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A semi-analytical approach for evaluating effects of amplified spontaneous emission on characteristics of *Q*-switched lasers

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Abstract. **Possible effects of amplified spontaneous emission on output pulse characteristics of a *Q*-switched laser are discussed within the framework of a semi-analytical approach. It is shown that output energy decreases almost exponentially with average path length of the spontaneously emitted photons which in turn depends on geometrical specification and active medium properties as well as on optical finishing of the surfaces (for solid-state lasers). Optimal coupling dependence on the average path length is also investigated and shown to increase with average path length increment.**

Keywords: *Q-switched laser, amplified spontaneous emission, rate equations.*

1. Introduction

In analysing and designing *Q*-switched lasers, the particle approach is usually used in which photons are considered as particles interacting with media. In this approach, a set of nonlinear coupled equations known as rate equations is solved [1]. To date, analytical and numerical solutions of rate equations have been used to elucidate the system evolution under various operational conditions [2]. Among these solutions, the results presented by Hofer et al. [3] seem to be a fully analytical solution to *Q*-switched laser description. They reported an integral formula, using Lambert function, for the temporal behaviour of the photon density inside the cavity [3]. However in almost all proposed methods, the initial photon density is ignored which may not be true all the time. In fact, neglecting the initial photon density must be done more carefully because in high gain and/or highly pumped media, the initial photon density can be quite high with noticeable directionality because of amplified spontaneous emission (ASE) [4–9].

To evaluate ASE effects, some case study based on numerical calculation has been performed [10]. Li et al. [10] pointed out the significance of the initial photon density produced in the up-conversion phase and presented a modified set of rate equations taking ASE into account. They introduced an experimentally determined factor to describe spontaneous photon amplification in the gain media and reported ASE-induced changes in the pulse width, output energy, repetition rate and average power in a passively *Q*-switched $Nd:YVO_4$ laser. More recently, M.V. Bogdanovich et al. [11] studied ASE-induced peculiarities in a diode-pumped Yb, Er: borosilico-phosphate glass laser using the rate equation approach with the ASE flux included. Although these numerical approaches could be very useful for practical purposes, they generally provide insufficient insight into the problem of ASE-induced changes. Thus, simplified techniques are demanded to draw a better picture of the problem.

Here we intend to propose a simple approach to the solution of the problem of fast *Q*-switched lasers, taking ASE into account. This approach is especially suitable for a four-level solid-state gain medium with rod geometry. Discussion is organised to answer two major questions. (i) Are the ASE mechanisms affecting the *Q*-switched laser output through depleting inversion in the pumping phase unique or can the ASE flux also affect the laser output directly? (ii) How reflections and internal refractions can affect the *Q*-switched laser output? According to our knowledge, detailed discussion of these problems has not been reported yet.

2. Population inversion process in the presence of ASE

In a gain medium with no resonator or very high intracavity losses, no laser output exists but amplification of spontaneous photons takes place [12]. Thus, for exact prediction of the upper level evolution, it is necessary to consider this amplified photon flux known as ASE. There are some features related to ASE, which make this problem so complex that only state of the art spatiotemporal rate equations coupled with statistical computation (Monte Carlo method for example) enables one to exactly predict the population inversion process [13].

Among these features, random nature of spontaneous emission [14], wavelength dependent angle of reflection/refraction from boundaries, complexity due to line narrowing and path calculations are of main importance. However some powerful analytical methods have been reported which approximately describe inversion as well as energy extraction processes [15]. Barnes et al. [16] have also proposed an analytical approach governing inversion dynamics which we briefly describe here. In active media with a broadband fluorescence spectrum, like in most of the solid-state lasers, spontaneous photons are emitted from any position in the active medium, at any variety of frequencies, and can have any directions. These photons will be amplified in the gain medium as they propagate and deplete inversion due to stimulated emission.

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One way of considering these photons in population inversion dynamics is averaging them in space and frequency. Unfortunately, the amplification factor $\exp(\sigma_e N z)$ [where σ_e is the effective emission cross section, N is the population density of the upper laser level (supposed to be independent of position), and z is the path length of a spontaneously emitted photon], depends on spectral and spatial terms which cannot be averaged easily. This problem mainly originates from the frequency dependent refraction/reflection from boundaries which affects the path length and propagation direction of the emitted photons. Hence, the photon frequency affects its path length and propagation direction; therefore, integrating over frequency and space cannot be separated. However, usually the path length and propagation direction dependences are relatively weak and separation is possible which enables us to find a closed form for the population density of the upper laser level. A series expansion of the amplification factor is also necessary in which only the first two terms must be kept. The series expansion is shown to converge because integration over frequency is in fact integration over a function of line shape $g(\nu - \nu_0)^k$ and increasing k causes the integral of this term to decrease.

