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Polarisation effects in gradient nano-optics

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Abstract. **The spectra of reflection of s- and p-polarised waves from gradient nanocoatings at arbitrary angles of incidence are found within the framework of two exactly solvable models of such coatings. To use the detected spectra in the visible and IR ranges, for different frequencies and coating thicknesses we present the wave reflection coefficients as functions of dimensionless frequencies related to the refractive index gradient of the coating material. It is shown that reflection from the gradient coatings in question is an order of magnitude weaker than reflection from uniform coatings, other parameters of radiation and the reflection system being equal. We report a new exactly solvable model illustrating the specific effect of gradient film optics – the possibility of non-reflective propagation of an s-wave through such a film (an analogue of the Brewster effect). The prospects are shown for the use of gradient nanostructures with different refractive index profiles to fabricate broadband non-reflective coatings.**

Keywords: *gradient nanofilms, reflection of waves, s and p polarisations, non-reflective wave propagation.*

1. Introduction

This paper deals with the processes of reflection of light at oblique incidence of radiation on a transparent gradient dielectric nanofilm. The physical foundations of these processes are due to the special mechanism of wave dispersion in nonuniform films, which is determined by a continuous distribution of the refractive index inside the film material [1]. The dispersion of waves in such media depends not only on the value of the refractive index n at a given point of the medium, but also on the values of n in the vicinity of this point; the spatial distribution of n is determined by a technique for nanofilm fabrication [2]. It is necessary to emphasise the fundamental difference of this mechanism from material dispersion related to the parameter $\partial^2 n/\partial\omega^2$ and from spatial dispersion of homogeneous media (the latter leads, as is known from the crystal optics and plasma physics [3], to small corrections to the refractive index of the order of $a/\lambda \ll 1$, where a is the lattice constant or mean free path of particles in the medium and λ is the wavelength). Away from the resonance frequencies of the medium, such effects accumulate slowly along the path of wave propagation at distances that make up many wavelengths. In contrast, the evolution of waves in gradient media has a number of features.

To study analytically the wave fields inside nanofilms, we should find exact solutions of Maxwell's equations with a certain spatial dependence of the refractive index $n(z) = n_0 U(z)$ without any assumptions about the slowness or smallness of spatial variations of the fields and the parameters of the medium [here n_0 is the value of the refractive index n at the medium boundary $z = 0$ and the dimensionless function $U(z)$ is assumed continuous]. At normal incidence of radiation on the film, polarisation effects are absent upon reflection, in which case the reflection spectra of gradient media for several exactly solvable models are found in [4]. When calculating the reflectance spectra for oblique incidence of electromagnetic waves there arise a number of effects caused by the polarisation structure of the wave fields:

(i) the vector field structure is different for s and p polarisations (these fields inside the medium are described by different equations);

(ii) the possibility of constructing the models of $U(z)$, providing accurate solutions for only one polarisation (e.g., only for s waves);

(iii) the set of exactly solvable models of $U(z)$, suitable for both s and p waves, is limited.

S- and p-waves inside gradient films with a dielectric constant $\varepsilon(z)$ can be conveniently described by introducing the generating functions Ψ_s and Ψ_p, corresponding to these polarisations. By selecting the normal to the layer as the z axis, and the projection of the wave vector to the layer surface as the y axis, we can write Maxwell's equations describing the polarisation structure of s-wave with the help of the electric component E_x of the wave field, parallel to the surface $z = 0$, and magnetic components H_y and H_z located in the incidence plane (y, z):

$$\frac{\partial E_x}{\partial z} = -\frac{1}{c}\frac{\partial H_y}{\partial t}, \quad \frac{\partial E_x}{\partial y} = \frac{1}{c}\frac{\partial H_z}{\partial t}, \quad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{\varepsilon(z)}{c}\frac{\partial E_x}{\partial t}, \qquad (1)$$

$$\mathrm{div}(\varepsilon \boldsymbol{E}) = 0, \quad \mathrm{div}(\mu \boldsymbol{H}) = 0. \qquad (2)$$

The components of the p-wave (H_x, parallel to the surface $z = 0$ and electric components E_y and E_z located in the plane of incidence) are still related by equation (2), but system (1) should be replaced:

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$$\frac{\partial H_x}{\partial z} = \frac{\varepsilon(z)}{c}\frac{\partial E_y}{\partial t}, \quad \frac{\partial H_x}{\partial y} = -\frac{\varepsilon(z)}{c}\frac{\partial E_z}{\partial t},$$
$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c}\frac{\partial H_x}{\partial t}. \tag{3}$$

