

STIMULATED EMISSION

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Elementary processes of stimulated emission

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Abstract. **We consider the simplest process of stimulated emission, i.e., evolution of an excited two-level atom interacting with a quantized electromagnetic field being initially in a one-photon state under different boundary conditions imposed on the field. We study the kinetics of evolution, spectral properties and modal composition of the radiation field emerging in such processes. It is shown that in the general case this field is only partly composed of doubly occupied modes.**

Keywords: *two-level atom, photon, spontaneous emission, stimulated emission, cavity, damping, identity of photons.*

1. Introduction

Let us start with the encyclopaedic definition: 'Stimulated emission, induced emission, emission of electromagnetic radiation by quantum systems as a result of interaction with incident radiation. Photons created in stimulated emission have the same frequency, direction of travel and polarisation as the photons triggering the emission' [1]. In the conventional interpretation, stimulated emission occurs when incident radiation interacts with quantum systems in excited states (below we will consider these systems to have a discrete energy spectrum and will call them the atoms).

The assertion of the second sentence in the quotation may be called the central dogma of quantum electronics. It is widely known: It can be found in the Nobel lectures (1964) of the founders of quantum electronics – Ch.H. Townes [2, p. 60], N.G. Basov [2, p. 93], A.M. Prokhorov [2, p. 110], – and in today's on-line encyclopaedias [3, 4], and in popular textbooks [5–7].

The theoretical basis of the central dogma is commonly assumed to originate from Dirac's results presented in his paper [8], where he constructed the generally accepted Hamiltonian scheme describing the interaction of atoms with a quantized electromagnetic field. A specific goal of paper [8] was the derivation of expressions for the Einstein coefficients A and B of the model in which the electromagnetic field has been quantized, i.e., the solution of the same (as that in Einstein's papers [9–11]) problem of the kinetics of the probability distribution over the stationary states of an atom under the action of a given broadband electromagnetic field (the Einstein–Dirac problem).

Due to the property of the matrix elements of the photon creation operator

$$\hat{a}^+ \big| N\big\rangle = \sqrt{N+1}\big| N+1\big\rangle, \tag{1}$$

in the first order of the perturbation theory the transition rate $\bar{N}$ from the upper energy state to the lower one increases, compared with the same rate in the absence of the field $\dot{W}_0$,

$$\dot{W} = (\bar{N}+1)\dot{W}_0, \tag{2}$$

where $\dot{W}$ is the average number of photons in the field modes that are resonant with the transition. Equation (2) coincides with the Einstein equation $\mathrm{d}W = (A + B\rho)\mathrm{d}t$ (ρ is the spectral density of radiation at the transition frequency). The increase in the transition rate is due to the processes leading to the emergence of the field modes, whose occupation exceeds unity. Because the photons of the same mode are identical in structure, we can draw the following conclusion: An increase in the rate of transitions from the excited state by an external field is due to the emission of photons that are identical to those in the initial state of the field.

Such a conclusion is correct within the framework of the formulated Einstein–Dirac problem about the evolution of an atom (in general, a system with a discrete spectrum) under the action of a broadband external field, whose spectral density is continuous and varies little in the intervals of the order of the natural linewidth. However, we should remember that in deriving formula (2), summation over the field modes is replaced by integration, and the value $\bar{N}(\omega)$ assumed constant is removed from the integral ([8], see also [12, § 29]). Thus, the Einstein–Dirac problem describes not an elementary process of interaction between a single photon and an atom but a more complex process that occurs in the presence of many photons with different frequencies.

By an elementary process of stimulated emission we mean the evolution of an atom interacting with a quantized electromagnetic field from the initial state in which the atom is excited, and the field is in a state with one photon in one of the field modes. The aim of this paper is to examine the kinetics and spectral properties of radiation of such processes under various boundary conditions imposed on the electromagnetic field.

We consider a system with a Hamiltonian

$$\hat{H} = \hat{H}_\mathrm{a} + \hat{H}_\mathrm{f} + \hat{V}, \tag{3}$$

where $\hat{H}_\mathrm{a}$ is the Hamiltonian of a two-level atom;

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$$\hat{H}_{\mathrm{f}} = \sum_{\mu} \hbar\omega_{\mu} \hat{a}_{\mu}^{+} \hat{a}_{\mu} \tag{4}$$

is the Hamiltonian of the quantized electromagnetic field; the subscript μ numbers the modes of the electromagnetic field; ω_{μ} is the frequency of the mode μ; $\hat{a}_{\mu}^{+}$ and $\hat{a}_{\mu}$ are the creation and annihilation operators of photons in the mode μ. The interaction operator, following [13, p. 261], is taken in the dipole approximation:

$$\hat{V} = \sum_{\mu} (\hat{v}_{\mu} + \hat{v}_{\mu}^{+}). \tag{5}$$

Here

$$\hat{v}_{\mu} = -\mathrm{i}\sqrt{\frac{2\pi\hbar\omega_{\mu}}{\mathcal{V}}}\, \hat{\boldsymbol{d}}\, \hat{a}_{\mu}^{+} \boldsymbol{E}_{\mu}(\boldsymbol{r}); \tag{6}$$