As a result, for a four-level system, using the concept of the averaged spontaneously emitted photon path length, and some other reasonable assumptions, a first order nonlinear differential equation is derived for the upper level evolution:

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\frac{n}{\tau_2} - \frac{\sigma_{ea} l_a n^2}{\tau_2} + W_p(n_{tot} - n), \quad (1)$$

where n and n_{tot} are, respectively, the upper level and total density; W_p is the pump rate; τ_2 is the upper level lifetime; σ_{ea} is the average effective emission cross section; and l_a is the average path length for the spontaneously emitted photon defined as:

$$l_a = \frac{1}{4\pi V_0}\int_0^{4\pi}\int_0^{V_0} z\,\mathrm{d}\Omega\,\mathrm{d}V\,. \quad (2)$$

This parameter contains geometrical shape as well as face reflectivity from the boundaries of the gain media of volume V_0 and its value directly affects the strength of the ASE flux in a medium. A novel experimental technique is also proposed for measuring this parameter by Barnes et al. [16] based on measuring the fluorescence decay curve. Up to now, we saw that ASE is taken into account in inversion dynamics by introducing a parameter known as the average path length. However, like every simplified model, applicability of this model is limited due to it assumptions and can be deduced from derivation of (1). Studying details of the approach reveals that errors from exclusion of higher terms in the amplification factor should be judged by the condition:

$$\frac{N\int_0^{4\pi}\int_0^{V_0} z^2\,\mathrm{d}\Omega\,\mathrm{d}V}{8\pi\sigma_{ea} l_a V_0}\int_0^{\infty}\sum_i\left(\frac{\pi\Delta\nu_i\sigma_{pi}}{2}\right)^2\beta_i g(\nu-\nu_i)^3\mathrm{d}\nu \ll 1, \quad (3)$$

where $\Delta\nu_i$ and σ_{pi} are the linewidth and peak emission cross section of the ith transition, respectively. One more point is necessary to be clarified about the performance of the model in regions where condition (3) is not valid. Since the series is convergent, inclusion of higher terms in calculations only causes the upper level density to decrease more severely. Hence, oscillation and/or incremental behaviour do not occur in upper level evolution and first approximation can be applied as an upper limit and used in the initial stage of designing. Now we return to (1) and present the solution to it:

$$n(t) = \frac{\zeta}{\beta}\left[\left(\frac{\Gamma e^{2\zeta\beta(t-t_0)} - 1}{\Gamma e^{2\zeta\beta(t-t_0)} + 1}\right) - \frac{\beta\Sigma}{2\zeta}\right], \quad (4)$$

where

$$\beta = \left(\frac{l_a\sigma_{ea}}{\tau_2}\right)^{1/2}; \quad \Sigma = \frac{\tau_2}{l_a\sigma_{ea}}\left(W_p + \frac{1}{\tau_2}\right);$$
$$\zeta = \left(W_p n_{tot} + \frac{l_a\sigma_{ea}\Sigma^2}{4\tau_2}\right); \quad \Gamma = \frac{2\zeta + \beta\Sigma + 2\beta n_{t0}}{2\zeta - \beta\Sigma - 2\beta n_{t0}}; \quad (5)$$

n_{t0} is the upper level density at time t_0.

The photon density inside the active medium can be calculated by considering the lifetime of the amplified spontaneous photons in a medium and calculating the density of the upper level with and without spontaneous emission in this time interval. Thus, neglecting nonradiative relaxation, the photon density can be expressed as:

$$\Phi_{ASE} = n_p(t + \delta t) - n(t + \delta t)$$
$$= n_{tot}(1 - e^{-W_p\delta t}) + n(t)e^{-W_p\delta t} - n(t + \delta t), \quad (6)$$