For the analysis of equations (1)–(3) it is expedient to express the components of the fields through the generating functions Ψ_s and Ψ_p. Then for s polarisation

$$E_x = -\frac{1}{c}\frac{d\Psi_s}{dt}, \; H_y = \frac{d\Psi_s}{dz}, \; H_z = -\frac{d\Psi_s}{\partial y} \tag{4}$$

and for p polarisation

$$H_x = \frac{1}{c}\frac{d\Psi_p}{dt}, \; E_y = \frac{1}{\varepsilon(z)}\frac{d\Psi_p}{dz}, \; E_z = -\frac{1}{\varepsilon(z)}\frac{d\Psi_p}{dy}. \tag{5}$$

Using expressions (4)–(5), we can reduce the system of equations (1) and (3) to two equations describing the s- and p-waves, respectively:

$$\frac{\partial^2 \Psi_s}{\partial z^2} + \left(\frac{\omega^2 n_0^2 U^2}{c^2} - k_y^2\right)\Psi_s = 0, \tag{6}$$

$$\frac{\partial^2 \Psi_p}{\partial z^2} + \left(\frac{\omega^2 n_0^2 U^2}{c^2} - k_y^2\right)\Psi_p = \frac{2}{U}\frac{dU}{dz}\frac{\partial \Psi_p}{\partial z}, \tag{7}$$

where $k_y = \omega n_1 \sin\delta/c$; n_1 is the refractive index of a homogeneous medium; and δ is the angle of incidence on the film.

Introducing a new variable η and new functions f_s and f_p:

$$f_s = \Psi_s\sqrt{U}, \; f_p = \frac{\Psi_p}{\sqrt{U}}, \; \eta = \int_0^z U(z_1)dz_1, \tag{8}$$

we can represent equations (6) and (7) for the s- and p-waves in the form

$$\frac{d^2 f_s}{d\eta^2} + f_s\left(K - \frac{U_{\eta\eta}}{2U} + \frac{U_\eta^2}{4U^2}\right) = 0, \tag{9}$$

$$\frac{d^2 f_p}{d\eta^2} + f_p\left(K + \frac{U_{\eta\eta}}{2U} - \frac{3U_\eta^2}{4U^2}\right) = 0. \tag{10}$$

Here $K = (\omega n_0/c)^2 - k_y^2/U^2$; $U_\eta = dU/d\eta$; $U_{\eta\eta} = d^2U/d\eta^2$. Equations (9) and (10) are valid for arbitrary distributions of refractive indices of $U(z)$ and angles of incidence δ.

The conditions of continuity of the field components on the surface of a nanofilm for s and p polarisations are different. Considering a plane wave

$$\Psi_0 = A_0 \exp\{i\omega[n_1(z\cos\delta + y\sin\delta)c^{-1} - t]\},$$

falling from a homogeneous medium with the refractive index n_1 at an angle δ on the film boundary $z = 0$ and introducing complex polarisation-dependent reflection coefficients R_s and R_p, we can write the continuity conditions for the s- and p-waves in the form

$$A_0(1 + R_s) = A_s\Psi_s|_{z=0},$$
$$\frac{i\omega n_1 A_0\cos\delta(1 - R_s)}{c} = A_s\frac{d\Psi_s}{dz}\Big|_{z=0},$$
$$\frac{i\omega n_1 A_0\sin\delta(1 + R_s)}{c} = ik_y A_s\Psi_s|_{z=0},$$
$$A_0(1 + R_p) = A_p\Psi_p|_{z=0}, \tag{11}$$
$$\frac{i\omega A_0\cos\delta(1 - R_p)}{cn_1} = \frac{A_p}{\varepsilon(z)}\frac{d\Psi_p}{dz}\Big|_{z=0},$$
$$\frac{i\omega n_1 A_0\sin\delta(1 + R_p)}{c} = ik_y A_p\Psi_p|_{z=0}.$$

If the film of thickness d is located on a uniform thick substrate with a refractive index n, the continuity conditions specified in the plane $z = d$ relate the components of the field in the film to the corresponding components of the field of a plane wave in the substrate:

$$\Psi = A\exp\{i\omega[(z\sqrt{n^2 - n_1^2\sin^2\delta} + yn_1\cos\delta)c^{-1} - t]\}.$$

The difficulty of solving equations (9) and (10), corresponding to the s and p polarisations, for a profile of the refractive index $n(z) = n_0U(z)$ within the coating limits the set of exactly solvable models of $U(z)$.

