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Calculation of the spectrum of whispering gallery modes in cylindrical resonators with perturbed boundary conditions

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Abstract. **The spectrum of whispering gallery modes for resonators with a small deformation of the boundary is calculated analytically. Cylindrical resonators with two different cross sections (segment close to a circle and segment close to a semicircle) are considered. The calculation is performed for resonators with metal boundaries, but the obtained result is a good approximation for dielectric resonators as well. The applicability limits of the found expressions for the spectra are analysed. It is shown that the spectra calculated using the obtained expressions coincide well with computer-calculated spectra. The perturbation-induces changes in the field distribution are qualitatively studied using numerical simulation.**

Keywords: *whispering gallery modes, cylindrical resonator, perturbation theory.*

1. Introduction

Whispering gallery mode (WGM) resonators have been extensively studied during the last two decades [1–3]. Recently [4], half-disk WGM lasers emitting in the mid-IR region were developed. The WGM resonators are characterised by a much higher Q factor than the resonators of other types. It is known [5], that the Q-factor in dielectric samples can exceed 10^8. In the case of semiconductor materials, it is possible to achieve a Q-factor of 10^4 [6]. Lasers based on WGM resonators are distinguished by a specific dynamic behaviour due to the geometry of mode fields and by positions of frequencies in the spectrum [7]. Semiconductor lasers also exhibit nontrivial optomechanical effects [8].

Deviations of disk and half-disk resonators from the ideal shape frequently occur upon fabrication. In particular, the cleavage of a half-disk resonator, as a rule, does not pass through the disk centre. Small chips can be formed on the side surfaces of disk resonators (Fig. 1). The aim of this work is to study the effect of small ($\alpha \approx \Delta R/R \ll 1$) defects on the mode structure of these resonators.

2. Statement of the problem

For simplicity, we will consider a resonator with perfectly conducting walls because it is known [9] that the solution for whispering gallery modes in a dielectric resonator with a high refractive index (the case of semiconductor lasers) only slightly differs from the solution for modes in a metal resonator filled with dielectric. This approximation well describes the mode structure, but, if it is necessary to take into account loss for radiation, the resonator boundaries must be taken as dielectric.

As is know [10], the solution of Maxwell's equations in a cylindrical region of the general form (not only in a circular cylinder) with perfectly conducting walls is reduced to the solution of the two-dimensional equation for the projection of the electric (TM mode) or magnetic (TE mode) field strength vectors on the cylinder axis. The corresponding component is described by the equation

$$\Delta\phi + k^2\phi = 0 \tag{1}$$

with the boundary conditions $\phi = 0$ for the TM mode and $\partial\phi/\partial n = 0$ for the TE mode. In the general case, the equation includes the refractive index of the material $n_{\rm m} \neq 1$ and k^2 must be replaced by $n_{\rm m}^2 k^2$, but we choose the speed of light such that $c/n_{\rm m} = 1$.

Note that the parameters k and $\lambda = 2\pi/k$ relate to the two-dimensional problem. At the same time, the total squared wavenumber in the three-dimensional space consists of two terms, $k_{\rm t}^2 = k_\zeta^2 + k^2$, where k_ζ and k are values of the same order.

As will be shown below, k for WGMs of imperfect resonators have the form

$$k = k_0(1 + \delta), \tag{2}$$

where k_0 is the dimensionless unperturbed transverse wavenumber and δ is a dimensionless small parameter weakly depending on the mode number. Therefore, the perturbed total wavenumber k'_t should be determined by the formula $k'_{\rm t} = k_{\rm t0}[1 + (k_0^2/k_{\rm t0}^2)\delta]$, where $k_{\rm t0}$ is the unperturbed total wavenumber.