$\mathcal{V}$ is the volume of the region containing the electromagnetic field; $\hat{\boldsymbol{d}}$ is the operator of the dipole moment; $\boldsymbol{E}_{\mu}(\boldsymbol{r})$ is the distribution of the electric field strength in the mode μ, normalised by the condition

$$\int \left| \boldsymbol{E}_{\mu}(\boldsymbol{r}) \right|^2 \mathrm{d}\boldsymbol{r} = \mathcal{V}. \tag{7}$$

2. Elementary process of stimulated emission in the cavity (single-mode case, damping is absent)

Let an atom interact with a single mode of the cavity field. In this case, the identity of all the photons is ensured by the construction, but the problem about the kinetics of stimulated emission remains open. In the absence of the field decay, this model was considered by Jaynes and Cummings [14]. The Hamiltonian (3) can be written in traditional form, using the rotating field approximation:

$$\hat{H}_{\mathrm{JC}} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\omega\hat{a}^{+}\hat{a} + \mathrm{i}\hbar\frac{\Omega}{2}(\hat{\sigma}_{+}\hat{a} - \hat{a}^{+}\hat{\sigma}_{-}), \tag{8}$$

where $\hat{\sigma}_i$ are the Pauli matrices $[\hat{\sigma}_{\pm} = (\hat{\sigma}_x \pm \mathrm{i}\hat{\sigma}_y)/2]$. To simplify the formulas, we restrict ourselves to the case of exact resonance, in which the frequencies of the atomic transition, ω_0, and the photon of the single mode, ω, coincide. The vacuum Rabi frequency is

$$\Omega = 2\zeta\sqrt{\frac{2\pi\omega}{\hbar\mathcal{V}}}\, d, \tag{9}$$

where d is the modulus of the matrix element of the component of the dipole moment of transition, and ζ is a dimensionless geometric factor that depends on the shape of the cavity, mode type and position of the atom in the cavity. For example, for the fundamental TM modes of a cylindrical cavity and an atom in the antinode of the field (on the axis of the cavity) $\zeta = 1.926$ [14].

If in the initial state the field is in the N-photon state, the model becomes a two-level system with the state vector $|\Psi\rangle = K|+, N\rangle + L|-, N+1\rangle$ (in the basic state $|+, N\rangle$ the atom is in the upper state, in the mode of N photons; and in the basic state $|-, N+1\rangle$ the atom is in the lower state, in the mode of $N+1$ photons). The equations of motion for the amplitudes have the form

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \sqrt{N+1}\,\frac{\Omega}{2}L, \quad \frac{\mathrm{d}L}{\mathrm{d}t} = -\sqrt{N+1}\,\frac{\Omega}{2}K. \tag{10}$$

Solutions of system (10) describe the exchange of energy between the atom and the field, occurring at the N-photon Rabi frequency $\tilde{\Omega} = \sqrt{N+1}\,\Omega$. For the initial conditions $K(0) = 1$, $L(0) = 0$, the probability of finding an atom in the upper state is

$$W_{+}(t) = \frac{1}{2}(1 + \cos\sqrt{N+1}\,\Omega t). \tag{11}$$

Consider, for example, an ammonia molecule (the transition frequency, $\omega_0 = 1.50\times10^{11}\,\mathrm{s}^{-1}$; the matrix element of the dipole matrix, $d = 1.47\times10^{-18}$ CGS units) in a cylindrical cavity of length $L = 10$ cm tuned to the fundamental mode frequency ω_0. Then, for the vacuum Rabi frequency, we obtain from (9) $\Omega = 62.8\,\mathrm{s}^{-1}$. It is interesting to compare this quantity with the rate Γ_{s} of spontaneous radiative transition in a free molecule; for the selected parameter values we have

$$\Gamma_{\mathrm{s}} = \frac{4d^2\omega_0^3}{3\hbar c^3} = 3.41\times10^{-7}\mathrm{s}^{-1}. \tag{12}$$

The presence of a cavity tuned to the transition frequency increases, in our example, the rate of energy transfer from the atom to the field by about 10^8 times.

3. Elementary process of stimulated emission in the cavity (single-mode case, damping is present)

The cavities of the first masers had Q-factors in the range from 1000 [15] to 12000 [16]. Having taken a larger value, under the conditions of our example we obtain the rate of the field decay

$$\kappa = \frac{\omega}{2Q} = 6.25\times10^{6}\mathrm{s}^{-1} \approx 10^{5}\Omega, \tag{13}$$

i.e., in this case, the 'atom' (a molecule of ammonia) in the cavity is a strongly damped system, and the Jaynes–Cummings model (8) is not applicable to it even approximately.

Progress in experimental facilities and methods (use of high-Q superconducting cavities, use of Rydberg atoms with a large dipole transition matrix element, etc.) made it possible to reach the region of small damping $\kappa \ll \Omega$. This was first done in the microwave range by Meschede et al. ($\kappa = 0.18\Omega$ [17]), and in the optical range by Kimble ($\kappa = 0.14\Omega$ [18]). It is therefore of interest to study the influence of damping on the kinetics of stimulated emission in a cavity in a wide range of values of the ratio $\eta = \kappa/\Omega$.