To show the sensitivity of the processes of wave reflection by a gradient coating to the form of the profile $U(z)$, in Section 2 we consider two simple models of $U(z)$ [5]:

$$U_1(z) = (1 + z/L)^{-1}, \; U_2(z) = \exp(-z/L), \tag{12}$$

which are characterised by one free parameter – the characteristic length L determining the spatial scale of the change in the refractive index. Although the values of the model functions $U_1(z)$ and $U_2(z)$ on the film boundary ($z = 0$) are equal, the values of their gradients at $z = 0$ are also equal and the differences at equal values of the ratio z/L for the considered thicknesses of nanofilms are small [thus, for example, $z/L = 0.3$, $U_1(0.3)/U_2(0.13) = 1.04$], the regimes of the field propagation defined by these profiles can vary significantly. The regimes of wave propagation through the coatings with profile (12) at arbitrary angles of incidence are described in the framework of these models with the help of exact analytical solutions of the corresponding equations. In Section 2 we present the advantages of gradient films when they are used as broadband non-reflective coatings. In contrast to papers [3] and [5] we show new physical effects that determine these advantages – artificial dispersion determined by the $U(z)$ profile, cutoff frequency of the waves and polarisation-independent regimes of wave tunnelling in gradient media. In Section 3 we demonstrate another property inherent in gradient films: unlike uniform films where only p-waves can propagate non-reflectively (Brewster effect [5]), gradient films allows for non-reflective propagation of s-waves. This analogue of the Brewster effect is given within the framework of a flexible exactly solvable model containing, in contrast to (12), three free parameters L, M and g:

$$U^2(z) = 1 - \frac{1}{g} + \frac{W^2(z)}{g},$$
$$W(z) = \left[\cos\left(\frac{z}{L}\right) + M\sin\left(\frac{z}{L}\right)\right]^{-1}. \tag{13}$$

When reducing the inhomogeneity of the refractive index ($L \to \infty$), distributions (12) and (13) degenerate to a constant value: $U \to 1$.

2. Broadband non-reflective coatings

Development of non-reflective thin-film coatings, which are effective in a wide spectral range at an arbitrary polarisation of the waves, attracts constant attention. The use of gradient films opens up new prospects for obtaining controlled reflection spectra in the specified wavelength range. The reflection spectra of these films with other parameters being equal exhibit significant dependence on the refractive index profile $U(z)$. We begin our analysis of profiles (12) with the expression for $U_1(z)$, and by substituting $U_1(z)$ in (8) we define the variable $\eta = L\ln(1 + z/L)$. Equations (9) and (10), defining the functions f_s and f_p, are in this case reduced to the Bessel equation:

$$\frac{d^2 f_{s,p}}{dx^2} + \frac{1}{x}\frac{df_{s,p}}{dx} + f_{s,p}\left(q^2_{s,p} - \frac{l^2_{s,p}}{x^2}\right) = 0, \quad x = \frac{\eta}{L}, \tag{14}$$

$$q^2 = -\left(\frac{\omega L \sin\delta}{c}\right)^2, \quad l^2_{s,p} = \frac{1}{4}\left(1 - \frac{1}{u^2}\right),$$
$$u = \frac{\Omega}{\omega}, \quad \Omega = \frac{c}{2n_0 L}. \tag{15}$$

Here Ω is a characteristic frequency associated with the medium inhomogeneity. Linearly independent solutions of equation (14) at $q^2 < 0$ are, as is known, the Bessel functions of the imaginary argument I_l and Macdonald functions K_l [6]. In studying the field in a layer of finite thickness d we should use their linear combinations: $\vartheta_l^{(1,2)} = I_l \pm iK_l$. Using the solutions of equation (14) and notations (8) we can write the generating functions for s- and p-waves:

$$\Psi_s = \left(1 + \frac{z}{L}\right)^{1/2} f_s, \quad \Psi_p = \left(1 + \frac{z}{L}\right)^{-1/2} f_p, \tag{16}$$

$$f_{s,p} = [\vartheta_l^{(1)}(\zeta) + Q_{s,p}\vartheta_l^{(2)}(\zeta)], \quad \zeta = \frac{\omega(L+z)\sin\delta}{c}. \tag{17}$$

