NONLINEAR OPTICAL PHENOMENA

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Spatial profile reconstruction of individual components of the nonlinear susceptibility tensors $\hat{\chi}^{(3)}(z, \omega', \omega', -\omega, \omega)$ and $\hat{\chi}^{(3)}(z, 2\omega \pm \omega', \pm\omega', \omega, \omega)$ of a one-dimensionally inhomogeneous medium

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***Abstract*. We have proved for the first time and proposed an algorithm of unique spatial profile reconstruction of the components $\chi^{(3)}_{yyyy}$ of complex tensors $\hat{\chi}^{(3)}(z, \omega', \omega', -\omega, \omega)$ and $\hat{\chi}^{(3)}(z, 2\omega \pm \omega', \pm\omega', \omega, \omega)$, describing four-photon interaction of light waves in a one-dimensionally inhomogeneous plate, whose medium has a symmetry plane m_y that is perpendicular to its surface. For the media with an additional symmetry axis 2_z, 4_z, 6_z or ∞_z that is perpendicular to the plate surface, the proposed method can be used to reconstruct about one-fifth of all independent components of the above tensors.**

Keywords: *cubic susceptibility, one-dimensionally inhomogeneous medium, inverse problem, reflection coefficient, transmission coefficient, conversion factor.*

Reconstruction of the spatial dependence of nonlinear optical properties of one-dimensionally inhomogeneous structures is becoming a popular practical problem [1 – 3]. We proposed for the first time [4] a method for unique reconstruction of profiles of some components of the cubic nonlinearity tensor $\hat{\chi}^{(3)}(z, \omega, -\omega, \omega, \omega)$ in a one-dimensionally inhomogeneous plate. It was assumed that its dielectric properties vary only along the z axis, that is perpendicular to two parallel flat surfaces of the plate, and are arbitrary frequency-dependent. In this paper we prove that a similar method can be used to uniquely determine the coordinate dependence of the complex components of the tensor $\hat{\chi}^{(3)}(z, \omega', \omega', -\omega, \omega)$, $\hat{\chi}^{(3)}(z, 2\omega - \omega', -\omega', \omega, \omega)$ and $\hat{\chi}^{(3)}(z, 2\omega + \omega', \omega', \omega, \omega)$, responsible for frequently used in practice four-photon nonlinear interactions of two waves with different frequencies [5]. Such a reconstruction can be implemented using two series of experiments on the interaction between a plate and signal waves with frequencies ω_1 and ω_2, incident on the plate at different angles within a certain range of angles, in the presence of a high-power wave with the frequency ω_3; in this case, $\omega_3 = 0.5(\omega_1 + \omega_2)$ or $\omega_3 = 0.5(\omega_2 - \omega_1)$.

Consider a plate, which borders linear homogeneous isotropic nonabsorbing and nondispersive media with the real permittivity ε_0 along the planes $z = z_1$ and $z = z_2$ ($z_2 > z_1$). We assume that the point symmetry groups of the various layers of a one-dimensionally inhomogeneous plate are such that one of their common elements of symmetry is a symmetry plane perpendicular to the surfaces of the plate. Let us direct the axis $x \perp z$ along this symmetry plane. Suppose that a low-intensity s-polarised plane signal wave propagating in the positive or negative direction of the z axis is incident on a plate at an angle α_1. In the first case, its electric field strength is equal to $E_{1+}\boldsymbol{e}_y \exp\{\mathrm{i}[\omega_1 t - k_x x - k_{1z}(z - z_1)]\} + \text{c.c.}$ (for $z < z_1$), and in the second case, it is equal to $E_{1-}\boldsymbol{e}_y \exp\{\mathrm{i}[\omega_1 t - k_x x + k_{1z}(z - z_2)]\} +$ c.c (for $z > z_2$). Here, $\boldsymbol{e}_y$ is the unit vector perpendicular to the incidence plane; $k_x = k_{01}\sin\alpha_1$; $k_{1z} = k_{01}\cos\alpha_1$; $k_{01} = \omega_1\sqrt{\varepsilon_0}/c$; c is the speed of light in a vacuum. Suppose, moreover, that a plane high-power fundamental wave with frequency ω_3 falls on a plate, perpendicular to its surface in the positive direction of the z axis; for $z < z_1$ the electric field vector of this wave is equal to $E_0\boldsymbol{e}_y \exp\{\mathrm{i}[\omega_3 t - k_{03}(z - z_1)]\} + \text{c.c.}$, where $k_{03} = \omega_3\sqrt{\varepsilon_0}/c$. In other words, we consider simultaneously two independent problems. In the first problem a high-power and signal waves fall on the same side of the plate under study (subscript ‘plus’). In the second problem they fall on the opposite sides of the plate (subscript ‘minus’). In the experiment use can be made of any of these measurement schemes, and if necessary to obtain results with high accuracy, measurements can be carried out using both schemes.

We assume for definiteness that the frequencies ω_3 and ω_1, where $\omega_1 < 2\omega_3$, and nonlinear dielectric properties of the plate medium are such that if the plate is exposed to a high-power wave with frequency ω_3 and to a fairly weak signal wave with frequency ω_1, only three waves effectively interact in the medium, namely one high-power wave $E_{\mathrm{f}}(z)\boldsymbol{e}_y \exp(\mathrm{i}\omega_3 t) + \text{c.c.}$ and two weak waves – an original signal wave $E_{\mathrm{s}1\pm}(z)\boldsymbol{e}_y \exp[\mathrm{i}(\omega_1 t - k_x x)] + \text{c.c.}$ and a new wave $E_{\mathrm{s}2\pm}(z)\boldsymbol{e}_y \exp[\mathrm{i}(\omega_2 t + k_x x)] + \text{c.c.}$, produced in a nonlinear medium. Hereinafter, $\omega_2 = 2\omega_3 - \omega_1$. The new wave $E_{\mathrm{s}2\pm}(z)\boldsymbol{e}_y \exp[\mathrm{i}(\omega_2 t + k_x x)] + \text{c.c.}$ arises as a result of nonlinear interaction of a high-power and signal waves, the interaction being described by the cubic susceptibility tensor $\hat{\chi}^{(3)}(z, \tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$.

Our assumption, in particular, means that the nonlinear interaction of a new and high-power waves, affecting the

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propagation of the original signal wave, does not lead to any noticeable generation of the waves with other frequencies. Then, taking into account self-action of a high-power wave and weak signal and new waves in the linear approximation in the amplitudes, for the selected polarisation of the incident waves due to the presence of the local symmetry plane m_y in the medium, only the y component of electric induction vector in the plate is different from zero [6]:

$$D_{y\pm} = \left[\varepsilon_{yy}(z,\omega_3) + 4\pi\chi^{(3)}_{yyyy}(z,\omega_3,-\omega_3,\omega_3,\omega_3)|E_f|^2\right]E_f$$
$$\times \exp(i\omega_3 t) + \left\{\left[\varepsilon_{yy}(z,\omega_1) + 8\pi\chi^{(3)}_{yyyy}(z,\omega_1,\omega_1,-\omega_3,\omega_3)|E_f|^2\right]\right.$$
$$\times E_{s1\pm} + 4\pi\chi^{(3)}_{yyyy}(z,\omega_1,-\omega_2,\omega_3,\omega_3)E_f^2 E^*_{s2\pm}\}$$
$$\times \exp[i(\omega_1 t - k_x x)] + \{[\varepsilon_{yy}(z,\omega_2)$$
$$+ 8\pi\chi^{(3)}_{yyyy}(z,\omega_2,\omega_2,-\omega_3,\omega_3)$$
$$\times |E_f|^2]E_{s2\pm} + 4\pi\chi^{(3)}_{yyyy}(z,\omega_2,-\omega_1,\omega_3,\omega_3)E_f^2 E^*_{s1\pm}\}$$
$$\times \exp[i(\omega_2 t + k_x x)] + \text{c.c.} \quad (1)$$

