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Elliptically polarised cnoidal waves in a medium with spatial dispersion of cubic nonlinearity

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Abstract. **We present new specific analytic solutions of a system of nonlinear Schrödinger equations, corresponding to elliptically polarised cnoidal waves in an isotropic gyrotropic medium with spatial dispersion of cubic nonlinearity and second-order frequency dispersion under the conditions of formation of the waveguides of the same type for each of the circularly polarised components of the light field.**

Keywords: *cubic nonlinearity, spatial dispersion, nonlinear Schrödinger equations, elliptic polarisation, cnoidal waves.*

Possible emergence of elliptically polarised cnoidal waves, whose orthogonal components of the electric field vector are expressed in terms of Jacobi elliptic functions, is widely discussed in solving various problems of nonlinear optics (see, for example, [1–7]). For isotropic optically active media with local cubic nonlinearity, this possibility was discussed in [7]. In this case, however, nonlocal component of the nonlinear optical response, whose magnitude is very significant in gyrotropic media, was not taken into account. Nonlocality essentially affects the nature of self-focusing of beams [8, 9], pulse compression [10, 11], and other regimes of propagation [12] of elliptically polarised laser radiation in optically active media. Due to the nonlocality of the nonlinear response in media with anomalous frequency dispersion, consideration of the effects of self- and cross-modulation in the system of truncated equations for slowly varying amplitudes of circularly polarised waves [8–12], which form a system of nonlinear Schrödinger equations (NSE), leads to solutions in the form of elliptically polarised solitary waves [12]. At the same time, the problem of existence of periodic analytical solutions of this system, i.e., elliptically polarised cnoidal waves, remains virtually uninvestigated. This is primarily due to the nonintegrability of the NSE system describing nonlinear propagation of waves in media with spatial dispersion of cubic nonlinearity [13].

In this paper, we present for the first time the derived specific solutions of the NSE system describing propagation of laser radiation in an isotropic medium with spatial dispersion of cubic nonlinearity and second-order frequency dispersion. The solutions have the form of elliptically polarised cnoidal waves with the amplitudes of the two circularly polarised components $A_\pm(z,t) = A_x \pm \mathrm{i}A_y$ being connected by some relation. We discuss the physical meaning of this relation and present the dependence of the intensity $I(z,\ t) = (|A_+|^2 + |A_-|^2)/2$ of cnoidal waves of these types, degrees of ellipticity $M(z,\ t) = (|A_+|^2 - |A_-|^2)/(2I)$ and angle of rotation $\Psi(z,t) = 0.5\arg(A_+A_-^*)$ of the major axis of the polarisation ellipse on the longitudinal coordinate z and time t in the intrinsic (running) coordinate system. We also determine the conditions for the existence of these solutions.

In the absence of diffraction, self-action of laser radiation in a nonlinear isotropic gyrotropic medium with spatial dispersion of cubic nonlinearity and second-order frequency dispersion is described by the NSE system [9–12]:

$$\frac{\partial A_\pm}{\partial z} - \frac{\mathrm{i}k_2}{2}\frac{\partial^2 A_\pm}{\partial t^2} + \mathrm{i}\Big[\mp\rho_0 + \Big(\frac{\sigma_1}{2}\mp\rho_1\Big)|A_\pm|^2 + \Big(\frac{\sigma_1}{2}+\sigma_2\Big)|A_\mp|^2\Big]A_\pm = 0. \tag{1}$$

Here, $\sigma_{1,2} = 2\pi\omega^2\chi_{1,2}/(kc^2)$; $k_2 = \mathrm{d}^2k/\mathrm{d}\omega^2$ is a constant characterising the frequency dispersion of the medium; $\rho_{0,1} = 2\pi\omega^2\gamma_{0,1}/c^2$; $\chi_1 = 2\chi^{(3)}_{xyxy}$; $\chi_2 = \chi^{(3)}_{xxyy}$. In these relation, $\hat{\chi}^{(3)}(\omega; -\omega, \omega, \omega)$ is a local cubic nonlinearity tensor symmetric with respect to permutation of two last indexes. Pseudoscalar constants of linear and nonlinear gyration γ_0 and γ_1 define the nonzero tensor components of nonlocal linear and nonlinear optical susceptibilities, respectively.