For simplicity in expressions for the functions Ψ_s and Ψ_p (16) we have omitted the phase factors $\exp[i(k_y y - \omega t)]$; the values of Q_s and Q_p in (17) are determined from the conditions of continuity at the air–film ($n_1 = 0$, $z = 0$, $U_1 = 1$) and film–substrate interfaces [$z = d$, $U_1 = U_m = (1 + d/L)^{-1}$]. In the latter case, for the convenience of calculations the variable ζ can be written as

$$\zeta_m = \frac{\sin\delta}{2un_0 U_m}. \tag{18}$$

Substituting expressions (16) and (17) into (4) and (5) allows us to write explicitly the formulas for the components of the fields. Complex reflection coefficients for s- and p-waves are found from equations (11) using a standard procedure. For example, for an s-wave the coefficient R_s is determined by the equation

$$\frac{1 + R_s}{1 - R_s} = \frac{iB\Psi_s(\zeta_0)}{\Psi_s'(\zeta_0)}, \quad \zeta_0 = \frac{\omega L \sin\delta}{c}, \quad B = 2\zeta_0,$$
$$\Psi_s'(\zeta_0) = \frac{\omega L \sin\delta}{c}\frac{d\Psi_s}{d\zeta}\bigg|_{\zeta=\zeta_0} \tag{19}$$

[the coefficient R_p is obtained from (11) in the same way]. These data are used to construct the reflection spectra $|R_{s,p}(u)|^2$. Note that the functions $f_{s,p}$ (17) for both polarisations are equal, which greatly reduces the computation time, whereas the difference between Ψ_s and Ψ_p is determined only by the degree of the factor $(1 + z/L)$ (16). At high frequencies ($\omega > \Omega$, $u < 1$) the indices of the functions ϑ_l in (17) are imaginary numbers, since $l^2_{s,p} < 0$; in the low-frequency part of the spectrum ($\omega < \Omega$, $u > 1$) the indices of these functions are real numbers ($l^2_{s,p} > 0$). To calculate the functions ϑ_l under the above values of the indices $l_{s,p}$, use is made of the expressions in the form of infinite series or integral representations [6]; therefore, fields (17) are studied below using numerical methods.

Examples of reflection spectra of $|R(u)|^2$ in the low frequency region are shown in Fig. 1. To show the effect of the gradient structure on the spectra of $|R(u)|^2$, it is useful to compare them with the reflection spectra of homogeneous films; thus, the parameters of the incident waves (polarisation, frequency ω and angle of incidence δ), parameters of the reflection structures (refractive indices n_0 and n) and film thickness d are identical in both cases. Complex reflection coefficients of s- and p-waves from homogeneous films deposited on a substrate have the form

$$R_s = \frac{M_s\cos\delta - r_1 - it(\cos\delta - r_1 M_s)}{M_s\cos\delta + r_1 - it(\cos\delta + r_1 M_s)}, \tag{20}$$

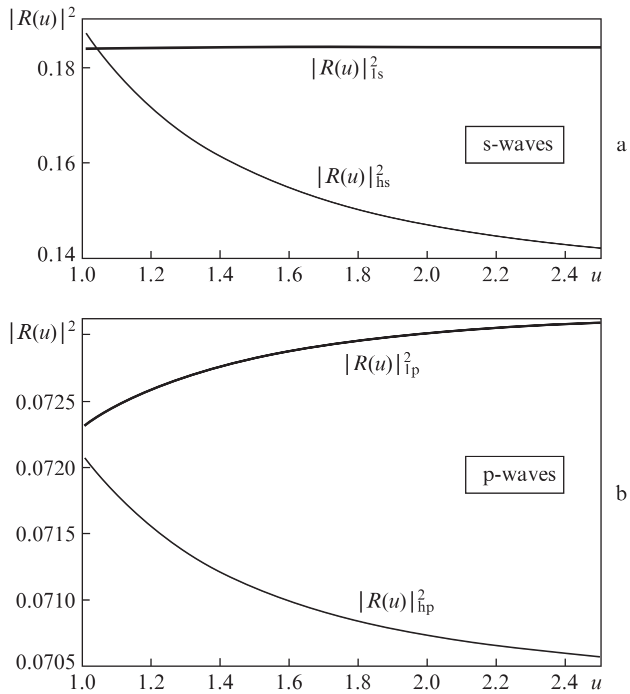

Figure 1. Influence of the gradient structure of nanocoatings with a refractive index profile $N(z) = n_0(1 + z/L)^{-1}$ on the reflection spectra of (a) s- and (b) p-waves incident at angle $\delta = 45°$ for $n_0 = 1.9$, and the refractive index of the substrate $n = 1.5$. Due to the smallness of the reflection coefficient, for gradient coatings $|R(u)|^2_{gs}$ and $|R(u)|^2_{gp}$ for a given dimensionless frequency u (15) and polarisation of incident waves (as compared with the corresponding coefficients of homogeneous coatings $|R(u)|^2_{hs}$ and $|R(u)|^2_{hp}$ we present in the Figure auxiliary curves $|R(u)|^2_{1s}$ and $|R(u)|^2_{1p}$ determining the reflection coefficients of gradient coatings as $|R(u)|^2_{gs} = 0.09|R(u)|^2_{1s}$, $|R(u)|^2_{gp} = 0.12|R(u)|^2_{1p}$.