Dynamics of a two-level atom bound to a single mode of the quantized electromagnetic field in a cavity with damping was first investigated by Sachdev [19]. The model was based on an equation for the density matrix of the system

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{\mathrm{i}}{\hbar}[\hat{H}_{\mathrm{JC}}, \hat{\rho}] - \kappa(\hat{a}^{+}\hat{a}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}^{+} + \hat{\rho}\hat{a}^{+}\hat{a}). \tag{14}$$

Hereafter we restrict ourselves to the case of zero temperature of the cavity. Instead of considering the equations for the density matrix elements we can construct a system of equations for the moments

$$A_p = \langle(\hat{a}^{+})^p \hat{a}^p\rangle, \quad p \geqslant 0, \quad A_0 = 1, \tag{15}$$

$$B_p = \frac{1}{2}\langle(\hat{a}^{+})^p \hat{\sigma}_z \hat{a}^p\rangle, \quad p \geqslant 0, \quad B_0 = \frac{1}{2}\langle\hat{\sigma}_z\rangle, \tag{16}$$

$$C_p = \left\langle (\hat{a}^+)^p \hat{a}^{p-1}\hat{\sigma}_- + \hat{\sigma}_+ (\hat{a}^+)^{p-1}\hat{a}^p \right\rangle, \quad p > 0. \tag{17}$$

For spontaneous emission ($N = 0$), this system takes the form

$$\frac{\mathrm{d}B_0}{\mathrm{d}t} = \frac{\Omega}{2} C_1, \qquad \frac{\mathrm{d}A_1}{\mathrm{d}t} = -\frac{\Omega}{2} C_1 - 2\kappa A_1,$$

$$\frac{\mathrm{d}B_1}{\mathrm{d}t} = \frac{\Omega}{4} C_1 - 2\kappa B_1, \quad \frac{\mathrm{d}C_1}{\mathrm{d}t} = -\Omega B_0 - 2\Omega B_1 - \frac{\Omega}{2} - \kappa C_1 \tag{18}$$

(A_p, B_p, $C_p \equiv 0$ при $p \geqslant 2$). System (18) permits an analytic solution [19], but it is cumbersome, and we restrict our consideration to asymptotics for the limiting cases. In the case of weak damping ($\kappa \ll \Omega$), the probability of finding an atom in the upper state, $W_+ = B_0 + 1/2$, varies according to the law

$$W_+(t) \approx \frac{\mathrm{e}^{-\kappa t}}{2}(1 + \cos \Omega t), \tag{19}$$

i.e., performs Rabi oscillations (11) with the vacuum frequency, damping at a rate equal to half the rate of decay of the field energy without interaction with an atom. This can be easily interpreted: On average the system stores energy only half the time in a damping field mode. In the case of large damping ($\kappa \gg \Omega$), the population decreases exponentially

$$W_+(t) \approx \exp\left(-\frac{\Omega^2}{2\kappa} t\right). \tag{20}$$

The exponent in (20) can be written as

$$\frac{\Omega^2}{2\kappa} = \frac{4d^2\omega^3}{3\hbar c^3} \frac{3\zeta^2 Q \lambda^3}{4\pi^2 \mathcal{V}} = \Gamma_s \frac{3\zeta^2 Q \lambda^3}{4\pi^2 \mathcal{V}}, \tag{21}$$

where $\lambda = 2\pi c/\omega$ is the wavelength of resonant radiation. This result was first obtained by Purcell from the phenomenological model in which the geometrical factor had a universal value $\zeta = 1$ [20].

The simplest case of stimulated emission ($N = 1$) is described by the seventh-order system that is similar to (18):

$$\frac{\mathrm{d}B_0}{\mathrm{d}t} = \frac{\Omega}{2} C_1, \qquad \frac{\mathrm{d}A_1}{\mathrm{d}t} = -\frac{\Omega}{2} C_1 - 2\kappa A_1,$$

$$\frac{\mathrm{d}B_1}{\mathrm{d}t} = \frac{\Omega}{4} C_1 - 2\kappa B_1 + \frac{\Omega}{2} C_2, \quad \frac{\mathrm{d}C_1}{\mathrm{d}t} = -\Omega B_0 - 2\Omega B_1 - \frac{\Omega}{2} - \kappa C_1,$$

$$\frac{\mathrm{d}A_2}{\mathrm{d}t} = -\Omega C_1 - 4\kappa A_2, \qquad \frac{\mathrm{d}B_2}{\mathrm{d}t} = \frac{\Omega}{2} C_2 - 4\kappa B_2, \tag{22}$$

$$\frac{\mathrm{d}C_2}{\mathrm{d}t} = -2\Omega B_1 - 2\Omega B_2 - \Omega - 3\kappa C_2$$

(A_p, B_p, $C_p \equiv 0$ at $p \geqslant 3$). The general analytical solution of system (22) is impossible, but it can be easily solved numerically. Figure 1 shows the time dependences of the probability of finding an atom in an excited state for the spontaneous [$N(0) = 0$] and simplest stimulated [$N(0) = 1$] cases.