Substituting (1) into the wave equation for the electric field strength and equating separately the terms that do not depend on the coordinate x and the terms that are proportional to $\exp[i(\omega_1 t - k_x x)]$ and $\exp[i(\omega_2 t + k_x x)]$, after some transformations we obtain

$$\frac{d^2E_f}{dz^2} + 0.5\omega_3^2[\varepsilon_{yy}(z,\omega_3)\varepsilon_{n3}(z)]\frac{E_f}{c^2} = 0, \quad (2)$$

$$\frac{d^2E_{s1\pm}}{dz^2} + \left[\frac{\omega_1^2\varepsilon_{n1}(z)}{c^2} - \lambda\right]E_{s1\pm} + \frac{\omega_1^2 r_{12}(z)E^*_{s2\pm}}{c^2} = 0,$$
$$\frac{d^2E^*_{s2\pm}}{dz^2} + \left[\frac{\omega_2^2\varepsilon^*_{n2}(z)}{c^2} - \lambda\right]E^*_{s2\pm} + \frac{\omega_2^2 r^*_{21}(z)E_{s1\pm}}{c^2} = 0, \quad (3)$$

where $\lambda = k_x^2$; $r_{12}(z) = 4\pi\chi^{(3)}_{yyyy}(z,\omega_1,-\omega_2,\omega_3,\omega_3)E_f^2(z)$; $r_{21}(z) = 4\pi\chi^{(3)}_{yyyy}(z,\omega_2,-\omega_1,\omega_3,\omega_3)E_f^2(z)$; $\varepsilon_{nk}(z) = \varepsilon_{yy}(z,\omega_k) + 8\pi\chi^{(3)}_{yyyy}(z,\ \omega_k,\omega_k,-\omega_3,\omega_3)|E_f(z)|^2$; $k = 1,\ 2,\ 3$.

Now suppose that instead of a signal wave with frequency ω_1, the same plate is exposed to a signal wave with frequency $\omega_2 = 2\omega_3 - \omega_1$ at an angle α_2 such that $k_{02}\sin\alpha_2 = k_x$, where $k_{02} = \omega_2\sqrt{\varepsilon_0}/c$. The electric field strength of the signal wave is equal to $E_{2+}\boldsymbol{e}_y\times \exp\{i[\omega_2 t - k_x x - k_{2z}(z - z_1)]\} + \text{c.c.}$ as it propagates in the positive direction of z axis ($z < z_1$) and to $E_{2-}\boldsymbol{e}_y\times \exp\{i[\omega_2 t - k_x x + k_{2z}(z - z_2)]\} + \text{c.c.}$ as it propagates in the negative direction of z axis ($z > z_2$). Here, $k_{2z} = k_{02}\cos\alpha_2$ and we assume that $k_x < k_{02}$.

It follows from the previously formulated hypothesis that the nonlinear interaction of this signal wave with the high-power wave $E_f(z)\boldsymbol{e}_y\exp(i\omega_3 t) + \text{c.c.}$ propagating in the plate will lead to the emergence of a new weak wave with frequency $\omega_1 = 2\omega_3 - \omega_2$, whose electric field strength is $E_{s1\pm}(z)\boldsymbol{e}_y\exp[i(\omega_1 t + k_x x)] + \text{c.c.}$ In this case, the equations, describing the change in the values of $E_{s1\pm}(z)$ and $E^*_{s2\pm}(z)$ in the plate, will still have the form of (3).

Thus, when the signal wave $E_{sq\pm}(z)$ with frequency ω_q propagates in the plate, the resultant nonlinear interaction leads to the emergence of a new wave $E_{sl\pm}(z)$ with frequency ω_l. Hereinafter, $q = 1, 2$ and $l = 2$ at $q = 1$ and $l = 1$ at $q = 2$, i.e., $l = 1 + \delta_{q1}$, where δ_{ij} is the Kronecker delta. The new wave arising in the plate continues to spread in the adjacent homogeneous linear media in the form of a wave $E^{(l1)}_{s\pm}\boldsymbol{e}_y\exp\{i[\omega_l t + k_x x + k_{lz}(z - z_1)]\} + \text{c.c.}$ in the region $z < z_1$ and in the form of a wave $E^{(l2)}_{s\pm}\boldsymbol{e}_y\exp\{i[\omega_l t + k_x x - k_{lz}(z - z_2)]\} + \text{c.c.}$ in the region $z > z_2$. At the same time, the quantities $E^{(l1)}_{s\pm}$, $E^{(l2)}_{s\pm}$ and $E_{sl\pm}(z)$ on the plate surfaces meet the Maxwell boundary conditions:

$$E_{sl\pm}(z_1) = E^{(l1)}_{s\pm},\quad \left.\frac{dE_{sl\pm}}{dz}\right|_{z=z_1} = ik_{lz}E^{(l1)}_{s\pm},$$
$$E_{sl\pm}(z_2) = E^{(l2)}_{s\pm},\quad \left.\frac{dE_{sl\pm}}{dz}\right|_{z=z_2} = -ik_{lz}E^{(l2)}_{s\pm}.$$

Thus, this paper describes simultaneously four different situations. Propagation of a signal and new waves in these situations is described by the system of equations (3), but the boundary conditions in each of them are different. Below we present the boundary conditions for each of the situations under study. If the signal wave with frequency ω_1 is incident on the plate from the region $z < z_1$, then the boundary conditions have the form

$$E_{s1+}(z_1) = (1 + R_{1+})E_{1+},\quad E_{s1+}(z_2) = T_{1+}E_{1+},$$
$$\left.\frac{dE_{s1+}}{dz}\right|_{z=z_1} = -ik_{1z}(1 - R_{1+})E_{1+},$$
$$\left.\frac{dE_{s1+}}{dz}\right|_{z=z_2} = -ik_{1z}T_{1+}E_{1+},$$
$$E^*_{s2+}(z_1) = \left(E^{(21)}_{s+}\right)^* \equiv G^{(21)}_+E_{1+}, \quad (4.1)$$
$$E^*_{s2+}(z_2) = \left(E^{(22)}_{s+}\right)^* \equiv G^{(22)}_+E_{1+},$$
$$\left.\frac{dE^*_{s2+}}{dz}\right|_{z=z_1} = -ik_{2z}G^{(21)}_+E_{1+},$$
$$\left.\frac{dE^*_{s2+}}{dz}\right|_{z=z_2} = ik_{2z}G^{(22)}_+E_{1+}.$$

In the case when the signal wave with frequency ω_1 is incident on the plate from the region $z > z_2$, the boundary conditions are given by the expressions

$$E_{s1-}(z_1) = T_{1-}E_{1-},\quad E_{s1-}(z_2) = (1 + R_{1-})E_{1-},$$
$$\left.\frac{dE_{s1-}}{dz}\right|_{z=z_1} = ik_{1z}T_{1-}E_{1-},$$
$$\left.\frac{dE_{s1-}}{dz}\right|_{z=z_2} = ik_{1z}(1 - R_{1-})E_{1-},$$
$$E^*_{s2-}(z_1) = \left(E^{(21)}_{s-}\right)^* \equiv G^{(21)}_-E_{1-}, \quad (4.2)$$
$$E^*_{s2-}(z_2) = \left(E^{(22)}_{s-}\right)^* \equiv G^{(22)}_-E_{1-},$$
$$\left.\frac{dE^*_{s2-}}{dz}\right|_{z=z_1} = -ik_{2z}G^{(21)}_-E_{1-},$$

$$\left.\frac{\mathrm{d}E^*_{\mathrm{s}2-}}{\mathrm{d}z}\right|_{z=z_2} = \mathrm{i}k_{2z}G^{(22)}_{-}E_{1-}.$$