3. Calculation of modes in the case of a half-disk resonator

Let us consider the case of a half-disk resonator (Fig. 1a). It is always possible to choose a system of units in which the disk radius is $R = 1$. Let us introduce cylindrical coordinates (r, φ) so that the straight part of the half-disk boundary corresponds the angles $\varphi = 0$ and $\varphi = \pi$. Then, the solution of Eqn (1) for a half-disk with a unit radius will be

$$\phi_{mn} = A_{mn} J_m(p_{mn} r)\sin(m\varphi), \;\; k_{mn} = p_{mn}. \tag{3}$$

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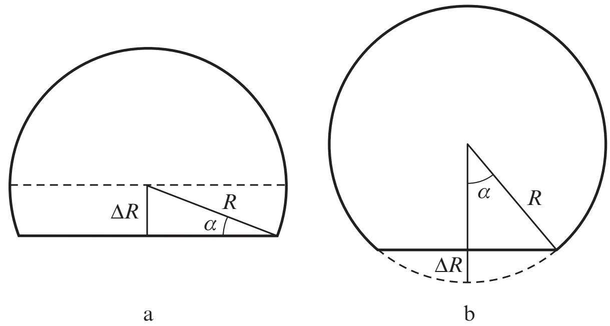


Figure 1. Considered imperfect regions close to (a) a half-disk and (b) a disk.

Here, J_m is the Bessel function of the mth order and p_{mn} is the nth zero of the Bessel function of the mth order (m is a natural number). The A_{mn} coefficient is chosen so that $\int_A |\phi_{mn}|^2 \mathrm{d}S = 1$ (integration is performed over the entire half-disk). From this, we obtain

$$A_{mn} = \left(\frac{\pi}{2}\int_0^1 J_m(p_{mn}r)^2 r\,\mathrm{d}r\right)^{-1/2} = \left(\frac{\sqrt{\pi}}{2}\,|\,J_{m-1}(p_{mn})\,|\right)^{-1}.$$

At $m \gg 1$ and $n \ll m$, the Bessel function considerably differs from zero only at $r \approx 1$, i.e., solution (3) is concentrated near the half-disk boundary. These modes are the whispering gallery modes for the given resonator.

The solution of (1) for the region shown in Fig. 1a can be found by conformal mapping of this region onto a semicircle. If the function $w(z)$, where $z = x + \mathrm{i}y$, performs this transformation, then Eqn (1) can be written in the form

$$\Delta\psi = -k^2\psi\,|\,\mathrm{d}w/\mathrm{d}z\,|^2. \tag{4}$$

Here, the operator Δ and the functions $|\,\mathrm{d}w/\mathrm{d}z\,|^2$ and ψ are considered in the coordinates for the half-disk region. If the region in Fig. 1a is close to a half-disk, then there exists a near-trivial transformation $w(z) \approx z + \alpha g(z)$ with a constant small parameter α. Then, $|\,\mathrm{d}w/\mathrm{d}z\,|^2 \approx 1 + \alpha f(z)$. Thus, with an accuracy to the terms of first-order smallness in α, Eqn (4) takes the form

$$\Delta\psi + k^2\psi = -\alpha k^2\psi f. \tag{5}$$

This is the equation with a small right-hand side for the simple half-disk region with the above boundary conditions $\psi = 0$. If $\alpha = 0$, then the solutions ϕ_{mn} are known. For (5), one can use the standard perturbation theory [11], which, with an accuracy to the terms of the first-order smallness in α, leads to

$$k^2 = k_{mn}^2\left(1 - \alpha\int_A f\phi_{mn}\phi_{mn}^*\,\mathrm{d}S\right). \tag{6}$$