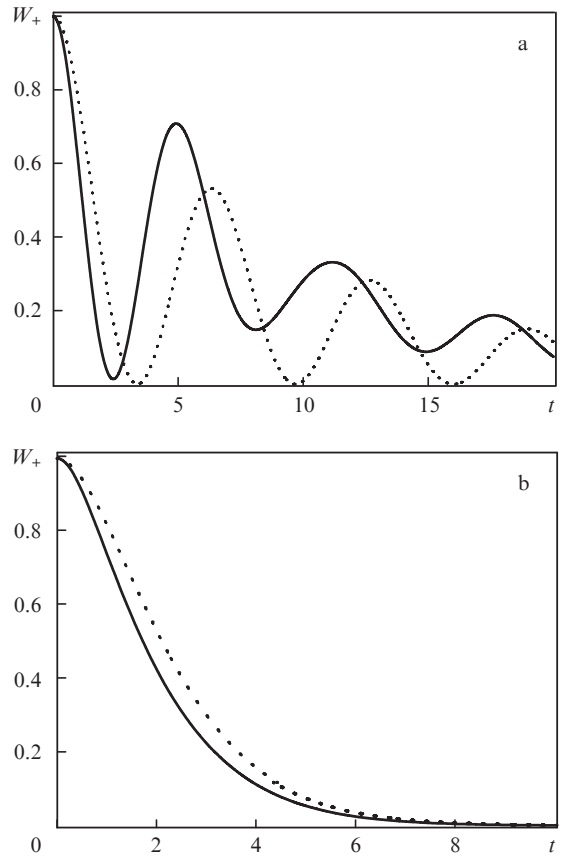

Figure 1. Time dependences of the probabilities W_+ of an initially excited two-level atom, interacting with a resonant damped mode of the electromagnetic field, to reside in the excited state (in units of Ω^{-1}) for the cases of small damping $\kappa = 0.1\ \Omega$ (a) and boundary damping $\kappa = \Omega$ (b). The dashed curves show the spontaneous emission (the initial state of the field $|0\rangle$), solid curves show the elementary stimulated emission (the initial state of the field $|1\rangle$).

We define the effective rate of emission Γ by the equation

$$\frac{1}{\Gamma} = \int_0^\infty W_+(t)\mathrm{d}t. \tag{23}$$

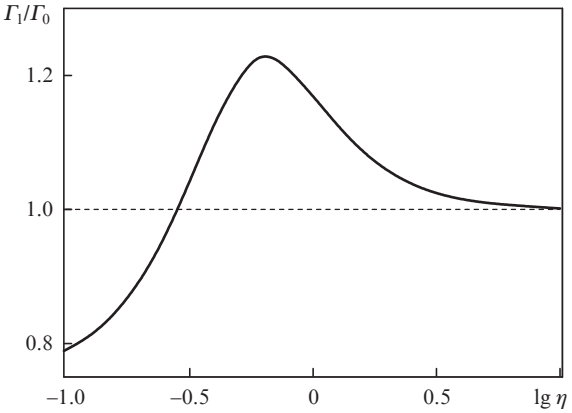

Figure 2. Ratio of the effective rate of the elementary stimulated emission (Γ_1) and spontaneous emission (Γ_0) of a two-level atom with a single resonant field mode vs. $\lg\eta = \lg(\kappa/\Omega)$.

The dependence of the ratio of the effective rate of the simplest stimulated (Γ_1) and spontaneous (Γ_0) emissions on $\eta = \kappa/\Omega$ is shown in Fig. 2.

One can see that in the case of small damping, the stimulated emission occurs slower than the spontaneous emission; at a vanishingly low damping the ratio Γ_1/Γ_0 tends to 0.75. If $\eta > 0.28$, the stimulated emission is faster than the spontaneous emission. The maximum ratio Γ_1/Γ_0 is achieved at $\eta = 0.65$ and equals 1.23.

4. Elementary process of stimulated emission in the cavity (two-mode case, damping is absent)

Consider now the problem in which the atom can interact with two modes of the cavity (without damping). Let one of them have the frequency $\omega_1 = \omega_0$, equal to the transition frequency, while the second atom has a small detuning, $\omega_2 = \omega_0 + \Delta$, $|\Delta| \ll \omega_0$. We will call them the resonant and nonresonant modes, respectively, and the vacuum Rabi frequencies Ω for them are assumed the same. Then, in the approximation of the rotating field the system can be regarded as being in a superposition of the five states closest in energy:

$$|\Psi\rangle = A|+10\rangle + B|+01\rangle + C|-20\rangle + D|-02\rangle + E|-11\rangle. \quad (24)$$

The system of equations for the amplitudes has the form

$$\frac{dA}{dt} = \sqrt{2}\frac{\Omega}{2}C + \frac{\Omega}{2}E e^{-i\Delta t}, \quad \frac{dB}{dt} = \sqrt{2}\frac{\Omega}{2}D e^{-i\Delta t} + \frac{\Omega}{2}E,$$

$$\frac{dC}{dt} = -\sqrt{2}\frac{\Omega}{2}A, \quad \frac{dD}{dt} = -\sqrt{2}\frac{\Omega}{2}B e^{i\Delta t}, \quad (25)$$

$$\frac{dE}{dt} = -\frac{\Omega}{2}A e^{i\Delta t} - \frac{\Omega}{2}B.$$

The initial conditions corresponding to the simplest process are as follows: $A(0) = 1$, and all the other amplitudes at the initial moment of time are equal to zero. In this problem, we consider the appearance of states with two photons as a process of copying of the initial photon. The solution of system (25) is of quasi-periodic character (without definite final state), and of greatest interest are the time-average probabilities. The control parameter of system (25) is $\beta = |2\Delta|/\Omega$ – the detuning of the nonresonant mode, referred to the half of the vacuum Rabi frequency.