$$R_{\rm p} = \frac{M_{\rm p} n^2 \cos\delta - r_1 - {\rm i}t(n^2\cos\delta - r_1 M_{\rm p})}{M_{\rm p} n^2 \cos\delta + r_1 - {\rm i}t(n^2\cos\delta + r_1 M_{\rm p})}, \qquad (21)$$

$$M_{\rm s} = \frac{r_1}{r_2},\ M_{\rm p} = M_{\rm s}\frac{n^2}{n_0^2},\ r_1 = \sqrt{n_0^2 - \sin^2\delta},$$
$$r_2 = \sqrt{n^2 - \sin^2\delta},\ t = \tan\left(\frac{\omega d r_1}{c}\right). \qquad (22)$$

To compare the coefficients $R_{\rm s}$ and $R_{\rm p}$ for homogeneous and gradient films at a given frequency, we should substitute into formula (20)–(22) the values of $t = \tan(\omega d r_1/c)$ expressed through u and $U_{\rm m}$:

$$t = \tan\left[\left(\frac{1 - U_{\rm m}}{2uU_{\rm m}}\right)\left(1 - \frac{\sin^2\delta}{n_0^2}\right)^{1/2}\right]. \qquad (23)$$

We note the essential features of the spectra of reflection from gradient nanofilms with the refractive index profile $U_1(z)$ (Figs 1a and 1b):

(i) the dispersion of the film caused by the inhomogeneity of the refractive index determines the characteristic frequency Ω differentiating the spectral intervals of high and low frequencies, to which correspond different spatial structures of the fields inside the film (the frequency Ω is the same for s- and p-waves);

(ii) in a wide spectral range the reflection coefficient of gradient films with respect to the power is an order of magnitude less than the same coefficient for a uniform film at the same frequencies and polarisations of the incident waves and coinciding values of film thicknesses and refractive indices n_0 and n;

(iii) gradient coatings with the profile $U_1(z)$ may significantly reduce reflection though their thickness may be less than the wavelength of the reflected wave. Using the definition of the variable u (15), we can write the ratio between the wavelength λ and thickness d, corresponding to any value of the variable u:

$$\frac{d}{\lambda} = \frac{1 - U_{\rm m}}{4\pi u n_0 U_{\rm m}}. \qquad (24)$$

According to Fig. 1a at a frequency $u = 1.5$ the reflection coefficient of the s-wave from the gradient nanofilm is $|R(u)|^2 = 0.0165$. This means that the above value of $|R(u)|^2$ characterises reflection of the waves from various structures, whose parameters satisfy relation (23). Thus, for the parameters corresponding to Fig. 1, it follows from (23) that $d = 0.007\lambda$; in this case, incidence of IR radiation ($\lambda = \lambda_0 = 14.28$ μm) at an angle $\delta = 45°$ on a gradient film of thickness $d = d_0 = 100$ nm is characterised by the reflection coefficient $|R(1.5)|^2 = 0.0165$. The same value of $|R(u)|^2_{\rm gs}$ determines the reflection coefficient of the wave with $\lambda = m\lambda_0$ from the film of thickness $d = md_0$, where m is an arbitrary positive number. Replacement of this gradient film by a uniform film of the same thickness according to Fig. 1a will increase the reflection coefficient by an order of magnitude. Such gradient films are of interest for the development of non-reflective nanocoatings for the IR range.

The fields of s- and p-waves in the film with an exponential refractive index profile $U_2(z)$ (12) are described by the same equation (14) as in the model for $U_1(z)$, but unlike the latter use is made of other parameters:

$$x = \exp\left(-\frac{z}{L}\right),\ q^2 = \left(\frac{\omega n_0 L}{c}\right)^2,\ l_{\rm s}^2 = \left(\frac{\omega L \sin\delta}{c}\right)^2. \qquad (25)$$

It should be emphasised that the value q^2 (25) corresponding to the model $U_2(z)$ is always positive as compared to the model $U_1(z)$, where q^2 is always negative. Because of this difference, linearly independent solutions of equation (14) are represented by Hankel functions $H_l^{(1,2)} = J_l \pm {\rm i}N_l$ (where J_l and N_l are the Bessel and Neumann functions [6]), and the generating functions for s-waves in the model can be written in the form

$$\Psi_{\rm s} = H_l^{(1)}(\zeta) + Q_{\rm s} H_l^{(2)}(\zeta),\ \zeta = qx, \qquad (26)$$

The generating function for the p-wave is different from (26):

$$\Psi_{\rm p} = \exp\left(-\frac{z}{L}\right)[H_l^{(1)}(\zeta) + Q_{\rm p} H_l^{(2)}(\zeta)],$$
$$l = l_{\rm p} = \left[1 + \left(\frac{\omega L \sin\delta}{c}\right)^2\right]^{1/2}. \qquad (27)$$

The variable ζ in the function $\Psi_{\rm p}$ is defined in (26). The reflection coefficients are calculated by substituting the functions $\Psi_{\rm s}$ and $\Psi_{\rm p}$ in the boundary conditions (11) and by using numerical methods for the calculation of expressions containing Hankel functions with arbitrary real indices.