When the signal wave with frequency ω_2 falls on the plate from the region $z < z_1$, they have the form

$$E_{\mathrm{s}1+}(z_1) = E^{(11)}_{\mathrm{s}+} \equiv \left(G^{(11)}_{+}E_{2+}\right)^*,$$

$$E_{\mathrm{s}1+}(z_2) = E^{(12)}_{\mathrm{s}+} \equiv \left(G^{(12)}_{+}E_{2+}\right)^*,$$

$$\left.\frac{\mathrm{d}E_{\mathrm{s}1+}}{\mathrm{d}z}\right|_{z=z_1} = \mathrm{i}k_{1z}\left(G^{(11)}_{+}E_{2+}\right)^*,$$

$$\left.\frac{\mathrm{d}E_{\mathrm{s}1+}}{\mathrm{d}z}\right|_{z=z_2} = -\mathrm{i}k_{1z}\left(G^{(12)}_{+}E_{2+}\right)^*, \tag{4.3}$$

$$E^*_{\mathrm{s}2+}(z_1) = \left(1 + R^*_{2+}\right)E^*_{2+}, \quad E^*_{\mathrm{s}2+}(z_2) = T^*_{2+}E^*_{2+},$$

$$\left.\frac{\mathrm{d}E^*_{\mathrm{s}2+}}{\mathrm{d}z}\right|_{z=z_1} = \mathrm{i}k_{2z}\left(1 - R^*_{2+}\right)E^*_{2+},$$

$$\left.\frac{\mathrm{d}E^*_{\mathrm{s}2+}}{\mathrm{d}z}\right|_{z=z_2} = \mathrm{i}k_{2z}T^*_{2+}E^*_{2+}.$$

In the latter case, when the signal wave with frequency ω_2 is incident on the plate from the region $z > z_2$, the boundary conditions are given by the formulas

$$E_{\mathrm{s}1-}(z_1) = E^{(11)}_{\mathrm{s}-} \equiv \left(G^{(11)}_{-}E_{2-}\right)^*,$$

$$E_{\mathrm{s}1-}(z_2) = E^{(12)}_{\mathrm{s}-} \equiv \left(G^{(12)}_{-}E_{2-}\right)^*,$$

$$\left.\frac{\mathrm{d}E_{\mathrm{s}1-}}{\mathrm{d}z}\right|_{z=z_1} = \mathrm{i}k_{1z}\left(G^{(11)}_{-}E_{2-}\right)^*,$$

$$\left.\frac{\mathrm{d}E_{\mathrm{s}1-}}{\mathrm{d}z}\right|_{z=z_2} = -\mathrm{i}k_{1z}\left(G^{(12)}_{-}E_{2-}\right)^*, \tag{4.4}$$

$$E^*_{\mathrm{s}2-}(z_1) = T^*_{2-}E^*_{2-}, \quad E^*_{\mathrm{s}2-}(z_2) = \left(1 + R^*_{2-}\right)E^*_{2-},$$

$$\left.\frac{\mathrm{d}E^*_{\mathrm{s}2-}}{\mathrm{d}z}\right|_{z=z_1} = -\mathrm{i}k_{2z}T^*_{2-}E^*_{2-},$$

$$\left.\frac{\mathrm{d}E^*_{\mathrm{s}2-}}{\mathrm{d}z}\right|_{z=z_2} = -\mathrm{i}k_{2z}\left(1 - R^*_{2-}\right)E^*_{2-}.$$

Here R_{q+} and R_{q-} are the amplitude coefficients of reflection of signal waves E_{q+} and E_{q-} from the plate; T_{q+} and T_{q-} are the amplitude coefficients of transmission of these waves through the plate; $G^{(l1)}_{\pm} \equiv (E^{(l1)}_{\mathrm{s}\pm})^*/E_{q\pm}$, $G^{(l2)}_{\pm} \equiv (E^{(l2)}_{\mathrm{s}\pm})^*/E_{q\pm}$ are the conversion coefficients of the signal wave $E_{q\pm}$. The latter characterise the conversion efficiency of the signal wave with frequency ω_q into two waves with frequency ω_l, propagating on opposite sides of the plate. Given the linearity of boundary conditions (4) with respect to $E_{1\pm}$ and $E^*_{2\pm}$, and the linearity of the system of equations (3) with respect to $E_{\mathrm{s}1\pm}(z)$ and $E^*_{\mathrm{s}2\pm}(z)$, we find that all introduced coefficients are independent of $E_{q\pm}$. Recall that a high-power fundamental wave in all four cases falls onto the plate in the positive direction of the z axis.

If the dependences $\varepsilon_{\mathrm{n}1}(z)$, $\varepsilon_{\mathrm{n}2}(z)$, $r_{12}(z)$ and $r_{21}(z)$ are known, then by solving system (3), (4) we can uniquely calculate the coefficients $R_{q\pm}$, $T_{q\pm}$, $G^{(l1)}_{\pm}$, $G^{(l2)}_{\pm}$ for any angles of incidence of a plane wave signal with frequency ω_q and, hence, solve the direct problem. We are interested in a more complex inverse problem: determination of $\varepsilon_{\mathrm{n}1}(z)$, $\varepsilon_{\mathrm{n}2}(z)$, $r_{12}(z)$ and $r_{21}(z)$ for a layer of given thickness by eight amplitude complex coefficients of reflection, transmission and conversion of signal waves, namely, R_{q+}, T_{q+}, $G^{(qv)}_{+}$ or R_{q-}, T_{q-}, $G^{(qv)}_{-}$ ($v = 1, 2$), known for a certain interval of angles of incidence. In Appendix 1 we prove that if such an inverse problem has a solution then it is unique. At the same time, components $\chi^{(3)}_{yyyy}(z,\omega_1,\omega_1,-\omega_3,\omega_3)$, $\chi^{(3)}_{yyyy}(z,\omega_2,\omega_2,-\omega_3,\omega_3)$, $\chi^{(3)}_{yyyy}(z,\omega_1,-\omega_2,\omega_3,\omega_3)$ and $\chi^{(3)}_{yyyy}(z,\omega_2,-\omega_1,\omega_3,\omega_3)$ can be reconstructed, in particular, by finding a single zero minimum of a specially constructed functional on test functions, describing the coordinate dependence of the dielectric properties of the investigated plate. Principles of construction of such a functional are described in detail in [4], and its form is given in Appendix 2. In this case, the profiles $\varepsilon_{yy}(z,\omega_k)$ ($k = 1, 2, 3$) of the linear permittivity of the medium (the reconstruction method of such profiles was proposed in [7] and tested in a numerical experiment in [8]) and the distribution of the electric field $E_{\mathrm{f}}(z)$ of a high-power wave in the medium [4] are considered to be known. Note that changing two of the three frequencies or all the three frequencies (ω_1, ω_2 and ω_3), we can obtain information not only about the spatial profile, but also about the frequency dispersion of the components $\chi^{(3)}_{yyyy}$ of the tensors $\hat{\chi}^{(3)}(z,\omega',\omega',-\omega,\omega)$ and $\hat{\chi}^{(3)}(z,2\omega-\omega',-\omega',\omega,\omega)$.