We introduce the notation $p = \overline{|C|^2}$, $q = \overline{|E|^2}$, $r = \overline{|D|^2}$ for the time-average probabilities of the states in which the field contains two photons. The dependences of these variables on the logarithm of the parameter β are shown in Fig. 3.

At $\beta \to 0$, when the mode frequencies approach each other, the field consisting of two photons, on average, with a probability of 3/8 contains two photons, which are identical to the initial photon; with a probability of 1/4 – two photons in different modes; and with a probability of 3/8 – two photons in a mode, which differs from the initial one. Thus, if the detuning is negligible, then the time-average probability Υ of finding a field with two photons in a state with a doubly occupied mode is $\Upsilon(2) = 0.75$. With the growth of β the quantity p, proportional to the probability of finding two photons that are identical to the initial one, mostly increases, tending to the limiting value 1/2, which is characteristic of the single-mode case [see (11)]. The value of q at small β increases, reaching a maximum $\max q = 0.204$ at $\beta = 1.0$, and then generally decreases at $\beta \gg 1$ as $q \approx \beta^{-2}$. The value of $r(\beta)$ monotonically decreases: at large detunings $r \approx \beta^{-4}$. It is interesting to note minor violations of monotonicity $p(\beta)$ and $r(\beta)$ at $\beta \approx 2$.

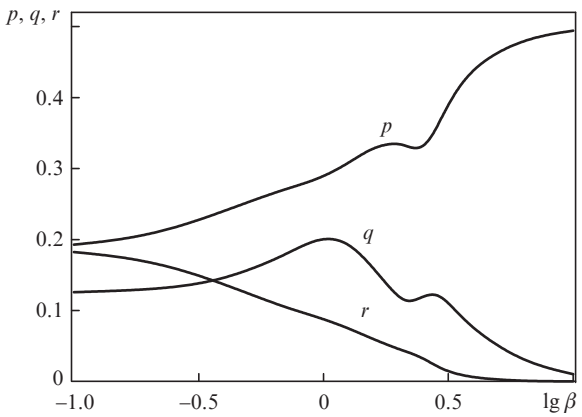

Figure 3. Average probabilities of population of the modes vs. $\lg\beta = \lg(|2\Delta|/\Omega)$. Curve p shows two photons in the resonant mode, q – one photon in each of the modes, r – two photons in the nonresonant mode.

The average frequency can be used as a measure of the frequency deviation of photons from the initial value. Then, the relative frequency error of copying will be given by the expression

$$\delta_\omega(\beta) = \frac{\Omega}{2\omega_0}F(\beta), \quad F(\beta) = \beta\frac{q + 2r}{p + r + q}. \quad (26)$$

Figure 4 shows the plot of the function $F(\beta)$. One can see that it reaches a maximum, equal to 0.97, at $\beta = 2.82$. Therefore, when an atom interacts with two modes of the field with close frequencies in the cavity the frequency error of copying a photon δ_ω can reach $\sim\Omega/2\omega_0$. Under conditions of the example used above $\delta_\omega = 2.09\times10^{-10}$. Note also that because the considered modes can have fields with a completely different structure, there can be no simple relationship between the field distributions in the modes of triggering and induced photons.

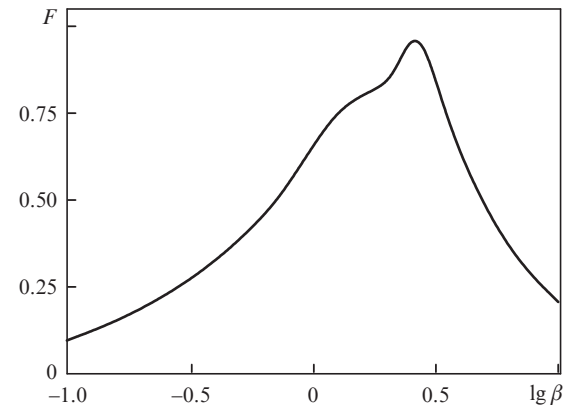

Figure 4. Dependence of the function F (26), proportional to the frequency error of copying, δ_ω, on $\lg\beta = \lg(2\Delta/\Omega)$.