Because in the general case system (1) is nonintegrable, we impose an additional constraint on the desired solutions in the form of a second-order integral, i.e., a linear relationship between the intensities of the two circularly polarised components of the propagating wave:

$$\delta_+|A_+(z,t)|^2 + \delta_-|A_-(z,t)|^2 = \delta_0, \tag{2}$$

where the constants $\delta_{0,\pm}$ will be defined below. Note that when condition (2) is fulfilled, expressions in square brackets in (1) are also linearly related. In fact, they describe changes in the refractive index due to local and nonlocal cubic nonlinearity. Therefore, when the waves described by the required solutions propagate in the medium, for each of the components $A_\pm(z,t)$ there will be formed, due to nonlinearity, nonlinear waveguides of the same type, whose profiles differ only in scale factors.

We will use below a standard procedure of separation of variables, assuming $A_\pm(z,\ t) = R_\pm(t)\exp(\mathrm{i}z\kappa_\pm)$. We emphasise that the phases of circularly polarised components of the field

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increase linearly with increasing coordinate z, whereas the amplitudes $R_\pm(t)$ are real and depend only on time. Due to (2), the constants $\kappa_\pm$ become the eigenvalues of two independent equations, following directly from (1):

$$\frac{d^2 R_\pm}{dt^2} - \frac{2}{k_2}\left[(\kappa_\pm \mp \rho_0 + \delta_0\delta_\mp^{-1}) + \left(\frac{\sigma_1}{2} \mp \rho_1 - \frac{\delta_\pm}{\delta_\mp}\right)R_\pm^2\right]R_\pm = 0. \quad (3)$$

All possible real periodic solutions to the two obtained ordinary differential equations can be expressed in terms of Jacobi elliptic functions [2, 14]: $\mathrm{sn}(vt, \mu)$, $\mathrm{cn}(vt, \mu)$ and $\mathrm{dn}(vt, \mu)$, where v is an arbitrary real scale factor, and μ is a modulus of the elliptic functions. The relations between the elliptic functions [15] allow us to express $\delta_{0,\pm}$ via six quantities: $\kappa_\pm$, v, μ and the maximum values of $R_\pm$. Two of them are free parameters of problem (3). As latter ones, it is convenient to choose v and μ because it is the square of the Jacobi elliptic function that defines the profile of the nonlinear waveguide for each of the circular polarised waves. Note that a similar situation occurred for stable orthogonally polarised multicomponent cnoidal waves in photorefractive media [3]. All possible pairwise combinations of elliptic functions produce a family of physically different specific solutions of problem (1)–(3) in the form of nine cnoidal waves. For convenience, we denote them by first letters of Jacobi elliptic functions entering into the expressions for $A_+(z, t)$ and $A_-(z, t)$, i.e., ss, cc, dd, sc, cs, sd, ds, cd and dc.

The solutions sc, cs, sd and ds are possible if the inequalities $\rho_1 > 0$, $k_2(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2) < 0$ and $-\rho_1 < \sigma_2 < \rho_1$ or inequalities $\rho_1 < 0$, $k_2(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2) > 0$ and $\rho_1 < \sigma_2 < -\rho_1$ are fulfilled. If the parameters of a nonlinear gyrotropic isotropic medium satisfy the conditions $\sigma_2 > 0$, $-\sigma_2 < \rho_1 < \sigma_2$ and $k_2(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2) > 0$ or the conditions $\sigma_2 < 0$, $\sigma_2 < \rho_1 < -\sigma_2$ and $k_2(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2) < 0$, the solution ss is realised. Finally, the solutions cd, dc, cc and dd are possible if the inequalities $\sigma_2 > 0$, $k_2(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2) < 0$ and $-\sigma_2 < \rho_1 < \sigma_2$ or inequalities $\sigma_2 < 0$, $k_2(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2) > 0$ and $\sigma_2 < \rho_1 < -\sigma_2$ are valid.

In the optical frequency range the size of the region of manifestation of the optical response nonlocality is significantly smaller than the wavelength, and $|\rho_1| \ll |\sigma_{1,2}|$. Therefore, the solutions sc, sd, cs and ds cannot be realised. Below we present the expressions for the components $A_\pm(z, t)$, corresponding to physically realisable (in the case of gyrotropic media) solutions to problem (1), (2), as well as for the intensities (which correspond to these solutions) of the propagating cnoidal waves, degrees of ellipticity of their polarisation ellipses and angles of rotation of the major axes of these ellipses. For the solutions such as cd and dc we have:

$$A_\pm(z,t) = \left\{\mu v\left[-\frac{k_2(\sigma_2 \mp \rho_1)}{\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2}\right]^{1/2}\right\}\mathrm{cn}(vt,\mu)\exp\left[\pm iz\rho_0 + izv^2 k_2 \frac{\mu^2(\sigma_2 \mp \rho_1)(\sigma_1 \mp 2\rho_1) - (\rho_1^2 - \sigma_2^2 \mp \rho_1\sigma_1 \mp 2\rho_1\sigma_2)}{2(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2)}\right], \quad (4)$$

