{n-1}^2(p) - J_{n+1}^2(p)}{(n\omega)^2 - \omega_{\mathrm{r}}^2} \approx \frac{J_0^2(p) - J_2^2(p)}{\omega^2 - \omega_{\mathrm{r}}^2}. \quad (13)$$

As a result, the desired quantities are found from the system of transcendental equations:

$$S^{-1} = 4M_1^2\sin^2(\omega\tau_{\mathrm{d}}/2), \quad (14)$$

$$\omega^2(\sin\chi_1 + R\cos\chi_1) + 2M_1\omega\sin(\omega\tau_{\mathrm{d}})$$
$$= 4M_1^2\sin^2(\omega\tau_{\mathrm{d}}/2)(R\cos\chi_1 - \sin\chi_1), \quad (15)$$

$$\frac{M_1}{M}\exp[\mathrm{i}(\chi - \chi_1)] = I_0\left(\frac{2x_1\sin(0.5\omega\tau_{\mathrm{d}})}{\cos\chi_1}\right.$$
$$\left.\times\sqrt{\left[\exp(\mathrm{i}\chi_1) + \mathrm{i}\frac{M_1}{\omega}\sin(\omega\tau_{\mathrm{d}})\right]^2 + \left[\mathrm{i}\frac{2M_1}{\omega}\sin^2(0.5\omega\tau_{\mathrm{d}})\right]^2}\right). \quad (16)$$

In deriving these formulas, we excluded the coefficients α, β by using the expressions

$$\alpha\cos\chi_1 = \sin\chi_1 + 2(M_1/\omega)\sin(\omega\tau_{\mathrm{d}}),$$
$$\beta\cos\chi_1 = -2(M_1/\omega)\sin(\omega\tau_{\mathrm{d}}/2). \quad (17)$$

Equation (14) determines the relationship of the frequency with the oscillation amplitude $\omega[M_1, S^{-1}(x_1)]$ and does not depend on the effective feedback phase. Equation (15), by contrast, explicitly contains both the feedback parameters $\omega(M_1,\chi_1)$. In turn, the effective feedback factors are calculated using equation (16), which uses coefficients (17), obtained by solving a linear problem for the perturbations with respect to the unknown nonlinear solution. Only perturbations with parameters ω, α, β have a zero growth rate, while others (because we search for a steady-state periodic solution) are unstable and decay with time. Therefore, the coefficients obtained by solving a linear problem can be used in deriving the formula for effective feedback. Thus, ω, M_1, χ_1, and the parameter x_1 which determines the size of the real part of oscillations, are calculated from the complete system of equations (14)–(16).

These equations are analysed for $P = 0.8$, $M = 0.02$, $T = 1000$, $\tau_{\mathrm{d}} = 40$ and $\chi = -209°$, corresponding to bifurcation destroying steady-state generation. The feedback phase χ corresponds to a jump 0–2 on the stability diagram of steady states (Fig. 2). The performed numerical calculations revealed three solutions for $0 < x_1 < 3$, of which only one is stable, and can be compared with results of direct dynamic calculations. The stable solution is in good agreement with the oscillation

Table 1. Parameters of nonlinear oscillations and effective feedback.

Parameter	Stable solution	Unstable solution
Oscillation frequency ω	$1.05\omega_r$	$1.67\omega_r$
Oscillation amplitude x_1	0.58	2.9
Effective feedback ratio M_1/M	0.47	1.7
Effective feedback phase $\chi_1 - \chi$	132.8°	39°

frequency 1.07 ω_r, obtained from dynamics calculations in the vicinity of the selected bifurcation point in the regime of periodic oscillations (Fig. 2). Table 1 shows the parameters of oscillations and effective feedback of stable and unstable solutions.

Note that the effective feedback parameters of the mode differ quite significantly from those for unperturbed values: $M_1/M = 0.47$, $\chi_1 - \chi = 132.8°$. This leads to the formation of the oscillations markedly different in shape from the harmonic oscillations. For the same reason, the bifurcation in this regime follows a scenario that differs from the Hopf bifurcation. Dynamic calculations show that first we observe the growth of small harmonic oscillations at a frequency $1.4\omega_r$ [before the bifurcation point, these oscillations decay (Fig. 3)]. At this point, the roots of equations (14) and (15) are different and equal to $1.23\omega_r$ and $1.33\omega_r$, respectively. Only after a specific form of oscillations is achieved (Fig. 3) (with a gradual change in their repetition rate), the roots of both equations become equal. This type of solution is preserved with a further change in the feedback phase (Fig. 2) up to the value $\chi = -190°$, where the period-doubling bifurcation takes place.