5. Elementary process of stimulated emission in free space

Consider now the scattering of a photon (we will call it initial and denote as $|1_0\rangle$, and its frequency – as ω) by an excited atom in free space. As usual, we assume that the field is in a cube with edge length L ($L^3 = \mathcal{V}$) and obeys the periodic boundary conditions on the faces of the cube, and at the end of calculations passes to the limit $\mathcal{V} \to \infty$. For a virtual cavity cube the distribution of the electric field $\boldsymbol{E}_\mu(\boldsymbol{r})$ is given by the function $\boldsymbol{e}_\mu \exp(-\mathrm{i}\boldsymbol{k}_\mu \boldsymbol{r})$, where $\boldsymbol{e}_\mu$ is the unit vector of the polarisation of the mode μ, and $\boldsymbol{k}_\mu$ is its wave vector; however, in the dipole approximation used by us we can simply put $\boldsymbol{E}_\mu(\boldsymbol{r}) = \boldsymbol{e}_\mu$.

In the first order of the perturbation theory, the process of emission of a photon with a frequency ω_1 by an excited atom in the presence of an initial photon with a frequency ω is described by a standard Weisskopf–Wigner solution [21] (see also [22], § 18): The dynamic interaction of the incident photon and atom is absent, and the combinatorial interaction increases the transition matrix element by $\sqrt{2}$ times for a single mode. In the limit $\mathcal{V} \to \infty$, this matrix element becomes infinitesimally small and introduces changes neither in the kinetics of the transition, which is exponential,

$$W_+(t) = \exp(-\Gamma_s t), \tag{27}$$

nor in the spectral distribution of radiation, which is Lorentzian,

$$G(\omega_1) = \frac{1}{\pi}\frac{\gamma}{(\omega_0 - \omega_1)^2 + \gamma^2} \tag{28}$$

(here $\gamma = \Gamma_s/2$ is half the rate of spontaneous emission), and is independent of the initial photon frequency ω.

The observable effects, in which the incident photon interacts dynamically with the atom, causing a transition in it, are described by the third order of the perturbation theory. As the two-level atom can undergo only a sequence of transitions $|+\rangle \to |-\rangle \to |+\rangle \to |-\rangle$, then in the rotating field approximation a process is possible, in which an atom emits a photon into mode μ in the presence of the initial photon, and then absorbs the initial photon with frequency ω, and, finally, emits a second photon in mode ν. The process in which the second transition is accompanied by absorption of the photon emitted with frequency ω_μ, will not depend on the characteristics of the initial photon; it describes the radiative correction to the spontaneous emission of an atom. By the law of conservation of energy, the frequencies of the transition and photons are related by

$$\omega_0 + \omega = \omega_\mu + \omega_\nu. \tag{29}$$

In the future we will be interested only in processes involving photons with frequencies close to the resonance frequency ($|\omega_\mu - \omega_0| \ll \omega_0$), so that everywhere except the resonance denominators, we assume the frequencies of all three photons to be identical and equal to ω_0. Taking into account the linewidth finiteness, the third-order composite matrix element has the form

$$\tilde{V}^{(3)} = \mathrm{i}\left(\frac{2\pi\hbar\omega_0}{\mathcal{V}}\right)^{3/2} \sum_{\mu,\nu}\left[\frac{\langle -1_\mu 1_\nu |\hat{\boldsymbol{d}}\boldsymbol{e}_\nu \hat{a}_\nu^+| +1_\mu\rangle}{(\omega_0 - \omega + \mathrm{i}\gamma)} \times \right.$$

$$\left. \times \left[\frac{\langle +1_\mu |\hat{\boldsymbol{d}}\boldsymbol{e}_0 \hat{a}_0| -1_0 1_\mu\rangle\langle -1_0 1_\mu |\hat{\boldsymbol{d}}\boldsymbol{e}_\mu \hat{a}_\mu^+| +1_0\rangle}{(\omega_0 - \omega_\mu + \mathrm{i}\gamma)}\right]\right]. \tag{30}$$

The rate of the transition process in the elementary interval on the energy surface is constant and is given by the formula

$$\mathrm{d}\dot{W} = \frac{2\pi}{\hbar}|\tilde{V}^{(3)}|^2 \mathrm{d}\rho(E_k), \tag{31}$$

where $\mathrm{d}\rho(E_k)$ is the differential of the density of final states in an elementary interval of the integrals of motion (which differ from the energy) on the energy surface [8].

Let us find the spectral cross section of the photons emitted in this process, which is proportional to the probability that the frequency of a photon lies in the range $\mathrm{d}\omega_1$ near ω_1:

$$\mathrm{d}\sigma = \mathcal{V}\frac{\mathrm{d}\dot{W}}{c}. \tag{32}$$

Depending on whether the photons are emitted in an occupied mode or in empty modes, the differential of the density of final states has a different order in $\mathcal{V}$. The main contribution is made by processes in which photons are emitted in different modes (μ and ν). For them, the differential of the density of final states is

$$\mathrm{d}\rho = \mathcal{V}\frac{\omega_1^2}{\hbar(2\pi c^3)}\mathrm{d}\Omega_1 \mathcal{V}\frac{\omega_2^2}{\hbar(2\pi c^3)}\mathrm{d}\Omega_2 \mathrm{d}(\hbar\omega_1). \tag{33}$$

If a photon of the mode μ is emitted to the mode of the initial photon, then the matrix element receives a factor $\sqrt{2}$, but the modes in (30) are summed only over the index of the final state of the mode ν,

$$\mathrm{d}\rho = \mathcal{V}\frac{\omega_1^2}{\hbar(2\pi c^3)}\mathrm{d}\Omega_1, \tag{34}$$

and the corresponding contribution to the rate of the process at $\mathcal{V} \to \infty$ becomes vanishingly small. Similarly, negligibly small are the contributions from other cases in which combinatorial interaction takes place. Thus, statistically overwhelming cases, in which both photons are emitted in the unoccupied modes, make a nonvanishing contribution to the spectral cross section of the elementary process of radiation upon scattering of a photon by an excited atom at $\mathcal{V} \to \infty$.