where $\delta t = 2l_a/c$ is the averaged photon lifetime [12]; n_p is the upper level density due to pump only; and n is the actual upper level density [equation (4)]. So far, it has been assumed that no considerable feedback exists and as a result no oscillation occurs within the population inversion process. Switching on high Q causes initiation of oscillation by spontaneous photons existing inside the cavity. Of course, all of these photons do not contribute to the coherent pulse generation phase due to their different frequencies. If we assume that the photon flux has a bandwidth of $\Delta\nu$, $m = \delta\pi\nu^2 V_R\Delta\nu/c^3$ oscillation modes would generated within the resonator volume V_R, where ν is the laser optical frequency. If we suppose the laser output to have m_{las} modes, the fraction of ASE photons contributing to the lasing process can be written in the form:

$$\Phi_0 = \Phi_{ASE}\frac{\chi m_{las}}{m}, \quad (7)$$

where χ is a factor originated from random directionality of ASE photons and is the ratio of photons existing in the laser spatial modes to the total number of spontaneously emitted photons. This factor appears since photons in all directions are averaged with an equal weight in calculations, but laser spatial modes (resonator mode structure) actually determine the fraction of photons contributing to the laser output. This factor depends on the geometry of the active medium as well as resonator mode structure and its magnitude in a longer active medium is often higher but nevertheless smaller than unity. It is worth mentioning that the bandwidth of amplified spontaneous emission depends on such a parameter as total gain [17].

Based on the derived formulas, the amount of initial inversion and photon density can be calculated and used as initial conditions in coherent pulse generation phase. This phase will be studied in the next section.

3. Coherent pulse generation phase

In this phase, intracavity loss is reduced suddenly to a minimum value preparing necessary conditions for oscillation. Considering a step shape for the loss profile as discussed in [1–3], time evolution of photon and population inversion density in this phase can be described by the equations:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \Phi\left(c\sigma n\frac{l}{L} - \frac{\varepsilon_{\min}}{t_{\mathrm{R}}}\right), \tag{8}$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\gamma c\sigma\Phi n, \tag{9}$$

where l is the length of the gain media; L is the optical length of the resonator; c is the speed of light; σ is the stimulated emission cross section; $\varepsilon_{\min}$ is the minimum fractional loss per round trip in the cavity; γ is the inversion reduction factor supposed to be equal to unity; and t_{R} is the cavity transit time. Pumping and relaxation of the upper level are neglected during Q-switching, because they are very slow compared with Q-switched pulse duration. Eliminating time in (8) and (9) leads to the equation:

$$\Phi(n) = \Phi_0 + \frac{l}{ct_{\mathrm{R}}}(n_{\mathrm{i}} - n) + \frac{\varepsilon_{\min}}{c\sigma t_{\mathrm{R}}}\ln\frac{n}{n_{\mathrm{i}}}. \tag{10}$$

Here Φ_0 is a part of the amplified spontaneous emission contributing to lasing. It is related to the initial upper level density and n_{i} is related also to the resonator mode structure, gain medium geometry and spectroscopic parameters. Although this term is very small in comparison with other terms, laser operation cannot be initiated without it. Substituting (10) in (9) leads to the following first order nonlinear equation describing the temporal behaviour of the population inversion density:

$$\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}t} - \frac{l\sigma}{t_{\mathrm{R}}}n + \frac{\varepsilon_{\min}}{t_{\mathrm{R}}}\ln\frac{n}{n_{\mathrm{i}}} + \frac{l\sigma n_{\mathrm{i}}}{t_{\mathrm{R}}} = -c\sigma\Phi_0. \tag{11}$$

By changing the variable, $u = \ln(n/n_{\mathrm{i}})$, equation (11) reduces to the differential equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = \sum_{s=2}^{\infty} B_s u^s - c\sigma\Phi_0, \tag{12}$$

where series expansion of $\exp(u)$ is used in right-hand side of the equation and $A = (\varepsilon_{\min} - \sigma l n_{\mathrm{i}})/t_{\mathrm{R}}$, $B_s = \sigma l n_{\mathrm{i}}/(t_{\mathrm{R}} s!)$.

This nonlinear first order equation can be solved if only the first term of the expansion dominates the others. Although, this condition imposes certain limitations and is valid only in those cases when initial inversion is not far above the threshold inversion (i.e., high repetition rate Q-switched lasers), a fully analytical solution for inversion dynamics is possible only in this regime. Thus, keeping the first term of the series expansion, Eqn (12) has a closed form solution:

$$n = n_{\mathrm{i}}\exp\left[\frac{-\xi}{\eta\coth(\xi\eta(t - t_{\mathrm{i}})) + A/2\xi}\right], \tag{13}$$

where ξ and η are variables defined as:

$$\xi = (c\sigma\Phi_0)^{1/2},\ \eta = \left(B + \frac{A^2}{4\xi^2}\right)^2. \tag{14}$$