In our case, mapping of the half-disk (Fig. 1a) to a semicircle can be performed using the function

$$z = \left[\left(\frac{b+w}{b-w}\right)^{\pi/2d} - 1\right]\left[\left(\frac{b+w}{b-w}\right)^{\pi/2d} + 1\right]^{-1}, \tag{7}$$

where $d = 2\arctan[(1 + \sin\alpha)/b]$ and $b = \cos\alpha$. Expanding function (7) in series with an accuracy to the terms of first-order smallness in α, we find

$$w = z + \frac{\alpha}{\pi}(1 - z^2)\ln\left(\frac{1+z}{1-z}\right).$$

Then, $f(r,\varphi)$ in expression (5) takes the form

$$f(r,\varphi) = \frac{2}{\pi}\left[2 - r\cos\varphi\ln\left(\frac{1 + 2r\cos\varphi + r^2}{1 - 2r\cos\varphi + r^2}\right)\right.$$
$$\left. + 2r\sin\varphi\arctan\left(\frac{2r\sin\varphi}{1 - r^2}\right)\right].$$

The integral in (6) is

$$\int_A f\phi_{mn}\phi_{mn}^*\,\mathrm{d}S = \int_0^1\int_0^\pi A_{mn}^2 J_m(p_{mn}r)^2\sin^2(m\varphi)f(r,\varphi)\,r\,\mathrm{d}\varphi\,\mathrm{d}r. \tag{8}$$

Since $\sin^2(m\varphi) = {}^1\!/_2[1 - \cos(2m\varphi)]$ and the function $\cos(2m\varphi)$ at large m rapidly oscillates compared to the slowly varying $f(r,\varphi)$, then, upon angular integration, the term $\cos(2m\varphi)$ can be neglected. In this case, integral (8) is equal to $4/\pi$. Substituting this result into (6), we derive the final formula for the wavenumber of the given mode (in common units):

$$k = k_0(1 + \delta),\ \ \delta = -2\alpha/\pi, \tag{9}$$

where $k_0 = p_{mn}/(n_\mathrm{m}R)$ is the unperturbed wavenumber of the given mode.

The TE waveguide mode relates to the problem with homogeneous Neumann boundary conditions $\delta\phi/\delta n = 0$. Exactly the same considerations as in the case of the TM mode yield an expression coinciding with (9) but with different formula for the unperturbed wave number, $k_0 = \eta_{mn}/(n_\mathrm{m}R)$, where η_{mn} is the nth zero of the m-order Bessel function derivative.

At $m \to \infty$, the eigenvalues of whispering gallery modes can be calculated from the condition that the phase incursion is a multiple of π, i.e., $kn_\mathrm{m}(\pi + 2\alpha)R = m\pi$; then,

$$k = \frac{m}{n_\mathrm{m}R}\left(1 - \frac{2\alpha}{\pi}\right). \tag{10}$$

In formula (9), $p_{mn} \approx m$ at $m \to \infty$, and expressions (9) and (10) coincide. Thus, formula (9) is confirmed in the limiting case.

For applicability of formula (9), the eigenvalues of Eqn (5) for an imperfect resonator must be well described by the first approximation of the perturbation theory with respect to parameter α. This condition is fulfilled if the fields of modes corresponding to a given eigenvalue in perturbed and unperturbed resonators differ insignificantly. From the diffraction theory, it is known that the latter occurs when the size of a defect of the boundary parallel to the Poynting vector must be much smaller than the wavelength. In the described case, the characteristic size of a perturbation that qualitatively change the field is described by the formula $\Delta R = R(1 - \cos\alpha) \approx {}^1\!/_2R\alpha^2$. As a result, we obtain the equivalent conditions

$$\alpha \ll \sqrt{\lambda/R},\ \ \alpha \ll \sqrt{2\pi/m}. \tag{11}$$

For $m = 600$ we have $\alpha < 0.1$.

Table 1 compares the wavenumbers found by formula (9) (k_f) and by computer calculation (k_c). One can see that, even for a relatively small $m = 100$, formula (9) at $\Delta R/R = \alpha = 0.26$ yields an error of 10%, which confirms applicability of formula (9) even at rather strong deviations from the semicircle shape.

Table 1. Imperfect half-disk. Mode wavenumbers obtained by numerical computer calculation (k_c^2) and by formula (9) (k_f^2) at different $\Delta R/R \approx \alpha$.