Until now, we have assumed that the medium forming a layer has only a symmetry plane m_y that is perpendicular to its surface. We now consider media with the symmetry axis 2_z, 4_z, 6_z or ∞_z. Without changing polarisation of a high-power pump wave, we will change polarisation by 90° and rotate the plane of the signal wave incidence by 90°:

$$E_{1\pm}\boldsymbol{e}_x \exp\{\mathrm{i}[\omega_1 t - k_y y \mp k_{1z}(z - z_{1,2})]\} + \mathrm{c.c.}$$

or

$$E_{2\pm}\boldsymbol{e}_x \exp\{\mathrm{i}[\omega_2 t - k_y y \mp k_{2z}(z - z_{1,2})]\} + \mathrm{c.c.},$$

where $k_y = k_{01}\sin\alpha_1 = k_{02}\sin\alpha_2$. Then, in the above formulated approximations, the expression for the nonzero components of the electric induction vector will have the form [6]:

$$\begin{aligned} D_{x\pm} = {} & \big\{\big[\varepsilon_{xx}(z,\omega_1) + 8\pi\chi^{(3)}_{xxyy}(z,\omega_1,\omega_1,-\omega_3,\omega_3)|E_{\mathrm{f}}|^2\big]E_{\mathrm{s}1\pm} \\ & + 4\pi\chi^{(3)}_{xxyy}(z,\omega_1,-\omega_2,\omega_3,\omega_3)E_{\mathrm{f}}^2E^*_{\mathrm{s}2\pm}\big\}\exp[\mathrm{i}(\omega_1 t - k_y y)] \\ & + \big\{\big[\varepsilon_{xx}(z,\omega_2) + 8\pi\chi^{(3)}_{xxyy}(z,\omega_2,\omega_2,-\omega_3,\omega_3)|E_{\mathrm{f}}|^2\big]E_{\mathrm{s}2\pm} \\ & + 4\pi\chi^{(3)}_{xxyy}(z,\omega_2,-\omega_1,\omega_3,\omega_3)E_{\mathrm{f}}^2E^*_{\mathrm{s}1\pm}\big\} \\ & \times\exp[\mathrm{i}(\omega_2 t + k_y y)] + \mathrm{c.c.}, \end{aligned}$$

$$\begin{aligned} D_{y\pm} = {} & \big[\varepsilon_{yy}(z,\omega_3) + 4\pi\chi^{(3)}_{yyyy}(z,\omega_3,-\omega_3,\omega_3,\omega_3)|E_{\mathrm{f}}|^2\big]E_{\mathrm{f}} \\ & \times\exp(\mathrm{i}\omega t) + \mathrm{c.c.} \end{aligned}$$

Because the parameters of a high-power pump wave remained unchanged, equation (2) and the dependence of $E_f(z)$ do not change compared with the previous geometry. Boundary conditions (4) also retain their form. Equations for $E_{s1\pm}(z)$ and $E^*_{s2\pm}(z)$ still have the form of (3), but with the parameter λ replaced by $\tilde{\lambda} = k_y^2$, the coefficients $\varepsilon_{nq}(z)$ replaced by $\tilde{\varepsilon}_{nq}(z) = \varepsilon_{xx}(z, \omega_q) + 8\pi\chi^{(3)}_{xxyy}(z, \omega_q, \omega_q, -\omega_3, \omega_3) \times |E_f(z)|^2$, and $r_{ql}(z)$ replaced by $\tilde{r}_{ql}(z) = 4\pi\chi^{(3)}_{xxyy}(z, \omega_q, -\omega_l, \omega_3, \omega_3)E_f^2(z)$ ($q = 1, 2$; $l = 1 + \delta_{q1}$).

Therefore, by measuring new coefficients of transmission (R_{1-}, R_{2-}), reflection (T_{1-}, T_{2-}) and conversion [$G_-^{(qv)}$, $v = 1, 2$] for each of the signal waves in some interval of angles of incidence, we can uniquely reconstruct the dependences $\tilde{\varepsilon}_{n1}(z)$, $\tilde{\varepsilon}_{n2}(z)$, $\tilde{r}_{12}(z)$ and $\tilde{r}_{21}(z)$ and, hence, the profiles of the components $\chi^{(3)}_{xxyy}(z, \omega_1, \omega_1, -\omega_3, \omega_3)$, $\chi^{(3)}_{xxyy}(z, \omega_2, \omega_2, -\omega_3, \omega_3)$, $\chi^{(3)}_{xxyy}(z, \omega_1, -\omega_2, \omega_3, \omega_3)$ and $\chi^{(3)}_{xxyy}(z, \omega_2, -\omega_1, \omega_3, \omega_3)$:

$$\chi^{(3)}_{xxyy}(z, \omega_q, \omega_q, -\omega_3, \omega_3) = \tilde{\varepsilon}_{nq}(z) - \frac{\varepsilon_{xx}(z, \omega_q)}{8\pi|E_f(z)|^2},$$

$$\chi^{(3)}_{xxyy}(z, \omega_q, -\omega_l, \omega_3, \omega_3) = \frac{\tilde{r}_{ql}(z)}{4\pi E_f^2(z)}.$$

Recall that the spatial distribution of the electric field $E_f(z)$ of a high-power wave in the plate can be reconstructed using the method proposed in [4]. A change in two of the three frequencies or in all the three frequencies (ω_1, ω_2 and ω_3), which does not violate the equality $\omega_1 + \omega_2 = 2\omega_3$, makes it possible to study the frequency dispersion of the component $\chi^{(3)}_{xxyy}$ of the tensors $\hat{\chi}^{(3)}(z, \omega', \omega', -\omega, \omega)$ and $\hat{\chi}^{(3)}(z, 2\omega - \omega', -\omega', \omega, \omega)$. For media with the axis of symmetry of the lowest order (2_z), we can also reconstruct the profiles and to investigate the frequency dispersion of the components $\chi^{(3)}_{xxxx}$, $\chi^{(3)}_{yyxx}$ of these tensors. To do this, we should rotate the plate by 90° around the z axis and fully repeat all the above measurements. For media with the symmetry axis 4_z, 6_z or ∞_z, these extra measurements are not necessary, since for them $\chi^{(3)}_{xxxx} = \chi^{(3)}_{yyyy}$ and $\chi^{(3)}_{xxyy} = \chi^{(3)}_{yyxx}$ [6].

Until now we believed that $\omega_1 < 2\omega_3$. We can consider also the case $\omega_1 > 2\omega_3$. Then, $\omega_2 = \omega_1 - 2\omega_3$ and the interaction of three waves with frequencies ω_3, ω_1 and ω_2 is described by the components $\chi^{(3)}_{yyyy}$, $\chi^{(3)}_{xxxx}$, $\chi^{(3)}_{yyxx}$ or $\chi^{(3)}_{xxyy}$ (depending on the symmetry of the medium and the orientation of the planes of incidence of an s-polarised high-power and signal waves) of the tensors $\hat{\chi}^{(3)}(z, \omega_1, \omega_1, -\omega_3, \omega_3)$, $\hat{\chi}^{(3)}(z, \omega_2, \omega_2, -\omega_3, \omega_3)$, $\hat{\chi}^{(3)}(z, \omega_1, \omega_2, \omega_3, \omega_3)$ and $\hat{\chi}^{(3)}(z, \omega_2, \omega_1, -\omega_3, -\omega_3)$. Therefore, their profiles can be reconstructed as described above. Changing two of the three frequencies, or all the three frequencies (ω_1, ω_3, and ω_2), we can investigate the frequency dispersion of the corresponding components of the cubic nonlinearity tensors $\hat{\chi}^{(3)}(z, \omega', \omega', -\omega, \omega)$, $\hat{\chi}^{(3)}(z, 2\omega + \omega', \omega', \omega, \omega)$ and $\hat{\chi}^{(3)}(z, \omega' - 2\omega, \omega', -\omega, -\omega)$. In this case, the possibility of reconstructing from one to four components of these tensors is determined by the local spatial symmetry of the medium of the studied inhomogeneous plate. As is known, the spatial symmetry of one-dimensionally inhomogeneous media, strictly speaking, refers to one of ten classes (1, 2, m, mm2, 3, 4, 6, 3m, 4mm, 6mm) or to two limiting symmetry groups (∞, ∞m) [4]. Unfortunately, our method does not make it possible to determine and control the cubic nonlinearity of one-dimensionally inhomogeneous media with the symmetry classes 1, 2, 3, 4, 6 and ∞. For media with the symmetry m (more precisely m_y) or 3m, we can reconstruct only the component $\chi^{(3)}_{yyyy}$ of each of the above tensors. The components $\chi^{(3)}_{yyyy}$, $\chi^{(3)}_{xxyy}$, $\chi^{(3)}_{xxxx}$ and $\chi^{(3)}_{yyxx}$ of these tensors can be found for the media with the symmetry classes mm2. In addition, for media with the symmetry class 4mm, 6mm or ∞m, we can reconstruct the components $\chi^{(3)}_{yyyy} = \chi^{(3)}_{xxxx}$ and $\chi^{(3)}_{xxyy} = \chi^{(3)}_{yyxx}$. Finally, for media with the symmetry classes mm2, 4mm, 6mm or ∞m it is possible to reconstruct approximately one-fifth of all independent components of the cubic nonlinearity tensors $\hat{\chi}^{(3)}(z, \omega', \omega', -\omega, \omega)$ as well as $\hat{\chi}^{(3)}(z, 2\omega - \omega', -\omega', \omega, \omega)$ and (or) $\hat{\chi}^{(3)}(z, 2\omega + \omega', \omega', \omega, \omega)$.