However, as is well known [6], Hankel functions $H_l^{(1,2)}$ at half-integer values of l are expressed in terms of elementary functions. Minimum possible half-integer values of $l_{\rm s}$ and $l_{\rm p}$ are 1/2 and 3/2, respectively. In these cases, the calculations are simplified, thereby revealing an important physical result that expands the scope of application of the well-known Brewster law.

3. Analogy of Brewster's law for s-waves in a gradient film

Consider the simple case of s-wave reflection from a film with an exponential index profile $U_2(z)$ at $l_{\rm s} = 1/2$. The generating function $\Psi_{\rm s}$ in this case can be written as

$$\Psi_{\rm s} = \frac{A[\cos(qx) + Q\sin(qx)]}{\sqrt{x}}, \qquad (28)$$

where A is a constant. The equation defining $R_{\rm s}$ follows from the boundary conditions (11):

$$\frac{1 + R_{\rm s}}{1 - R_{\rm s}} = \frac{{\rm i}B[\cos q + Q\sin q]}{\cos q + 2q\sin q + Q[\sin q - 2q\cos q]},\ B = \cot\delta. \qquad (29)$$

The value of the constant B (29) is obtained from the expression for B (19) at $l_{\rm s} = 1/2$. The parameter Q in (29) is calculated from the boundary conditions at the boundary $z = {\rm d}$:

$$Q = \frac{2qx_{\rm m}\sin(qx_{\rm m}) - ({\rm i}B_{\rm m} - 1)\cos(qx_{\rm m})}{2qx_{\rm m}\cos(qx_{\rm m}) + (iB_{\rm m} - 1)\sin(qx_{\rm m})}, \qquad (30)$$

$$B_{\rm m} = \frac{2\omega L\sqrt{n^2 - \sin^2\delta}}{c},\ x_{\rm m} = \frac{d}{L},\ \delta = \delta_0. \qquad (31)$$

By substituting (30) and (31) into (29) we can obtain an explicit expression for $R_{\rm s}$.

It is important that at a certain set of parameters of the film and substrate the found reflection coefficient $R_{\rm s}$ may vanish. Thus, when a 800-nm s-wave is incident on a gradient nanofilm ($n_0 = 2.255$, $d = 20$ nm) with a refractive index profile $U_2(z)$, deposited on a substrate with $n = 1.53$, the reflection coefficient at an angle of incidence $\delta_0 = 70°$ vanishes, i.e., such an s-wave passes through the nanofilm without reflection and refraction. In contrast, a similar effect for the s-wave in uniform films, as seen from the definition of R_s (20), is impossible: the condition $R_s = 0$ is not met at any values of the angle δ. Non-reflective propagation of p-waves through uniform films, as seen from equation (21), is possible for the angles of incidence δ_B, determined by the condition $R_p = 0$:

$$\tan^2\delta_B = \frac{(nn_0)^2}{n^2 + n_0^2 - (nn_0)^2}. \quad (32)$$

In a particular case of $n_0 = 1$ corresponding to the incidence of a wave from air to a homogeneous medium, expression (32) transforms into the well-known formula for the Brewster angle: $\tan\delta_B = n$ [7].

Note that non-reflective propagation of s-waves through gradient films is possible even at other refractive index profiles $U(z)$. Consider, for example, a more complex monotonic profile (13) (Fig. 2) containing, in contrast to (12), three free parameters L, M and g. Curves (*1*) and (*2*) in Fig. 2 show how, preserving the depth and width of the $U_2(z)$ distribution, we can change the slope of the distribution. The solution to the wave equation (6) for distribution (13), which requires special transformations, is given in the Appendix. We present here the results of the solutions which show the possible non-reflective propagation of s-waves through the gradient layer (13). Thus, when a wave is incident from a dielectric medium with $n_1 = 1.433$ at angle $\delta = 60°$ on a nanofilm with profile (13) (thickness $d = 100$ nm, $n_0 = 1.9$, $M = 0.9632$, $g = 2.1$), deposited on a substrate with $n = 2.28$, reflection disappears for a wave with $\lambda = 485$ nm. In another case when the s-wave is incident from air ($n_1 = 1$) at angle $\delta = 45°$ on a nanostructure with parameters $n_0 = 1.9$, $M = 0.9632$, $g = 1.3$, $n = 2$ and $d = 100$ nm, a non-reflective regime arises for a wave with $\lambda = 693$ nm.