$\Delta R/\lambda$	$\Delta R/R$	k_f^2	k_c^2	$(k_c^2 - k_f^2)/k_c^2$ (%)
1.7	0.10	1.03×10^4	1.04×10^4	1.0
2.1	0.12	1.00×10^4	1.02×10^4	2.0
2.4	0.14	9.73×10^3	9.97×10^3	2.4
2.8	0.16	9.43×10^3	9.74×10^3	3.3
3.1	0.18	9.13×10^3	9.52×10^3	4.0
3.5	0.20	8.82×10^3	9.30×10^3	5.0
3.8	0.22	8.52×10^3	9.10×10^3	6.3
4.0	0.24	8.22×10^3	8.90×10^3	7.6
4.5	0.26	7.92×10^3	8.71×10^3	9.0

Note: Disk radius $R = 1$, $m = 100$, unperturbed eigenvalue $k_0^2 = 1.18\times10^4$; $\Delta R/\lambda$ is the ratio of the defect size to the wavelength.

4. Calculation of modes in the case of a disk resonator

Let us now turn to another, superficially similar problem, namely, consider a disk as an unperturbed region and a disk with a small cleavage as a perturbed one (Fig. 1b). The small angle α is related to the small (in radius, $\Delta R/R \ll 1$) defect by the formula $\Delta R/R = 1 - \cos\alpha \approx \alpha^2/2$.

This problem cannot be solved by conformal mapping because this mapping qualitatively changes the circle shape and, hence, is singular. The first term of the series for this transformation poorly describes the defect. Therefore, we will solve the second problem by the boundary perturbation method [11]. We will use this method for a degenerate case.

It is known that any solution $\psi(r)$ with the eigenvalue k from Eqn (1) for a perturbed region (lying inside an unperturbed region) with the boundary conditions $\psi(r) = 0$ satisfies within this region the exact integral equation

$$\psi(r) = \sum_q \phi_q(r)\left[\int_L \phi_q^*(r_0)\frac{\delta\psi(r_0)}{\delta n_0}\,\mathrm{d}l_0\right](k_q^2 - k^2)^{-1}, \qquad (12)$$

where the functions $\phi_q(r)$ with wavenumbers k_q are the solutions for the unperturbed region. If the perturbation is small, then $k \approx k_q$ for some q. The integration is performed over the line L, i.e., over the perturbed part of the boundary.

In our case, the modes ϕ_q are doubly degenerate, i.e., $k_1 = k_2$; $k_3 = k_4$, and so on, because of which the solution is sought in the form $\psi(r) = a_1\phi_1(r) + a_2\phi_2(r) + \delta\Psi(r)$. Here, $\delta\Psi(r) \to 0$ at $\alpha \to 0$. Substituting this expression into (12), we find in the first approximation

$$(k_1^2 - k^2)[a_1\phi_1(r) + a_2\phi_2(r)]$$
$$= \sum_{q'=1,2} \phi_{q'}(r)\oint_{S_0}\phi_{q'}^*(r_0)\left(a_1\frac{\delta\phi_1(r_0)}{\delta n_0} + a_2\frac{\delta\phi_2(r_0)}{\delta n_0}\right)\mathrm{d}l_0.$$

Multiplying this expression by ϕ_j^* ($j = 1,2$), integrating it over the unperturbed region, and taking into account the orthogonality of modes, we obtain a system of homogeneous algebraic equations, which has a solution if the determinant is equal to zero,

$$\begin{vmatrix} V_{11} + k^2 - k_0^2 & V_{12} \\ V_{21} & V_{22} + k^2 - k_0^2 \end{vmatrix} = 0, \qquad (13)$$

where

$$V_{ij} = \oint \phi_i^*(r_0)\frac{\delta\phi_j(r_0)}{\delta n_0}\,\mathrm{d}l_0. \qquad (14)$$

The solution of problem (1) for a disk with a unit radius and homogeneous boundary conditions $\phi = 0$ consists of the functions

$$\phi_{mn} = B_{mn}J_m(p_{mn}r)\exp(\pm \mathrm{i}m\varphi),\ \ k_{mn} = p_{mn}.$$

Here, one k_{mn} corresponds to the two orthogonal modes with the exponents $\pm\mathrm{i}m\varphi$. Similar to the previous case, we take $\int_A |\phi_{mn}|^2\mathrm{d}S = 1$. Then, $B_{mn} = A_{mn}/2$.