Thus, we have fully investigated the case when the frequencies of the used waves and nonlinear properties of the plate are such that in the case of incidence of a high-power wave with frequency ω_3 and of a relatively weak signal wave with frequency ω_1 on the plate, only one new wave is efficiently generated in it. In this case, the nonlinear interaction of the latter with a high-power wave affects the propagation of the signal wave, but does not lead to any noticeable generation of waves with other frequencies. The frequency of this new wave ω_2 can be equal to $2\omega_3 - \omega_1$ (at $\omega_1 < 2\omega_3$), $\omega_1 - 2\omega_3$ (at $\omega_1 > 2\omega_3$) or $\omega_1 + 2\omega_3$. Note that the last two cases are physically equivalent, since they are obtained one from another by replacing the indices $1 \leftrightarrow 2$.

The obtained results can be generalised to more complicated cases, when a medium with a cubic nonlinearity exhibits effective interaction of one high-power wave and three (or more) weak waves affecting propagation of each other only through interaction with a high-power wave, which does not lead to the generation of waves with other frequencies. Such a situation occurs, for example, if the medium exhibits possible effective interaction between a high-power wave with frequency ω_3 and three weak waves with frequencies ω_1, $2\omega_3 + \omega_1$ and $2\omega_3 - \omega_1$ (or $\omega_1 - 2\omega_3$) and there does not appear any noticeable generation of waves at other frequencies (e.g., at a frequency $4\omega_3 \pm \omega_1$). However, in this case, reconstruction of the corresponding components of the nonlinear susceptibility tensor $\hat{\chi}^{(3)}(z, \tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$ will require three series of measurements. In each of these series, for different angles of incidence it is necessary to determine the transmission and reflection coefficients of the signal wave with one of the three frequencies and four conversion coefficients of this signal wave into weak waves with other two frequencies. As a result, the number of necessary measurements increases more than twofold compared to the case described in detail in this paper.

Appendix 1. Proof of the uniqueness of the solution of the inverse electrodynamic problem

Recall that for a sufficiently wide class of functions $\varepsilon_{n1}(z)$, $\varepsilon_{n2}(z)$, $r_{12}(z)$ and $r_{21}(z)$ (piecewise continuous and bounded, or even only integrable [9]) the system of equations (3) has continuously differentiable solutions that we will sometimes write for brevity in the form of a column

$$\overset{\leftrightarrow}{\varphi}(z) = \begin{pmatrix} \varphi_h \\ \varphi_g \end{pmatrix} = \begin{pmatrix} E_{s1\pm}(z) \\ E^*_{s2\pm}(z) \end{pmatrix}.$$

Let the columns $\overleftrightarrow{\varphi}_m(z,\lambda)$, where $m = 1, 2, 3, 4$, be the solutions of system (3) with boundary conditions

$$\overleftrightarrow{\varphi}_1(z_1,\lambda) = \begin{pmatrix}1\\0\end{pmatrix},\quad \left.\frac{d\overleftrightarrow{\varphi}_1(z,\lambda)}{dz}\right|_{z=z_1} = \begin{pmatrix}0\\0\end{pmatrix},$$
$$\overleftrightarrow{\varphi}_2(z_1,\lambda) = \begin{pmatrix}0\\1\end{pmatrix},\quad \left.\frac{d\overleftrightarrow{\varphi}_2(z,\lambda)}{dz}\right|_{z=z_1} = \begin{pmatrix}0\\0\end{pmatrix},$$
$$\overleftrightarrow{\varphi}_3(z_1,\lambda) = \begin{pmatrix}0\\0\end{pmatrix},\quad \left.\frac{d\overleftrightarrow{\varphi}_3(z,\lambda)}{dz}\right|_{z=z_1} = \begin{pmatrix}1\\0\end{pmatrix}, \tag{A1.1}$$
$$\overleftrightarrow{\varphi}_4(z_1,\lambda) = \begin{pmatrix}0\\0\end{pmatrix},\quad \left.\frac{d\overleftrightarrow{\varphi}_4(z,\lambda)}{dz}\right|_{z=z_1} = \begin{pmatrix}0\\1\end{pmatrix}.$$

Then for any λ they form a fundamental system of solutions of equations (3), and the solutions $\overleftrightarrow{\varphi}_+^{(1)}(z,\lambda)$, $\overleftrightarrow{\varphi}_-^{(1)}(z,\lambda)$, $\overleftrightarrow{\varphi}_+^{(2)}(z,\lambda)$ and $\overleftrightarrow{\varphi}_-^{(2)}(z,\lambda)$ of four problems (3), (4) can be written as

$$\overleftrightarrow{\varphi}_\pm^{(q)}(z,\lambda) = C_{1\pm}^{(q)}\overleftrightarrow{\varphi}_1(z,\lambda) + C_{2\pm}^{(q)}\overleftrightarrow{\varphi}_2(z,\lambda) + C_{3\pm}^{(q)}\overleftrightarrow{\varphi}_3(z,\lambda) + C_{4\pm}^{(q)}\overleftrightarrow{\varphi}_4(z,\lambda), \tag{A1.2}$$

where $q = 1$ corresponds to the case when a signal wave with frequency ω_1 is incident on the plate, and $q = 2$ corresponds to the case when a signal wave with frequency ω_2 is incident on the plane; the plus sign corresponds to the incidence of the signal wave in the positive direction of z axis, and the minus sign – in the negative direction.