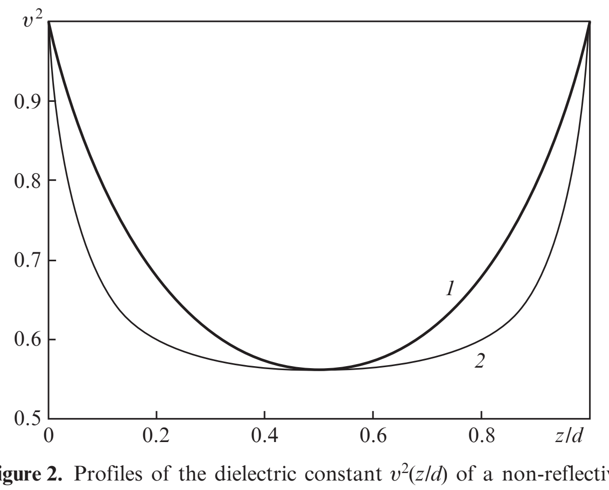

Figure 2. Profiles of the dielectric constant $v^2(z/d)$ of a non-reflective gradient film at $M = 0.9632$, $g = 1.1$ (*1*) and $M = 3.3627$, $h = 2.1$ (*2*).

4. Conclusions

Note the essential features of the discussed reflection spectra, depending on the distribution of the refractive index in the film thickness.

1. In gradient films the characteristic frequency Ω is formed, which is absent in uniform films; however, the spatial structures of high-frequency ($\omega > \Omega$) and low-frequency ($\omega < \Omega$) fields inside the film are different. Thus, for s(p)-waves the spatial structure of the field at $\omega > \Omega$ is formed monotonic (oscillating) changes in the amplitudes of the waves, and at $\omega < \Omega$ an opposite situation arises – fields are formed by oscillating (monotonic) modes.

2. Non-reflective propagation of an s-wave through a gradient film can be considered as an analogy of the Brewster effect, i.e., non-reflective propagation of p-waves through the interface of homogeneous media.

3. The nanofilms in question, characterised by a low reflection coefficient, may be of interest for the synthesis of broadband non-reflective coatings.

Appendix

To solve equation (6) with the refractive index profile (13), we introduce a new variable η and pass to a normalised variable ζ:

$$\eta = \int_0^z W(z_1)\,dz_1, \quad (A1)$$

$$\frac{\eta}{L} = \frac{1}{\sqrt{1+M^2}}\ln\left[\frac{1 + m\tan(z/2L)}{1 - m\tan(z/2L)}\right],$$
$$m_\pm = \sqrt{1+M^2} \pm M, \quad (A2)$$

$$\zeta = \frac{\eta\sqrt{1+M^2}}{L} - \zeta_0, \;\; \zeta_0 = \ln m_+,$$
$$\zeta|_{z=0} = -\zeta_0, \;\; \zeta|_{z=d} = \zeta_0. \quad (A3)$$

The function $W(z)$ (13) is expressed through the variable ζ:

$$W(\zeta) = \frac{\cosh\zeta}{\sqrt{1+M^2}}, \;\; W(\zeta_0) = W(-\zeta_0) = 1. \quad (A4)$$

By substituting the distribution $U_2(z)$ into the wave equation (6), by introducing a new function $f = \Psi\sqrt{W}$ and using the variable ζ, we write the equation for f in the form

$$\frac{d^2f}{d\zeta^2} + f\left(q^2 - \frac{\Lambda}{\cosh^2\zeta}\right) = 0, \quad (A5)$$

$$q^2 = \frac{\omega^2L^2n_0^2}{c^2g(1+M^2)} - \frac{1}{4}, \quad (A6)$$

$$\Lambda = \left(\frac{\omega L}{c}\right)^2\frac{\wp}{g} - \frac{1}{4}, \;\; \wp = n_0^2(g-1) - n_1^2 g\sin^2\delta. \quad (A7)$$

The general solution of equation (A5) can be expressed in terms of hypergeometric functions [8], but here we restrict ourselves to a special case of $\Lambda = 0$. Then equation (A5) is reduced to a simple form

$$\frac{d^2f_s}{d\zeta^2} + q^2f_s = 0, \;\; q^2 = \frac{1}{4}\left[\frac{n_0^2}{\wp(1+M^2)} - 1\right]. \quad (A8)$$