The spectral cross section can be easily reduced to an expression of form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega_1} = \frac{16}{9}\frac{d^6\omega_0^7}{\hbar^3 c^7} G(\omega_0, \omega, \omega_1, \gamma), \tag{35}$$

where the function G can be represented as the ratio of the fourth- and eighth-order homogeneous polynomials with respect to the arguments that have the character of the frequency:

$$G(\omega_0, \omega, \omega_1, \gamma) = \frac{P_4(\omega_0, \omega, \omega_1, \gamma)}{P_8(\omega_0, \omega, \omega_1, \gamma)}. \tag{36}$$

The expression for G, in general, is very cumbersome and we omit it here. In the case of the exact resonance ($\omega = \omega_0$), the function G has a simple form:

$$G(\omega_0, \omega_0, \omega_1, \gamma) = \left(\frac{2}{\Delta^2 + \gamma^2}\right)^2, \tag{37}$$

where $\Delta = \omega_1 - \omega_0$ is the detuning of the transition frequencies and of one of the emitted photons. The width of the emission line is close to the natural linewidth, but the shape is somewhat different (Fig. 5). For a Lorentzian line with small γ, half the total spectral power lies in the range $|\Delta| \leqslant \gamma$, and for the line of form (37), half the total spectral power lies in the range $|\Delta| \leqslant 0.442\gamma$.

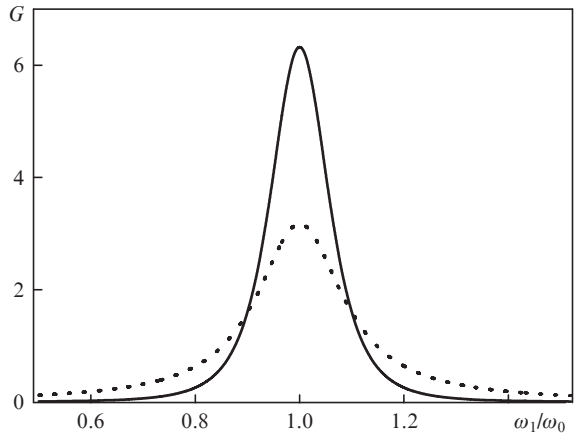

Figure 5. Shapes of the spectral distributions of the spontaneous emission (dotted curve) and the simplest stimulated emission (solid curve) in the case of resonance of the frequencies of the initial photon and transition. Half-width of the spontaneous emission line is $\gamma = 0.1$.

Note that the Mollow model [23], which describes the scattering of monochromatic radiation of frequency ω by a two-level system relaxing with the resonant frequency ω_0, leads to the same form of the spectral distribution. For the case of exact resonance ($\omega = \omega_0$), the power spectrum of the radiation in a weak field ($\Omega \ll \gamma$) has the form

$$P(\omega_1) = \frac{\Omega^2}{2\gamma^2}\left[2\pi\delta(\omega_1 - \omega_0) + \frac{2\Omega^2\gamma}{(\Delta^2 + \gamma^2)^2}\right]. \qquad (38)$$

Here, Ω is the Rabi frequency of the (classical) field incident on the system, and the detuning $\Delta = \omega_1 - \omega_0$. The first term describes the elastic scattering of single photons with energy conservation (the process is studied in detail by Heitler [22, p. 234]), whereas the second term, describing the scattering of pairs of photons with the energy exchange between them, has the same form as the dependence $G(\omega)$.

The total scattering cross section in the case of the exact resonance

$$\sigma = \frac{3}{\pi}\lambda^2 \qquad (39)$$

is twice as large as the resonance fluorescence cross section (see [13, p. 416]).

When the initial photon detunes from the resonance, the line shape (36), preserving the symmetry with the relative frequency $\bar{\omega} = (\omega_0 + \omega)2$, broadens, and at sufficiently large detunings splits roughly into two Lorentzian lines with intensity maxima at the frequencies of the transition and initial photon (Fig. 6).