When calculating matrix elements (14), one integrate only over the perturbed boundary regions, because for unperturbed regions $\phi_j = 0$. Then,

$$V_{11} = V_{22} = \int_{-\alpha}^{\alpha} B_{mn}^2 J_m(p_{mn}r)\left[J_m'(p_{mn}r)\,p_{mn}\cos\varphi + \frac{\mathrm{i}m\sin\varphi J_m(p_{mn}r)}{r}\right]\frac{\cos\alpha}{\cos^2\varphi}\,\mathrm{d}\varphi, \qquad (15)$$

where $J_m'(p_{mn}r) = (\mathrm{d}J_m(x)/\mathrm{d}x)|_{x=p_{mn}r}$. For integration over the boundary, we should use $r = \cos\alpha/\cos\varphi$. At $V_{12} = V_{21}^*$, the factor $\exp(\pm\mathrm{i}2m\varphi)$ appears in (15) under the integral sign. Expanding $J_m(p_{mn}r)$ in series near $r = 1$ with an accuracy to the terms of third-order smallness in α (assuming that $m\alpha$ is an arbitrary rather than a small parameter), we obtain from (13) the following final formula for δ from expression (2):

$$\delta^{\pm} = \frac{\alpha^3}{3\pi} \pm \frac{\Delta}{2}, \qquad (16)$$

where

$$\Delta = \frac{k^+ - k^-}{k_0} = \frac{1}{2\pi m^2}\left[\alpha\cos(2m\alpha) - \frac{\sin(2m\alpha)}{2m}\right]$$

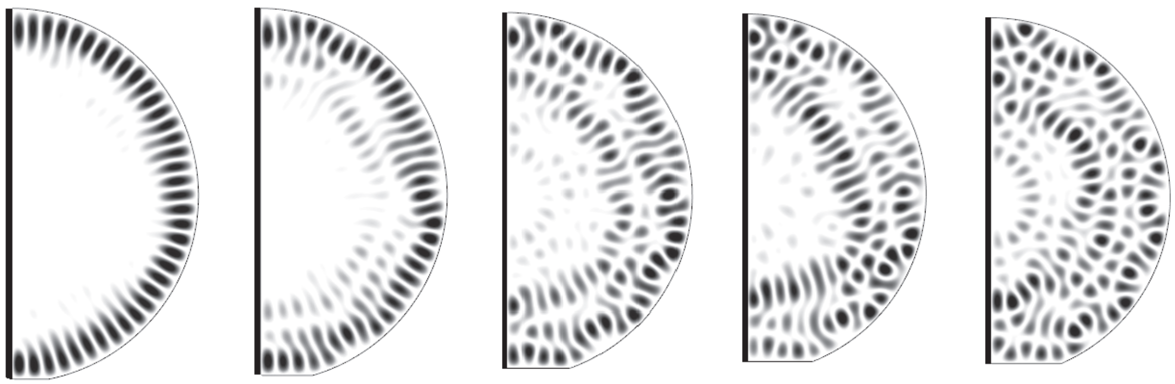
determines the wavenumber splitting.

Expression (16) includes neither the Bessel function nor its derivatives because they can be excluded after integration using the exact formula $J_{m-1}(p_{mn}) = J_m'(p_{mn})$.