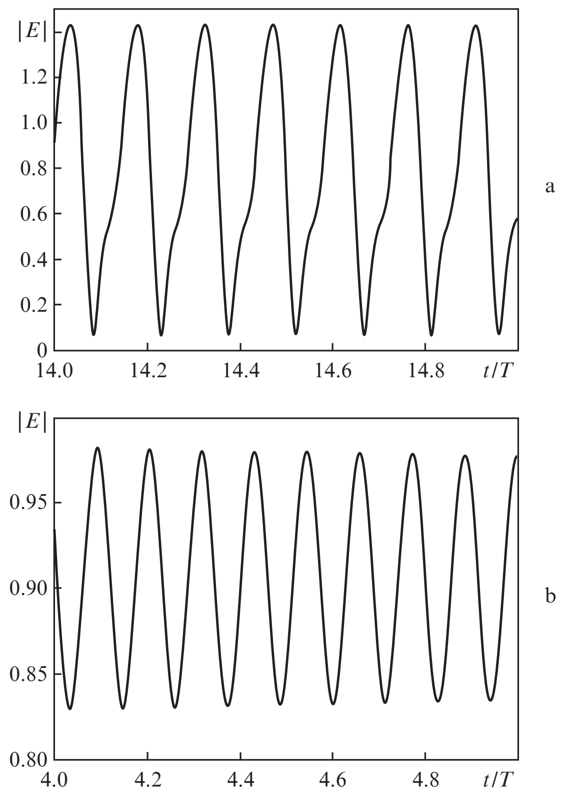

Figure 3. Behaviour of the laser field amplitude behind the bifurcation point (a) and decaying relaxation oscillations to the steady-state mode before the bifurcation, $\omega = 1.38\omega_r$ (b).

Unstable solutions obtained in the analysis of these equations are not realised. Table 1 shows the data for one of the unstable solutions with two roots with $\mathrm{Re}\gamma > 0$ of characteristic equation (20). The characteristics of this decision indicate that the phase shift of effective feedback is equal approximately to 39°. Another unstable solution has one root with a positive growth rate; its oscillation frequency ($1.36\omega_r$) is close to the initial (transient) oscillations that arise when crossing a bifurcation point. The amplitude of steady oscillations is also high: $x_1 = 1.78$.

Formulas (14)–(16), which were obtained in analytical calculations, allow one to generalise the results found previously in the analysis of the steady-state solution at the bifurcation point of equations (8), (9). The equations have two differences associated with the transition to periodic phase trajectories. The first difference consists in the fact that the effective feedback parameters (M_1,χ_1), according to (16), are adjusted to the established regime of nonlinear oscillations, and the second consists in the fact that there arises a dependence of the repetition rate of oscillations on amplitude (14).

For the Hopf bifurcations, the oscillation amplitude $x_1 = 0$, so that formulas (14), (15) and (8), (9) are identical. This limit is interesting, because it explains the emergence of two types of bifurcations from the steady state in the regime of periodic oscillations. From equations (8), (9) in this case, we can derive a bicubic equation with respect to the frequency of developing oscillations:

$$\omega^2(\omega^2 - \omega_r^2)(4M^2 + \omega_r^2 - \omega^2)$$
$$= M^2[\omega_r^2(\sin\chi - R\cos\chi) - 2\omega^2\sin\chi]^2. \quad (18)$$

The form of the cubic expression with respect to ω^2 in the left-hand side of the equation suggests that one of its roots is always negative (Fig. 4). The cubic equation can have either one or three real roots. Only in the latter case, we can expect the appearance of real oscillation frequencies (i.e., roots with $\omega^2 > 0$). The existence of real roots corresponds to the Hopf bifurcation with zero oscillations.