Consider the most important and easily implemented experimentally case $\lambda \in (0, \min\{k_{01}^2, k_{02}^2\})$, when k_{1z} and k_{2z} are real values. Substituting $\overleftrightarrow{\varphi}_\pm^{(q)}$ from (A1.2) in (4) and using (A1.1), we obtain the expressions for the coefficients $C_{m\pm}^{(q)}$:

$$C_{1+}^{(1)} = (1+R_{1+})E_{1+},\quad C_{1-}^{(1)} = T_{1-}E_{1-},\quad C_{2\pm}^{(1)} = G_\pm^{(21)}E_{1\pm},$$
$$C_{3+}^{(1)} = -ik_{1z}(1-R_{1+})E_{1+},\quad C_{3-}^{(1)} = ik_{1z}T_{1-}E_{1-},$$
$$C_{4\pm}^{(1)} = -ik_{2z}G_\pm^{(21)}E_{1\pm},\quad C_{1\pm}^{(2)} = \left(G_\pm^{(11)}E_{2\pm}\right)^*,$$
$$C_{2+}^{(2)} = [(1+R_{2+})E_{2+}]^*,\quad C_{2-}^{(2)} = (T_{2-}E_{2-})^*,$$
$$C_{3\pm}^{(2)} = ik_{1z}\left(G_\pm^{(11)}E_{2\pm}\right)^*,\quad C_{4+}^{(1)} = ik_{2z}[(1-R_{2+})E_{2+}]^*,$$
$$C_{4-}^{(2)} = -ik_{2z}(T_{2-}E_{2-})^*.$$

In addition, we find sixteen linear equations relating the elements of columns $\overleftrightarrow{\varphi}_m(z,\lambda)$ and their derivatives at point $z = z_2$ with coefficients $R_{q\pm}$, $T_{q\pm}$ and $G_\pm^{(qv)}$ ($q = 1, 2$, $v = 1, 2$):

$$T_{1-}f_1 + G_-^{(21)}f_2 = 1 + R_{1-},\quad \left(G_-^{(11)}\right)^*f_1 + T_{2-}^*f_2 = \left(G_-^{(12)}\right)^*,$$
$$T_{1-}f_3 + G_-^{(21)}f_4 = G_-^{(22)},\quad \left(G_-^{(11)}\right)^*f_3 + T_{2-}^*f_4 = 1 + R_{2-}^*, \tag{A1.3}$$
$$R_{1+}f_1 + G_+^{(21)}f_2 + f_5 = T_{1+},$$
$$\left(G_+^{(11)}\right)^*f_1 + R_{2+}^*f_2 + f_6 = \left(G_+^{(12)}\right)^*,$$
$$R_{1+}f_3 + G_+^{(21)}f_4 + f_7 = G_+^{(22)},\quad \left(G_+^{(11)}\right)^*f_3 + R_{2+}^*f_4 + f_8 = T_{2+}^*,$$

$$T_{1-}f_9 + G_-^{(21)}f_{10} = ik_{1z}(1-R_{1-}),$$
$$\left(G_-^{(11)}\right)^*f_9 + T_{2-}^*f_{10} = -ik_{1z}\left(G_-^{(12)}\right)^*,$$
$$R_{1+}f_9 + G_+^{(21)}f_{10} + f_{13} = -ik_{1z}T_{1+},$$
$$\left(G_+^{(11)}\right)^*f_9 + R_{2+}^*f_{10} + f_{14} = -ik_{1z}\left(G_+^{(12)}\right)^*, \tag{A1.4}$$
$$T_{1-}f_{11} + G_-^{(21)}f_{12} = ik_{2z}G_-^{(22)},$$
$$\left(G_-^{(11)}\right)^*f_{11} + T_{2-}^*f_{12} = -ik_{2z}(1-R_{2-}^*),$$
$$R_{1+}f_{11} + G_+^{(21)}f_{12} + f_{15} = ik_{2z}G_+^{(22)},$$
$$\left(G_+^{(11)}\right)^*f_{11} + R_{2+}^*f_{12} + f_{16} = ik_{2z}T_{2+}^*.$$

In (A1.3), (A1.4) we have used the notations

$$f_{1,5}(k_{1z}) \equiv \Psi_{1h}(\lambda) \pm ik_{1z}\Psi_{3h}(\lambda),$$
$$f_{2,6}(k_{2z}) \equiv \Psi_{2h}(\lambda) \mp ik_{2z}\Psi_{4h}(\lambda),$$
$$f_{3,7}(k_{1z}) \equiv \Psi_{1g}(\lambda) \pm ik_{1z}\Psi_{3g}(\lambda),$$
$$f_{4,8}(k_{2z}) \equiv \Psi_{2g}(\lambda) \mp ik_{2z}\Psi_{4g}(\lambda), \tag{A1.5}$$
$$f_{9,13}(k_{1z}) \equiv \Psi_{1hz}(\lambda) \pm ik_{1z}\Psi_{3hz}(\lambda),$$
$$f_{10,14}(k_{2z}) \equiv \Psi_{2hz}(\lambda) \mp ik_{2z}\Psi_{4hz}(\lambda),$$
$$f_{11,15}(k_{1z}) \equiv \Psi_{1gz}(\lambda) \pm ik_{1z}\Psi_{3gz}(\lambda),$$
$$f_{12,16}(k_{2z}) \equiv \Psi_{2gz}(\lambda) \mp ik_{2z}\Psi_{4gz}(\lambda),$$

where

$$\Psi_{mhz}(\lambda) = \left.\frac{d\varphi_{mh}(z,\lambda)}{dz}\right|_{z=z_2};\quad \Psi_{mgz}(\lambda) = \left.\frac{d\varphi_{mg}(z,\lambda)}{dz}\right|_{z=z_2};$$
$$\Psi_{mh}(\lambda) = \varphi_{mh}(z_2,\lambda);\quad \Psi_{mg}(\lambda) = \varphi_{mg}(z_2,\lambda).$$

From equations (A1.3), (A1.4) and the constancy of the Wronskian of (3), we can, in particular, obtain that $D_0 \equiv T_{1+}T_{2+}^* - \left(G_+^{(12)}\right)^*G_+^{(22)} = T_{1-}T_{2-}^* - \left(G_-^{(11)}\right)^*G_-^{(21)} \neq 0$ at $k_{1z}k_{2z} \neq 0$.

Thus, for $\lambda \in (0, \min\{k_{01}^2, k_{02}^2\})$ from equations (A1.3) we can find the functions $f_1(k_{1z})$, $f_2(k_{2z})$, $f_3(k_{1z})$ and $f_4(k_{2z})$:

$$f_1(k_{1z}) = \frac{T_{2-}^*(1+R_{1-}) - G_-^{(21)}\left(G_-^{(12)}\right)^*}{D_0},$$

$$f_2(k_{2z}) = \frac{T_{1-}\big(G_-^{(12)}\big)^* - (1+R_{1-})\big(G_-^{(11)}\big)^*}{D_0},$$
$$f_3(k_{1z}) = \frac{T_{2-}^* G_-^{(22)} - (1+R_{2-}^*)G_-^{(21)}}{D_0}, \tag{A1.6}$$
$$f_4(k_{2z}) = \frac{T_{1-}(1+R_{2-}^*) - G_-^{(22)}\big(G_-^{(11)}\big)^*}{D_0}.$$

On the other hand, the system of equations (3), obtained for nonnegative values of λ, can formally be considered at all, including complex, values of λ. For each fixed $z \in [z_1, z_2]$ its solutions $\varphi_{m\mathrm{h}}(z,\lambda)$ and $\varphi_{m\mathrm{g}}(z,\lambda)$ are the single-valued analytic functions of λ without any singular points in the final part of the plane, i.e., integer functions of λ [9, 10]. Hence, $\Psi_{m\mathrm{h}}$ and $\Psi_{m\mathrm{g}}$ are also integer functions of λ and, therefore, $k_{1z}^2 = k_{01}^2 - \lambda$ or $k_{2z}^2 = k_{02}^2 - \lambda$. The latter means that $\Psi_{m\mathrm{h}}$ and $\Psi_{m\mathrm{g}}$ are even integer functions of k_{1z} or k_{2z}, while $f_{1,3}$ and $f_{2,4}$ by the definitions (A1.5) are the integer functions of k_{1z} and k_{2z}, respectively. Given the parity of the functions $\Psi_{m\mathrm{h}}$ and $\Psi_{m\mathrm{g}}$ relative to $k_{1z,2z}$ from (A1.5) we obtain the relations:

$$\Psi_{1\mathrm{h},1\mathrm{g}}(\lambda) = \frac{f_{1,3}(k_{1z}) + f_{1,3}(-k_{1z})}{2},$$
$$\Psi_{3\mathrm{h},3\mathrm{g}}(\lambda) = \frac{f_{1,3}(k_{1z}) - f_{1,3}(-k_{1z})}{2\mathrm{i}k_{1z}},$$
$$\Psi_{2\mathrm{h},2\mathrm{g}}(\lambda) = \frac{f_{2,4}(k_{2z}) + f_{2,4}(-k_{2z})}{2}, \tag{A1.7}$$
$$\Psi_{4\mathrm{h},4\mathrm{g}}(\lambda) = \frac{-f_{2,4}(k_{2z}) + f_{2,4}(-k_{2z})}{2\mathrm{i}k_{2z}},$$

where $\lambda \equiv k_x^2 = k_{01}^2 - k_{1z}^2 = k_{02}^2 - k_{2z}^2$. Applying the results of [11] to system (3), we immediately obtain that to determine unambiguously $\varepsilon_{\mathrm{n}1}(z)$, $\varepsilon_{\mathrm{n}2}(z)$, $r_{12}(z)$ and $r_{21}(z)$ it is sufficient to know $\Psi_{m\mathrm{h}}$ and $\Psi_{m\mathrm{g}}$ on the whole complex plane of λ values.

Let the coefficients T_{1-}, R_{1-}, $G_-^{(2\nu)}$ ($\nu = 1, 2$) be known for some interval of angles of incidence $0 < \alpha_1^{(1)} \leqslant \alpha_1 \leqslant \alpha_1^{(2)} < \pi/2$ and the coefficients T_{2-}, R_{2-}, $G_-^{(1\nu)}$ – for some interval $0 < \alpha_2^{(1)} \leqslant \alpha_2 \leqslant \alpha_2^{(2)} < \pi/2$. In this case, $k_{01}\sin\alpha_1^{(1)} = k_{02}\sin\alpha_2^{(1)}$ and $k_{01}\sin\alpha_1^{(2)} = k_{02}\sin\alpha_2^{(2)}$. Then, using (A1.6), for real values of $k_{1z} \in [k_{01}\cos\alpha_1^{(2)}, k_{01}\cos\alpha_1^{(1)}]$ and $k_{2z} \in [k_{02}\cos\alpha_2^{(2)}, k_{02}\cos\alpha_2^{(1)}]$ we can find $f_{1,3}(k_{1z})$ and $f_{2,4}(k_{2z})$, which are the integer functions. This is sufficient for their unambiguous analytical continuation onto the whole complex plane of k_{1z} and k_{2z} values, respectively [10]. Knowing $f_{1,3}(k_{1z})$ and $f_{2,4}(k_{2z})$, and using (A1.7) we can find $\Psi_{m\mathrm{h}}(\lambda)$ and $\Psi_{m\mathrm{g}}(\lambda)$ for any λ, and thus uniquely determine $\varepsilon_{\mathrm{n}1}(z)$, $\varepsilon_{\mathrm{n}2}(z)$, $r_{12}(z)$ and $r_{21}(z)$. A similar result can be obtained using the known coefficients T_{1+}, R_{1+}, $G_+^{(2\nu)}$ and T_{2+}, R_{2+}, $G_+^{(1\nu)}$.

Appendix 2. Functional for the unique reconstruction of profiles of the coefficients $\varepsilon_{\mathrm{n}1}(z)$, $\varepsilon_{\mathrm{n}2}(z)$, $r_{12}(z)$ and $r_{21}(z)$ in the system of equations (3)

Suppose that for some range K of values k_x we exactly know the coefficients of transmission, reflection and conversion of the signal waves with frequencies ω_1 and ω_2 for a layer whose boundaries have coordinates $z = z_1$ and $z = z_2$. In other words, we know $T_{q+}(k_x)$, $R_{q+}(k_x)$, $G_+^{(q\nu)}(k_x)$ and (or) $T_{q-}(k_x)$, $R_{q-}(k_x)$, $G_-^{(q\nu)}(k_x)$. To reconstruct the coefficients $\varepsilon_{\mathrm{n}1}(z)$, $\varepsilon_{\mathrm{n}2}(z)$, $r_{12}(z)$ and $r_{21}(z)$ of the system of equations (3) we find eight solutions

$$\overset{\leftrightarrow}{\varphi}_p(z,\lambda) = \begin{pmatrix} \varphi_{\mathrm{h}p} \\ \varphi_{\mathrm{g}p} \end{pmatrix} \quad (p = 1, 2, 3, \ldots, 8)$$

of an auxiliary system of equations with four test functions $q_{ij}(z)$, coinciding with system (3) at $q_{11}(z) = \varepsilon_{\mathrm{n}1}(z)$, $q_{22}(z) = \varepsilon_{\mathrm{n}2}(z)$, $q_{12}(z) = r_{12}(z)$, $q_{21}(z) = r_{21}(z)$:

$$\frac{\mathrm{d}^2\varphi_\mathrm{h}}{\mathrm{d}z^2} + \left[\frac{\omega_1^2 q_{11}(z)}{c^2} - \lambda\right]\varphi_\mathrm{h} + \frac{\omega_1^2 q_{12}(z)\varphi_\mathrm{g}}{c^2} = 0,$$
$$\frac{\mathrm{d}^2\varphi_\mathrm{g}}{\mathrm{d}z^2} + \left[\frac{\omega_2^2 q_{22}^*(z)}{c^2} - \lambda\right]\varphi_\mathrm{g} + \frac{\omega_2^2 q_{21}^*(z)\varphi_\mathrm{h}}{c^2} = 0. \tag{A2.1}$$

Eight solutions (A2.1) we are interested in satisfy the boundary conditions:

$$\varphi_{\mathrm{h}1}(z_1) = a_{\mathrm{h}2}, \quad \left.\frac{\mathrm{d}\varphi_{\mathrm{h}1}}{\mathrm{d}z}\right|_{z=z_1} = b_{\mathrm{h}2},$$
$$\varphi_{\mathrm{g}1}(z_1) = a_{\mathrm{g}2}, \quad \left.\frac{\mathrm{d}\varphi_{\mathrm{g}1}}{\mathrm{d}z}\right|_{z=z_1} = b_{\mathrm{g}2},$$
$$\varphi_{\mathrm{h}2}(z_2) = a_{\mathrm{h}1}, \quad \left.\frac{\mathrm{d}\varphi_{\mathrm{h}2}}{\mathrm{d}z}\right|_{z=z_2} = b_{\mathrm{h}1},$$
$$\varphi_{\mathrm{g}2}(z_2) = a_{\mathrm{g}1}, \quad \left.\frac{\mathrm{d}\varphi_{\mathrm{g}2}}{\mathrm{d}z}\right|_{z=z_2} = b_{\mathrm{g}1},$$
$$\varphi_{\mathrm{h}3}(z_1) = a_{\mathrm{h}4}, \quad \left.\frac{\mathrm{d}\varphi_{\mathrm{h}3}}{\mathrm{d}z}\right|_{z=z_1} = b_{\mathrm{h}4},$$
$$\varphi_{\mathrm{g}3}(z_1) = a_{\mathrm{g}4}, \quad \left.\frac{\mathrm{d}\varphi_{\mathrm{g}3}}{\mathrm{d}z}\right|_{z=z_1} = b_{\mathrm{g}4},$$
$$\varphi_{\mathrm{h}4}(z_2) = a_{\mathrm{h}3}, \quad \left.\frac{\mathrm{d}\varphi_{\mathrm{h}4}}{\mathrm{d}z}\right|_{z=z_2} = b_{\mathrm{h}3},$$
$$\varphi_{\mathrm{g}4}(z_2) = a_{\mathrm{g}3}, \quad \left.\frac{\mathrm{d}\varphi_{\mathrm{g}4}}{\mathrm{d}z}\right|_{z=z_2} = b_{\mathrm{g}3}, \tag{A2.2}$$
$$\varphi_{\mathrm{h}5}(z_1) = a_{\mathrm{h}6}, \quad \left.\frac{\mathrm{d}\varphi_{\mathrm{h}5}}{\mathrm{d}z}\right|_{z=z_1} = b_{\mathrm{h}6},$$
$$\varphi_{\mathrm{g}5}(z_1) = a_{\mathrm{g}6}, \quad \left.\frac{\mathrm{d}\varphi_{\mathrm{g}5}}{\mathrm{d}z}\right|_{z=z_1} = b_{\mathrm{g}6},$$
$$\varphi_{\mathrm{h}6}(z_2) = a_{\mathrm{h}5}, \quad \left.\frac{\mathrm{d}\varphi_{\mathrm{h}6}}{\mathrm{d}z}\right|_{z=z_2} = b_{\mathrm{h}5},$$
$$\varphi_{\mathrm{g}6}(z_2) = a_{\mathrm{g}5}, \quad \left.\frac{\mathrm{d}\varphi_{\mathrm{g}6}}{\mathrm{d}z}\right|_{z=z_2} = b_{\mathrm{g}5},$$