For $t \gg t_{\mathrm{i}}$, the ultimate upper level density can be calculated from (13) as:

$$\eta_{\mathrm{f}} = n_{\mathrm{i}}\exp\left[\frac{-\xi}{\eta + A/2\xi}\right]. \tag{15}$$

The ultimate upper level densities predicted in our approach and [3] are plotted in Fig. 1. Also the relative error due to the stated approximation is plotted versus initial inversion density in Fig. 2, which shows that the solution presented in (15) has a narrow domain of validity and is valid for the near-threshold regime because the error grows rapidly far from the threshold.

By deriving (13), it follows that the influence of the initial photon density, which is a part of the amplified spontaneous photons having the chance of oscillation within the cavity, is taken into account. However, conclusions based on the anal-

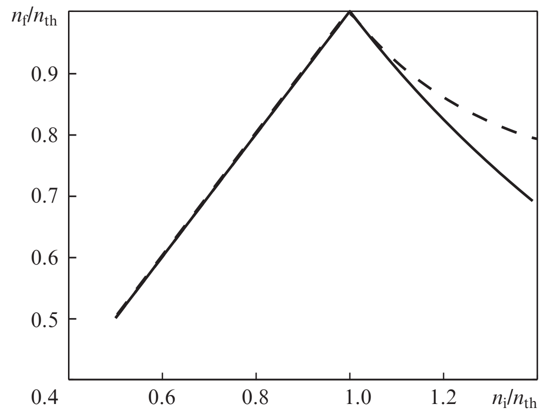

Figure 1. Final population inversion density n_{f} vs. initial population inversion density n_{i} in units of the threshold inversion density n_{th}. The dashed curve is deduced from (12) and the solid line is from Ref. [3].

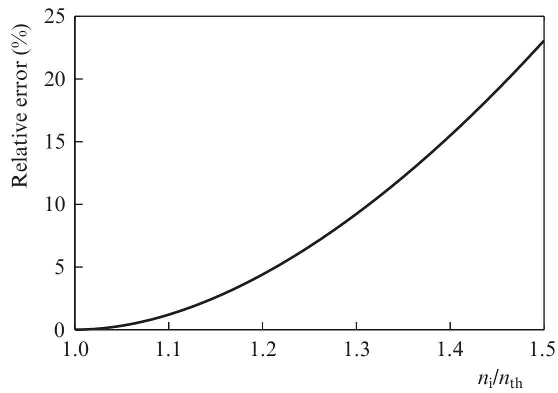

Figure 2. Relative error caused by approximate solution vs. ratio of the initial inversion level to the threshold inversion density.

ysis of equation (12) may not be exact because of the approximation used in deriving this expression. Thus, to perform an exact analysis, we solved (11) numerically with an appropriate initial condition determined in the previous section. The fifth order Runge – Kutta – Fehlberg algorithm is used here for numerical solution and general plots of the Q-switched pulse shape in presence of ASE with different average path lengths are shown in Fig. 3. According to this figure, ASE directly affects the Q-switched pulse through decreasing initial inversion. It decreases the peak photon density and delays building up of the photons inside the cavity.

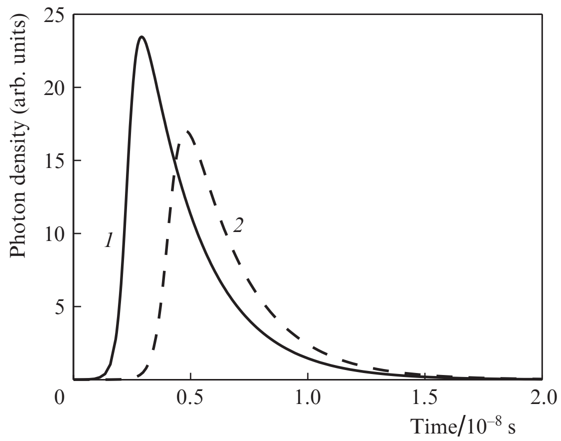

Figure 3. Photon density inside the resonator vs. time at $l_a = 5l$ (*1*) and $12l$ (*2*).