The error of the spectral copying of the initial photon δ_ω in the simplest process of stimulated emission in free space is determined by the ratio of the natural linewidth

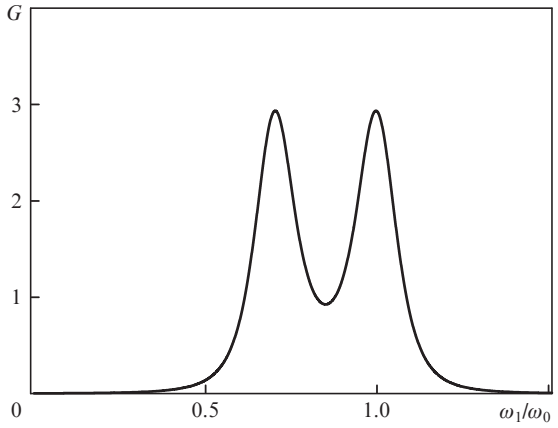

Figure 6. Shape of the spectral distribution of the simplest stimulated emission in the nonresonant case ($\omega_0 = 1$, $\omega = 0.7$, $\gamma = 0.1$).

$\gamma = \Gamma_s/2 = 2d^2\omega_0^3/(3\hbar c^3)$ [see (12)] to its central frequency ω_0. This relation can be written in the form

$$\delta_\omega = \frac{\gamma}{\omega} \sim \alpha^3\left(\frac{\omega_0}{\omega_{at}}\right)^2, \qquad (40)$$

where $\alpha = e^2/\hbar c$ is fine-structure constant; $\omega_{at} = me^4\hbar^{-3} = 4.13\times10^{16}$ s^{-1} is the atomic frequency scale. In the optical range the error of frequency copying, $\delta_\omega \sim 3\times10^{-9}$, is very small. On the contrary, the spatial characteristics of the initial photon are copied badly: The errors of copying the wave vector and polarisation vector are large, $\delta_k \sim 1$, $\delta_e \sim 1$. Two reasons lie behind this phenomenon. On the one hand, the atom that is a point one in the dipole approximation does not feel the spatial structure of the field – the dynamic interaction in free space is independent of the wave vector. On the other hand, the combinatorial interaction, leading to a small (by $\sqrt{2}$ times) increase in the matrix element for the emission of a photon in an occupied mode, in free space is suppressed by reducing the number of possible final states.

6. Conclusions

The above calculations show that the two features inherent in the Einstein–Dirac problem (namely, an increase in the transition rate and the emergence of multiply occupied modes of the quantized radiation field) are present to variable degrees in the elementary processes of stimulated emission. When interacting with a single field mode in the cavity, copying of the photon is exact [the time-average probability of finding a field with two photons in a state with a doubly occupied mode is $\Upsilon(1) = 1$], but in the presence of field decay the rate of the atomic transition to the lower state for stimulated emission may be either higher or lower than the transition rate for spontaneous emission. It follows from the problem with two cavity modes that copying of the initial photon is, in general, approximate, and its efficiency depends on the detuning of the second (nonresonant) mode. If the detuning is negligible, then the time-average probability of finding a field with two photons in a state with a doubly occupied mode is $\Upsilon(2) = 0.75$ (see Section 4). The calculations, which are not presented here, show that for the three resonant modes $\Upsilon(3) = 0.556$. There is no doubt that with a further increase in the number of resonant modes the probability of double occupation of the modes will continue to fall. In the limiting case of an infi-

nite number of modes, corresponding to emission in free space, the scattering of a photon by an excited atom does not lead to double occupation of the modes, although the spectral copying of the initial photon takes place with high accuracy.

Hence, it is necessary to clarify the definition of stimulated emission (such a need, based on other considerations, was already noted in [24]). We can start with a trivial fact: The presence of the initial photon alters the evolution of the atomic system, its kinetics and the mode composition of the emerging radiation.

If we follow Einstein and consider the change in *the kinetics* of the transition to be a defining characteristic of stimulated emission (the original term – Zustandsänderungen durch Einstrahlung, changing the state under irradiation [9–11]*), then the reference to the emission of identical photons should be removed from the definition of stimulated emission, because such emission may occur in part (see Section 4) or not occur at all (see Section 5). In addition to historical continuity, such a kinetic definition is more preferable as it describes the phenomenon in the terms of observables and is not tied to a specific model. In particular, it admits the possibility of describing stimulated emission with the help of classical models, which was known to Einstein [9–11] and found widespread use in the scientific literature (see, for example, [27–29]).

On the other hand, we can formally follow the central dogma and *define* stimulated emission as a process of emission of photons into occupied modes. In this case, the identity of the emitted photons to the initial photons will be guaranteed. But then, the change in the spectral properties of radiation of an excited atom in free space in interacting with an external photon (see Section 5) will be described as a new, third process, because it will be neither spontaneous nor stimulated emission. Occam's razor eliminates this approach.

It should also be noted that a completely incorrect statement about the exact copying of a single photon upon its scattering by the excited atom is everywhere (from school [30, 31] to university [5–7] textbooks) used to explain the high spatial directivity and quasi-monochromacity of laser radiation. Something must be done about it.

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*The translators of papers [9–11] for a collection of Einstein's papers [25] translated Einstrahlung (irradiation) everywhere as 'induced emission'. First, it immediately led to physically incorrect assertions – for example, that 'it can cause equally a decrease or increase in energy' (p. 390). Secondly, it is an anachronism: the term 'induced emission' was introduced by Van Vleck only in 1924 [26], and the equivalent term 'stimulated emission' – by Dirac in 1927 [8].