Similar considerations can be made for the TE mode. In this case, all the intermediate expressions are different due to different boundary conditions, but, after integration and simplifications, we obtain the same formula (16) with the only difference that the unperturbed eigenvalue of degenerate modes in the case of the TE mode is $k_0 = \eta_{mn}/(n_m R)$, where η_{mn} is the nth zero of the m-order Bessel function derivative. For calculation, we used the exact formula $J_m''(\eta_{mn}) = (m^2/\eta_{mn}^2 - 1)J_m(\eta_{mn})$.

With increasing α, the splitting Δ begins oscillate around zero with the frequency $2m$ and increases in amplitude. The oscillation envelope is easily found to be $|\Delta|_{\max} = (2\pi)^{-1}\alpha/m^2$. From this one can see that the maximum splitting slowly increases with increasing α and rapidly decreases with increasing m, because of which Δ for whispering gallery modes ($m \gg 1$) hardly can be determined by direct measurement of the spectrum, but, in some cases, the splitting can be measured by sending the beam through a nonlinear medium and measuring the difference frequency.

To estimate the applicability limits of formula (16), one can use the same diffraction considerations as in the case of a half-disk. Despite the use of the boundary perturbation



$\Delta R/\lambda$ 0.14 0.28 0.41 0.48 0.55

Figure 2. Computer-calculated distributions of parameter $|E_z|$ (module of the projection of the electric field vector on the z axis) for the TM mode ($m = 37$, $n = 1$) at different defect sizes $\Delta R/\lambda$ (shown under each of the structures). The darker spots correspond to stronger fields. The figure shows only halves of perturbed disks; the thick line shows the symmetry line.

theory in the case of a disk, the assumption of smallness of field changes in this case is also true. The expressions for the characteristic defect size (as well as the expressions for the applicability limits of this approach) coincide with the expressions for a half-disk (11).

Table 2 compares the wavenumbers $k_{\rm f}$ and $k_{\rm c}$ obtained using formula (16) and numerical computer calculations, respectively. The disk-shaped resonator is characterised by very strong mode field distortions at small defects. In this case, formula (16) gives a good approximation for eigenvalues even for strongly distorted fields, although the concept of whispering gallery modes may turn out to be inapplicable for these modes. Figure 2 shows how strongly the field changes at small (compared to the wavelength) deformations.

Table 2. Imperfect disk (notations are the same as in Table 1).

$\Delta R/\lambda$	$\Delta R/R$	$k_{\rm f}^2$	$k_{\rm c}^2$	$(k_{\rm c}^2 - k_{\rm f}^2)/k_{\rm c}^2$ (%)
0.55	0.10	1.251×10^3	1.253×10^3	0.2
0.67	0.12	1.258×10^4	1.257×10^4	0.1
0.78	0.14	1.266×10^3	1.264×10^3	0.2
0.83	0.15	1.274×10^3	1.266×10^3	0.6

Note: Disk radius $R = 1$, $m = 29$, $k_0^2 = 1.227\times10^3$. Eigenvalues are calculated for defect sizes at which the mode field still can be qualitatively attributed to whispering gallery modes.

5. Conclusions

The WGM spectra for half-disk- (9) and disk-shaped (16) resonators with specific perturbations are calculated. The formulas are obtained in the first approximation.

For a half-disk, the change in $k_{\rm f}$ linearly depends on the perturbation value, while the perturbation itself can exceed the wavelength. As is shown by computer calculations, the field is also resistant to the perturbation and remains distributed along the walls. Therefore, high-order WGMs are stable to the considered perturbation.

The spectrum for a disk resonator weaker depends on the linear dimensions of perturbation. However, the WGM field in this resonator depends on the considered perturbations much stronger. If the perturbation is larger than half-wavelength, the mode field is strongly distorted and redistributed over the resonator volume. In addition, indefinitely small boundary perturbation removes mode degeneration, because of which the running waves existing in a perfect disk transform into two standing waves with similar spectra, and running waves are difficult to observe experimentally.

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