Also in some situation the initial photon density may play a significant role, which is the subject that we are interested in. To evaluate the effects of the initial photon density, first we take a look at photon density evolution inside the cavity, by changing the amount of spontaneous photons contributing to the stimulated emission (with fixed initial inversion). Although the ASE photon density is strictly related to the initial inversion, expression (7) shows that the amount of photons contributing to the stimulated emission depends on other parameters. Thus, it is possible to change it independently of the initial inversion. Thus, the temporal position of the peak of the photon density shifts with the initial photon density mainly. Figure 4 shows the shift versus the initial photon density for $n_i = 8.3\times10^{17}$ cm^{-3}, $W_p = 5\times10^2$ s^{-1}, $l = 50$ mm and $L = 100$ mm in the Nd:YAG active medium.

As follows from [14, 18], the shift introduced by the initial photon density (ASE) is of statistical nature, and so it can affect the pulse train periodicity due to its random nature. For estimating the significance of the initial photon density, we rearrange (11) as:

$$\frac{1}{n}\frac{dn}{dt} - \frac{l\sigma}{t_R}n + \frac{\varepsilon_{\min}}{t_R}\ln\frac{n}{n_i} + \frac{l\sigma n_i}{t_R}(1+\alpha) = 0, \qquad (16)$$

where $\alpha = 2l\Phi_0/n_iL$. This differential equation is sensitive to α and the degree of this sensitivity depends mainly on the active medium and resonator length, stimulated emission cross section and minimum fractional loss per round trip in the cavity. For example, taking parameters similar to those used in Fig. 4, the pulse peak position seems to be nearly 3% while changing Φ_0 by six orders of magnitude. Of course, the sensitivity variation is not linear as is evident from Fig. 4. From above discussion one can estimate the amount of the initial photon density that can affect the peak position as:

$$\Phi_0 \approx \alpha n_i, \qquad (17)$$

where $2l/L$ is supposed to be in order of unity. On the other hand, near saturation, the initial photon density can be estimated from (6) as:

$$\Phi_{\rm ASE} = [n_{\rm tot} - n(t)](1 - e^{-W_p\delta t}) - \left[\delta t\left(\frac{dn}{dt}\Big|_{t=t_i}\right)\right]$$
$$\approx 2n_i l_a(1 + \sigma_{\rm ea} l_a n_i)/c\tau_2. \qquad (18)$$

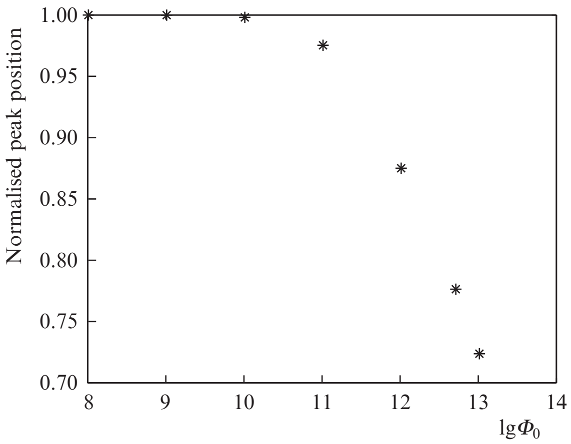

Figure 4. Relative shift of the photon density vs. logarithm of the initial photon density in the cavity.

Thus, noticeable effects of the initial photon density appear when the following condition is satisfied:

$$\frac{\chi m_{\rm las}}{m} \approx \frac{\alpha c\tau_2}{2l_a(1 + l_a\sigma_{\rm ea}n_i)}, \qquad (19)$$

where $m_{\rm las}$, m and χ are the same parameters as those used in (7). Now based on (19), one can judge about ignoring the initial photon density from calculations. Note that this is a theoretical criterion although the authors believe that condition (19) is rarely satisfied and for practical purposes we can safely ignore the effect of the initial photon density.

We should note that because of the broadband spectrum, ASE may affect the gain switching and hence indirectly affect the laser pulse characteristics which of course are not within the domain of this work.

4. Effects of average path length on energy and optimal coupling

In this section, the effect of the average path length of spontaneously emitted photons on the output energy is investigated numerically. Figure 5 shows the relative energy degradation due to increasing l_a/l. Thus, under same condition of active medium and pumping, reflectivity of boundaries may affect the Q-switched pulse drastically as it changes the average path length of the spontaneously emitted photons.