$$A_\mp(z,t) = \left\{v\left[-\frac{k_2(\sigma_2 \pm \rho_1)}{\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2}\right]^{1/2}\right\}dn(vt,\mu)\exp\left[\mp iz\rho_0 + izv^2 \times k_2 \frac{(\sigma_2 \pm \rho_1)(\sigma_1 \pm 2\rho_1) - \mu^2(\rho_1^2 - \sigma_2^2 \pm \rho_1\sigma_1 \pm 2\rho_1\sigma_2)}{2(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2)}\right], \quad (5)$$

$$I(t) = -v^2 k_2 \frac{(\sigma_2 \pm \rho_1)(1-\mu^2) + 2\mu^2\sigma_2\mathrm{cn}^2(vt,\mu)}{2(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2)}, \quad (6)$$

$$M(t) = \mp\frac{(\sigma_2 \pm \rho_1)(1-\mu^2) \pm 2\mu^2\rho_1\mathrm{cn}^2(vt,\mu)}{(\sigma_2 \pm \rho_1)(1-\mu^2) + 2\mu^2\sigma_2\mathrm{cn}^2(vt,\mu)}, \quad (7)$$

$$\Psi(z) = z\left[\rho_0 \pm v^2 k_2 \frac{(3\rho_1^2 + \sigma_1\sigma_2 - \sigma_2^2)(\mu^2 - 1)}{4(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2)}\right]. \quad (8)$$

In formulas (4)–(8) the upper sign corresponds to the solution cd, lower – to dc. Note that the linear dependence of Ψ on the coordinate z contains both linear and nonlinear characteristics of the medium.

Solutions of problem (1), (2) in the form of the three remaining functions (ss, cc and dd) are degenerate, i.e., they contain in the expressions for $A_+(z, t)$ and $A_-(z, t)$ identical elliptic functions

$$A_\pm(z,t) = \frac{\mu v[k_2(\sigma_2 \mp \rho_1)]^{1/2}}{(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2)^{1/2}}\mathrm{sn}(vt,\mu) \times \exp\left\{iz\left[\pm\rho_0 - \frac{k_2 v^2(\mu^2+1)}{2}\right]\right\}, \quad (9)$$

$$A_\pm(z,t) = \frac{\mu v[-k_2(\sigma_2 \mp \rho_1)]^{1/2}}{(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2)^{1/2}}\mathrm{cn}(vt,\mu) \times \exp\left\{iz\left[\pm\rho_0 + \frac{k_2 v^2(2\mu^2-1)}{2}\right]\right\}, \quad (10)$$

$$A_\pm(z,t) = \frac{v[-k_2(\sigma_2 \mp \rho_1)]^{1/2}}{(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2)^{1/2}}\mathrm{dn}(vt,\mu) \times \exp\left\{iz\left[\pm\rho_0 + \frac{k_2 v^2(2-\mu^2)}{2}\right]\right\}. \quad (11)$$

For each of these solutions, the degree of ellipticity is constant: $M(t) = -\rho_1/\sigma_2$ and the angle of rotation of the major axis of the polarisation ellipse depends linearly on the coordinate z and the coefficient of linear gyration: $\Psi(z) = \rho_0 z$. The time dependence of the intensities of cnoidal waves, corresponding to solutions of (9)–(11), are obtained by substituting sequentially the functions $-\mathrm{sn}^2(vt, \mu)$, $\mathrm{cn}^2(vt, \mu)$ and $\mathrm{dn}^2(vt, \mu)/\mu^2$ instead of F into the general expression $-\mu^2 v^2 k_2\sigma_2 F \times (\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2)^{-1}$.

Note in conclusion that we failed to find in the literature any references that the condition for formation of nonlinear waveguides, of the same type for the two components of the field, in a medium (i.e., waveguides, whose profiles differ only in scale factors) is equivalent to introducing a linear relationship between the intensities of these components (i.e., the second-order integral). The obtained new types of specific periodic solutions of the NSE system seem to be of interest not only for the applied problems of fibre optics and optics of media with spatial dispersion of cubic nonlinearity, but also for solving a sufficiently wide class of problems from other areas of physics [16]. All the mentioned specific periodic solutions have soliton asymptotics in the limit $\mu = 1$. In this case, the solutions cd, dc, cc and dd are transformed into a pair of 'bright' solitons, whereas the solutions ss – into a pair of 'dark' solitons, which coincide with solitary solutions obtained in [12]. This is a direct consequence of the fact that a linear relationship between the wave amplitudes considered in [12] is equivalent to condition (2) used by us at $\mu = 1$.

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