$$\varphi_{h7}(z_1) = a_{h8}, \quad \left.\frac{d\varphi_{h7}}{dz}\right|_{z=z_1} = b_{h8},$$

$$\varphi_{g7}(z_1) = a_{g8}, \quad \left.\frac{d\varphi_{g7}}{dz}\right|_{z=z_1} = b_{g8},$$

$$\varphi_{h8}(z_2) = a_{h7}, \quad \left.\frac{d\varphi_{h8}}{dz}\right|_{z=z_2} = b_{h7},$$

$$\varphi_{g8}(z_2) = a_{g7}, \quad \left.\frac{d\varphi_{g8}}{dz}\right|_{z=z_2} = b_{g7}.$$

Here, $a_{h1,h4} = T_{1\pm}$; $a_{h2,h3} = 1 + R_{1\pm}$; $a_{h5,h7} = (G_{\pm}^{(12)})^*$; $a_{h6,h8} = (G_{\pm}^{(11)})^*$; $b_{h1,h4} = \mp ik_{1z}T_{1\pm}$; $b_{h2,h3} = \mp ik_{1z}(1 - R_{1\pm})$; $b_{h5,h7} = -ik_{1z}(G_{\pm}^{(12)})^*$; $b_{h6,h8} = ik_{1z}(G_{\pm}^{(11)})^*$; $a_{g1,g3} = G_{\pm}^{(22)}$; $a_{g2,g4} = G_{\pm}^{(21)}$; $a_{g5,g8} = T_{2\pm}^*$; $a_{g6,g7} = 1 + R_{2\pm}^*$; $b_{g1,g3} = ik_{2z}G_{\pm}^{(22)}$; $b_{g2,g4} = -ik_{2z}G_{\pm}^{(21)}$; $b_{g5,g8} = \pm ik_{2z}T_{2\pm}^*$; $b_{g6,g7} = \pm ik_{2z}(1 - R_{2\pm}^*)$.

Consider now a nonnegative functional

$$G_n[\hat{q}] = \int_K dk_x \sum_{p=1}^{8} \left\{ \mu_{hp}\left|\varphi_{hp}(\tilde{d}_p) - a_{hp}\right|^2 + \mu_{gp}\left|\varphi_{gp}(\tilde{d}_p) - a_{gp}\right|^2 + \beta_{hp}\left|\left.\frac{d\varphi_{hp}}{dz}\right|_{z=\tilde{d}_p} - b_{hp}\right|^2 + \beta_{gp}\left|\left.\frac{d\varphi_{gp}}{dz}\right|_{z=\tilde{d}_p} - b_{gp}\right|^2 \right\} \tag{A2.3}$$

on the set of four test profiles $\hat{q}(z) = \{q_{11}(z),\ q_{12}(z), q_{21}(z), q_{22}(z)\}$, constructed in accordance with the principles described in detail in [4]. In (A2.3), $\tilde{d}_{1,3,5,7} = z_2$, $\tilde{d}_{2,4,6,8} = z_1$, and the weight coefficients μ_{hp}, μ_{gp}, β_{hp} and β_{gp} are any fixed nonnegative numbers. Moreover, if we only know T_{q+}, R_{q+}, $G_{+}^{(q\nu)}$ ($q = 1, 2;\ \nu = 1, 2$), then $\mu_{h1,h2,h5,h6} \neq 0$, $\mu_{g1,g2,g5,g6} \neq 0$, $\beta_{h1,h2,h5,h6} \neq 0$, $\beta_{g1,g2,g5,g6} \neq 0$, while the remaining weight coefficients are equal to zero. If only T_{q-}, R_{q-}, $G_{-}^{(q\nu)}$ are known, then, on the contrary, $\mu_{h3,h4,h7,h8} \neq 0$, $\mu_{g3,g4,g7,g8} \neq 0$, $\beta_{h3,h4,h7,h8} \neq 0$, $\beta_{g3,g4,g7,g8} \neq 0$, while the remaining weight coefficients are equal to zero.

The functional $G_n[\hat{q}]$ is a measure of the difference of the coefficients of transmission, reflection and conversion $\tilde{R}_{q\pm}$, $\tilde{T}_{q\pm}$, $\tilde{G}_{\pm}^{(q\nu)}$ for a layer with a set of profiles $\hat{q}(z)$ from the measured coefficients. Indeed, a comparison of formulas (4) at $E_{q\pm} = 1$ with (A2.2), (A2.3) shows that $G_n = 0$ only in the case of complete coincidence of the coefficients $\tilde{T}_{q+}(k_x)$, $\tilde{R}_{q+}(k_x)$ and $\tilde{G}_{+}^{(q\nu)}(k_x)$ and (or) $\tilde{T}_{q-}(k_x)$, $\tilde{R}_{q-}(k_x)$ and $\tilde{G}_{-}^{(q\nu)}(k_x)$ with the coefficients $T_{q+}(k_x)$, $R_{q+}(k_x)$, $G_{+}^{(q\nu)}(k_x)$ and (or) $T_{q-}(k_x)$, $R_{q-}(k_x)$, $G_{-}^{(q\nu)}(k_x)$ in the range of K. In Appendix 1 it was proved that this coincidence is only possible in one case. Thus, the reconstruction of $\varepsilon_{n1}(z)$, $\varepsilon_{n2}(z)$, $r_{12}(z)$ and $r_{21}(z)$ is reduced to finding a set of test functions $\hat{q}^{(0)}(z)$, which correspond to the only zero minimum of the functional $G_n[\hat{q}]$. Note that the profiles $\varepsilon_{yy}(z, \omega_q)$ of the linear permittivity of the medium and the spatial distribution of the electric field $E_f(z)$ of a high-power wave in the plate can be reconstructed using the method proposed in [7] and [4], respectively. Having found $\hat{q}^{(0)}(z)$ and knowing $\varepsilon_{yy}(z, \omega_q)$ and $E_f(z)$, we obtain

$$\chi_{yyyy}^{(3)}(z, \omega_q, -\omega_l, \omega_3, \omega_3) = \frac{q_{ql}^{(0)}(z)}{4\pi E_f^2(z)},$$

$$\chi_{yyyy}^{(3)}(z, \omega_q, \omega_q, -\omega_3, \omega_3) = \frac{q_{qq}^{(0)}(z) - \varepsilon_{yy}(z, \omega_q)}{8\pi |E_f(z)|^2},$$

where $q = 1, 2$; $l = 1 + \delta_{q1}$; $\omega_1 + \omega_2 = 2\omega_3$.

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