Equation (18) predicts the existence of a bifurcation that differs from the Hopf bifurcation when the system of equations (8), (9) turns inconsistent in the case of the real oscillation

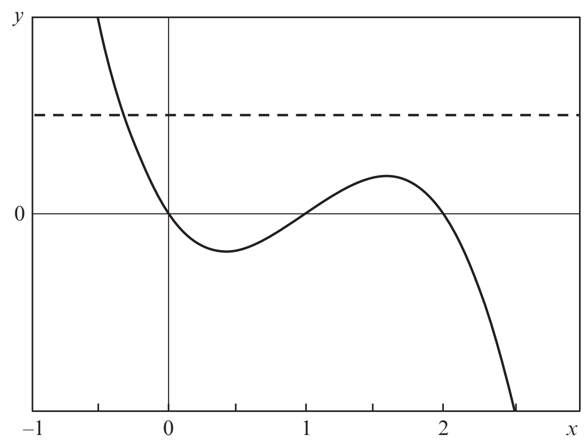

Figure 4. Illustration to the solution of equation (18) in the form $y = p$, $p > 0$ (dashed line) and $y = x(x - x_1)(x_2 - x)$, where $x = \omega^2$, $x_1 = \omega_r^2$, $x^2 = \omega_r^2 + 4M^2$ (solid curve). One root is always negative.

frequencies. Numerical studies describe these bifurcations as regimes with hard switching of the states.

Using the catastrophe theory, we can find the parameters which cause a change in the type of bifurcation in the case of a loss of the steady-state generation stability. The cusp catastrophe occurs along the separatrix and leads to a phase transition in the dynamic behaviour of the laser on different sides of the line. Formulas (14), (15) allow one to analyse the regimes for which characteristic equations (8), (9) become inconsistent. The consistency of their counterparts – equations (14), (15) – is provided by adjusting the effective feedback factors M_1, χ_1. Since there are two equations, we have two parameters to be estimated, namely, the oscillation frequency and amplitude x_1. If the nonlinear solution lies on the stable orbit, then any perturbation either decays or neutrally stable. In fact, only one combination of modes $\cos(\omega t)$ and $\cos[\omega(t - \tau_d/2)]$ is stable if it is present in the spectrum of a nonlinear solution, while others disappear in time. On this basis, effective feedback factors (16) are calculated using parameters (17), obtained from the linear problem for the perturbations.

From expressions (14), (15) we can derive a purely algebraic equation that is similar to (18):

$$\omega^2 S^{-1}(4M_1^2 - S^{-1})$$
$$= M_1^2[S^{-1}(R\cos\chi_1 - \sin\chi_1) - \omega^2(\sin\chi_1 + R\cos\chi_1)]^2. \quad (19)$$

If the definition of S [see (13)] takes into account the second term in the series expansion, equation (19) will be the fifth-degree equations with respect to the square of the frequency ω^2. Since according to (14), $0 \leqslant S^{-1} \leqslant 4M_1^2$, then the number of real solutions can be from zero to four; one root of ω^2 is always negative. The remaining roots appear in pairs with one root of the pair satisfying the condition $\omega\sin(\omega\tau_d)[S^{-1}(R\cos\chi_1 - \sin\chi_1) - \omega^2(\sin\chi_1 + R\cos\chi_1)] > 0$. Thus, the analysis of (19) predicts the existence (from zero to two) of the desired real oscillation frequencies for system (14), (15). One of the possible roots lies in the vicinity of the frequency ω_r, and the other – the frequency $\omega_r/2$ (period doubling regime).

5. Instability criterion

To determine the character of the nonlinear solution stability, we will take advantage of a linear equation for perturbations (4), using the parameters of the periodic orbit M_1, χ_1, p. The stability of such nonlinear solutions is determined by the sign of increments of growth of perturbations of types $\delta x = e^{\gamma t}$ and $\delta y = \alpha e^{\gamma t}$ along the phase trajectory. The parameters M_1, χ_1, p correspond to solutions for the problem for the spectrum of harmonic perturbations, found from (14)–(16):

$$f \equiv \Gamma^{-1} + (M_1/\gamma)^2(e^{-\gamma\tau_d} - 1)^2\gamma(\gamma + T^{-1})$$
$$- (M_1/\gamma)(e^{-\gamma\tau_d} - 1)[\Gamma^{-1}(\sin\chi_1 - R\cos\chi_1)$$
$$+ \gamma(\gamma + T^{-1})(\sin\chi_1 + R\cos\chi_1)] = 0, \quad (20)$$

where $\Gamma = \sum J_n^2(p)[(\omega_r + n\omega)^2 + \gamma(\gamma + T^{-1})]^{-1}$.

In the general case, the roots are complex, and so the stability of the solution corresponds, obviously, to the absence of roots with a positive real part in this equation. We can verify this by calculating the integral over a closed loop (11), which includes the imaginary axis and closes in the right half-plane around a circle of an infinitely large radius.