Consider the case $q^2 > 0$ when solutions to equation (A8) are described by forward and backward waves in the ζ-space; in this case, the generating function Ψ_s is expressed in terms of elementary functions:

$$\Psi_s = \left[\cos\left(\frac{z}{L}\right) + M\sin\left(\frac{z}{L}\right)\right]^{1/2}[\exp(iq\zeta) + Q\exp(-iq\zeta)]. \quad (A9)$$

By substituting the function Ψ_s into the boundary conditions (11), we calculate the complex reflection coefficient:

$$R_s = \frac{iB - M - 2iqY\sqrt{1+M^2}}{iB + M + 2iqY\sqrt{1+M^2}},$$

$$Y = \frac{\exp(-iq\zeta_0) - Q\exp(iq\zeta_0)}{\exp(-iq\zeta_0) + Q\exp(iq\zeta_0)}, \quad B = \frac{2\omega L n_1 \cos\delta}{c}. \quad (A10)$$

Values of ζ_0 and q are defined in (A3) and (A8), respectively. By calculating the parameter Q from the boundary conditions on the plane $z = d$, we can write the formula for R_s in the form:

$$R_s = \frac{\sigma_1 + i\sigma_2}{\chi_1 + i\chi_2},$$

$$\sigma_1 = t[BB_1 + M^2 - 4q^2(1+M^2)] - 4Mq\sqrt{1+M^2},$$

$$\sigma_2 = (B - B_1)(2q\sqrt{1+M^2} - Mt),$$

$$\chi_1 = t[BB_1 - M^2 + 4q^2(1+M^2)] + 4Mq\sqrt{1+M^2}, \quad (A11)$$

$$\chi_2 = (B + B_1)(2q\sqrt{1+M^2} - Mt),$$

$$t = \tan(2q\zeta_0), \quad B_1 = \frac{2\omega L\sqrt{n^2 - n_1^2\sin^2\delta}}{c}.$$

To find the conditions for non-reflective ($R_s = 0$) propagation of a wave through a coating, we put in (A11) $\sigma_1 = \sigma_2 = 0$; this condition is satisfied for the frequencies and angles related by the expression

$$BB_1 = M^2 + 4q^2(1 + M^2). \quad (A12)$$

By substituting the quantities B (A10) and B_1 (A11) into (A12), we obtain

$$\frac{\omega d}{c} = (\arctan M)\left(\frac{n_0^2 - \wp}{\wp n_1 \cos\delta\sqrt{n^2 - n_1^2\sin^2\delta}}\right)^{1/2}. \quad (A13)$$

The right-hand side of equation (A13), which is independent of the frequency ω and gradient coating thickness d, is determined by the coating parameters (n_0, M, L and g), refractive indices of homogeneous media surrounding the film (n_1 and n) and the angle δ of wave incidence. By setting these parameters we can find the dimensionless quantity ωdc^{-1} (A13), and knowing this value we can calculate the film thickness d providing non-reflective propagation of an s-wave with a frequency ω, incident on this nanostructure at angle δ. Examples of gradient film parameters calculated by these formulas and corresponding to non-reflective propagation in the case of incidence from a dielectric medium and from air are presented at the end of Section 3. Use of thicker films while preserving the values of all the parameters in the right-hand side of (A13) will lead to a shift in the frequency of non-reflective propagation to the IR region of the spectrum.

References

1. Shvartsburg A., Petite G. *Opt. Lett.*, **31**, 1127 (2006).
2. Volpian O.D., Kuz'michev A.I. *Otritsatel'noe prelomlenie voln* (Negative Refraction of Waves) (Kiev–Moscow: Avers, 2012).
3. Ginzburg V.L. *The Propagation of Electromagnetic Waves in Plasmas* (Oxford, New York: Pergamon press, 1961; Moscow: Nauka, 1970).
4. Shvartsburg A.B., Kuzmiak V., Petite G. *Phys. Rep.*, **452**, 34 (2007).
5. Wait J.R. *Electromagnetic waves in stratified media* (Oxford: Pergamon Press, 1970).
6. Abramowitz M., Stegun I. *Handbook of Mathematical Functions* (New York: Dover Publ., 1968).
7. Born M., Wolf E. *Principles of Optics* (London: Pergamon, 1970; Moscow: Nauka, 1973).
8. Shvartsburg A.B., Maradudin A.A. *Waves in Gradient Metamaterials* (Singapore: WSPC, 2013).