Increasing the average path length causes the amplified spontaneous flux to increase in the active medium and as a consequence the upper level to be depleted more severely. Thus, the available stored energy decreases and the output pulse energy falls down. However, the energy variation versus the average path length finds a complicated nonlinear form as is expected because of the way it contributes to upper laser level evolution [equation (1)]. Figure 5 illustrates that energy degradation could be suitably approximated by an exponential dependence in which two fitting parameters are enough to describe the variations. It should be noted that the predicted value for the output energy is more reliable in the case of smaller value of l_a/l. This point can be interfered directly from (3).

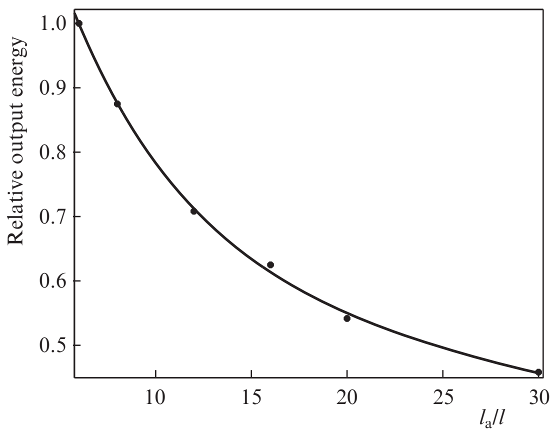

Figure 5. Relative output energy vs. average path length of spontaneously emitted photons and an exponential fit.

Optimal coupling is another important design parameter and we tried to investigate ASE-induced changes in this parameter. Combining the results of this paper with the existing formula for optimal reflectivity of the output coupler [1, 3], we have shown that the optimal value varies with the average path length as shown in Fig. 6. The parameters used in calculating the curve are: active medium, Nd:YAG; concentration, Nd = 1 at.%; $W_p = 5\times10^2$ s^{-1}; $l = 50$ mm; $L = 100$ mm; and round trip loss, $\delta = 0.05$ cm^{-1}.

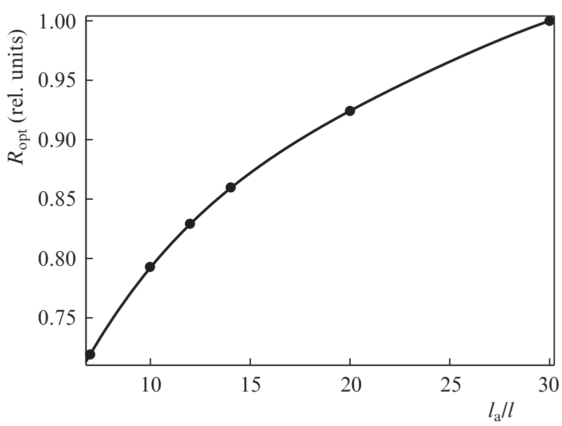

Figure 6. Relative optimal reflectivity of the output coupler, R_{opt}, with respect to R_{opt} at $l_a = 30l$ vs. average path length (points) and a fourth degree polynomial fit (solid curve).

Figure 6 reveals that the optimal output coupler reflectivity increases with increasing average path length, and this is mainly because of the reduced stored energy in the active medium. In addition, a nonlinear variation is observed as is expected from the nonlinear dependence of optimal coupling on the initial inversion level, on the one hand, and the nonlinear dependence of the initial inversion level on the average path length, on the other hand.

Note that the average path length of the spontaneously emitted photons is a function of geometry and reflectivity of the boundaries. Optical finishing of the surfaces (in solid-state lasers), type and quality of face coatings as well as housing reflectivity and pumping configuration affect this parameter. Fortunately, the experimental method is proposed for measuring the average path length [16] so that it can be measured once for a desired system. After that, one can predict and optimise the system in presence of ASE, using our presented method.

5. Conclusions

A method is presented to investigate the influence of ASE on characteristics of Q-switched lasers. A full analytical method is used for the near-threshold regime and a semi-analytical procedure is introduced for all cases. Using this method, we have shown that ASE affects the pulse characteristics via reducing the initial inversion level and increasing the initial photon density contributing to the stimulated emission process (under special circumstance derived in this paper). The conditions, under which the initial photon density can be neglected in calculations without considerable error, are clarified in terms of the laser oscillation modes and total resonance mode of the cavity besides the other parameters of the system such as pump rate and effective cross section.

Using the numerical approach, we also have shown that under the same condition of active medium and pumping, the reflectivity of boundaries may affect a Q-switched pulse drastically as it changes the average path length of the spontaneously emitted photons. We believe that the presented method can be used simply for accounting ASE for the Q-switched pulse characteristics evaluation at an initial stage of designing.

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