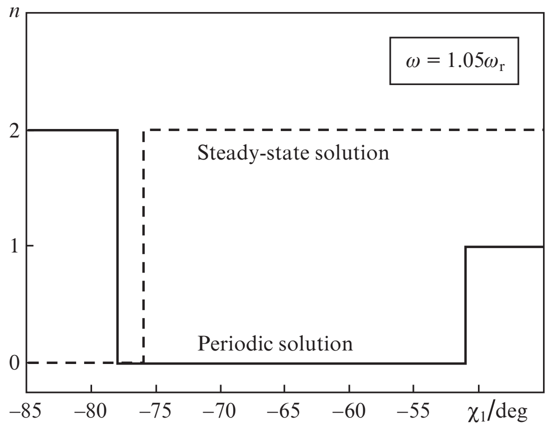

Figure 5. Number of unstable roots of equation (20) with the parameters $p = 0.53$, $M_1 = 0.47M$ and $\chi_1 - \chi = 132.8°$ in the vicinity of $\chi_1 = -76.2°$. The solution at the frequency $\omega = 1.05\omega_r$ is stable, because near the $\chi_1 = -76.2°$ the number of roots with $\mathrm{Re}\gamma > 0$ is zero (the instability emerges at $\chi_1 = -78°$). The dashed line shows the stability region for an alternative steady-state solution.

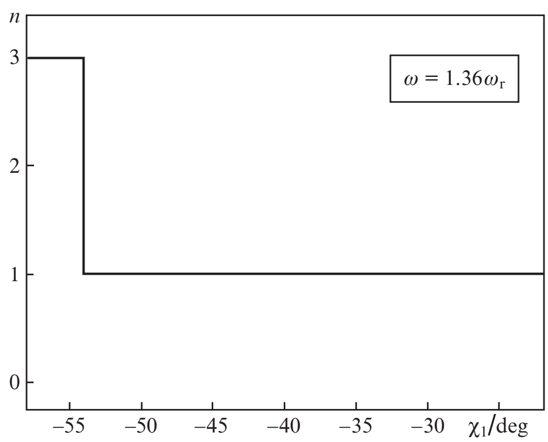

Figure 6. Number of unstable roots of equation (20) with the parameters $p = 0.96$, $M_1 = 1.3M$ in the vicinity of $\chi_1 = -50.6°$. The solution at the frequency $\omega = 1.36\omega_r$ is unstable, because near the $\chi_1 = -50.6°$ there already exists one root with $\mathrm{Re}\gamma > 0$.

Figures 5, 6 show the results [obtained by (14)–(16)] of the analysis of stability of two different solutions on the attractor near the bifurcation point from the steady state with $M = 0.02$ and $\chi = -209°$. The solutions differ in the oscillation amplitude: in the first solution $x_1 = 0.58$, and in the second $x_1 = 1.78$. In the numerical integration, the nonlinear solution with a lower modulation and frequency of oscillations $\omega = 1.05\omega_r$ is implemented (Fig. 5). In the vicinity of the effective feedback phase $\chi_1 = -76.2°$, the number of roots with $\mathrm{Re}\gamma > 0$ is equal to zero; however, the boundary of the stability region lies in close proximity ($\chi_1 = -78°$). In this interval, the hysteresis region is realised, i.e., two different types of solutions – both steady-state and periodic – can be implemented. In the latter case, the modulus of the effective feedback is half the feedback parameter M. Another solution (Fig. 6), obtained with the same feedback phase ($\chi = -209°$), has no stability because of the appearance of the root with $\mathrm{Re}\gamma > 0$. The effective feedback factors are indicated in the figure captions. Let us pay attention to the fact that they differ markedly from the feedback factor. If the feedback phase changes in the direction of the values $\chi = -190°$, we can find the period-doubling bifurcation. In this case, along with the frequency $\omega = \omega_r$ there appear small oscillations at half the frequency.

6. Conclusions

In this paper we have proposed a constructive solution to the stability problem at the point of transition to nonlinear periodic solutions. Using the model Lang–Kobayashi equations, describing the dynamics of a diode laser with delayed optical feedback, we have demonstrated a new method for analysing nonlinear dynamic solutions. This approach is not associated with direct integration of dynamic equations, and therefore cannot cover transitional regimes; however, it may be useful for analysing both steady-state regimes and solutions on periodic attractors. In the analysis we have used the linearisation of original equations applied directly on a periodic phase trajectory. Formally, the equations belong to the class of linear equations with periodic coefficients and, according to the Lyapunov theory, are reducible to an equivalent system of ordinary differential equations.

We have describes a method for finding the coefficients of the reduced system that is equivalent to the system of linear equations of the perturbation theory as applied to periodic regimes. In the first approximation, the dynamics can be described by one Lyapunov parameter through the introduction of the effective feedback factor. Formulas presented in this approximation make it possible to calculate the period of nonlinear oscillations and well describe the solution in the vicinity of the bifurcation destroying the steady-state stability, even if it is not the Hopf bifurcation but a bifurcation in the anharmonic regime. In analysing bifurcations with a hard switching to a periodic attractor of finite size, we have obtained not only the dynamic characteristics of the possible regime but also studied their stability. The results of numerical integration of the dynamic equations are in good agreement with those obtained with the help of the Lyapunov method.

The peculiarity of this approach is the reduction of dynamic equations with delay to the class of eigenvalue problems. The solution of the problem leads to algebraic transcendental equations. Despite the complicated form of the derived equations, the proposed method helps more fully explore the nature of possible nonlinear solutions, which exist in the given physical conditions. Beyond the bifurcation point, the method retains its validity but requires a more rigorous mathematical representation with application of special functions.

Appendix

System (2) was obtained from (1) under the assumption that the intensity is given by the formula $I(t,z) = |E(t,z)|^2$. Let us apply to system (2) the method of small perturbations to find the frequency spectrum. Because the functions depend on two variables, then the general form of the perturbation is determined by two increments:

$$n, \psi \propto \exp(\lambda\tau + \eta z) = \exp[\lambda(t - z) + \eta z]$$

$$= \exp[\lambda t + (\eta - \lambda)z].$$

In the laboratory coordinate system, the dependence on z is periodic, and with account for the periodic conditions, $(\eta - \lambda)\tau_{2L} = 2\pi k\mathrm{i}$. We will consider the effect of the feedback as an external force. In the case of resonance with the spectrum of eigenoscillations, possible is the amplitude of oscillation modes corresponding to different longitudinal modes (with the index k). However, in view of the relation $\tau_{2L} \ll \tau_d$, perturbations with $\eta = \lambda$ dominate, because other oscillations differ substantially in frequency. The spectrum of decaying eigenoscillations of the system in the absence of delayed feedback is found from the analysis of the characteristic equation for system (2):

$$\lambda^2 + \lambda(T_1^{-1} + 2\pi\mathrm{i}k/\tau_{2L}) + (2\pi\mathrm{i}k/\tau_{2L})T_1^{-1} + \omega_r^2 = 0.$$

In particular, when $k = 0$, the roots describe the usual relaxation oscillations with a frequency ω_r and slow decay: $\lambda = -(2T_1)^{-1} \pm \mathrm{i}\omega_r$. Analysis of the roots of the characteristic equation at nonzero k and the appropriate choice of the sign leads to the expression

$$\lambda \approx -T_1^{-1} + \mathrm{i}\omega_r^2\tau_{2L}/(2\pi k),$$

which allows one to specify the range of applicability of the zero-dimensional model. As is well known [1], the presence of delayed feedback leads to the development of oscillatory instabilities at a characteristic frequency close to ω_r. Perturbations at the other eigenfrequencies decay. Moreover, the dynamics of the solution in the numerical calculations of complete model (1) is very similar to that found in the zero-dimensional model (i.e., when $\eta = \lambda$). Thus, the behaviour of the one-dimensional model transforms into the behaviour of the zero-dimensional model while we study the regular dynamics with a relatively simple spectrum.

Controlling the system parameters, we can complicate the oscillation spectrum by performing a series of period-doubling bifurcations. If as a result, there arise resonances with the frequencies of the characteristic equation at $k \geqslant 1$, which are close to $\omega_r^2\tau_{2L}/(2\pi k)$, then the degeneracy is removed. Given that the transition to chaos occurs after a few period doublings, it is expected that the regular dynamics corresponds to the degenerate case. That is, until the oscillation frequency $\omega \gg \omega_r^2\tau_{2L}/(2\pi) \propto 1/T$, oscillations corresponding to different longitudinal modes of the internal cavity cannot be excited. Therefore, in the limit $\omega T \gg 1$, when the characteristic frequency of the arising oscillations far exceeds the inverse relaxation time of inversion, the dynamics of solutions of equations (2) will be close to the dynamics of the zero-dimensional model and the intensity distribution (inside the cavity) has no effect on the dynamics of oscillations excited by the delayed feedback.

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