

LASER BEAMS

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An Airy beam as a self-similar solution to the problem of slit laser beam propagation in a linear medium and in a photorefractive crystal with diffusion nonlinearity

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Abstract. **We have analysed self-similar solutions to the propagation problem of a slit beam with a plane wavefront in a linear medium and in a photorefractive crystal with diffusion nonlinearity. It is shown that in the latter case, despite the presence of the nonlinear term in the wave equation, the linear superposition principle holds true for the solutions of this class due to saturation. At the same time, the mirror symmetry violation of the wave equation for the transverse coordinate in the nonlinear case and the requirement to the spatial localisation modify one of the localised partial solutions (Airy beam) to the corresponding linear problem and prohibit the existence of other solutions of this class.**

Keywords: *wave equation, Airy beam, photorefractive crystal, diffusion nonlinearity, propagation problem, self-similar solution.*

1. Introduction

Propagation of a so-called slit radiation beam in a photorefractive crystal (PRC) with diffusion nonlinearity in the '1 + 1' dimensional approximation is described by a truncated wave equation [1]

$$\mathrm{i}\frac{\partial\phi(s,\xi)}{\partial\xi}+\frac{1}{2}\frac{\partial^2\phi(s,\xi)}{\partial s^2}+\gamma\frac{\partial|\phi(s,\xi)|^2}{\partial s}\frac{\phi(s,\xi)}{|\phi(s,\xi)|^2}=0. \quad (1)$$

In deriving expression (1) substitution $\boldsymbol{E}(x,z)=\boldsymbol{e}\phi(x,z)\exp(\mathrm{i}kz)$ allows one to single out a slow envelope $\phi(x,z)$ of the light wave field $\boldsymbol{E}(x,z)$ with the polarisation vector $\boldsymbol{e}$, which is directed along the x axis. By substituting the variables $s = x/x_0$ and $\xi = z/(kx_0^2)$, where x_0 is an arbitrary scale factor, we introduce the dimensionless coordinates. Here, $k = k_0n_e$ is the wavenumber; $k_0 = 2\pi/\lambda_0$; λ_0 is the wavelength of radiation in vacuum; n_e is the refractive index of an extraordinary wave in the PRC; $\gamma = (k_0^2x_0n_e^4r_{33})(K_BT/2e_0)$; r_{33} is the matrix component of the linear electro-optic effect, working in the chosen geometry (optical axis $\boldsymbol{c}$ of the PRC SBN is directed along the axis of the vector $\boldsymbol{e}$ and axis x); K_B is the Boltzmann constant; T is the temperature; and e_0 is the electron charge. Christodoulides and Carvalho [2] analyse the case of the diffusion nonlinearity [2], in which background illumination of the PRC is assumed negligibly small. The last approach is not fully correct and violated at those points of the x axis, where the diffusion field is zeroed.

Christodoulides and Coskun [1] have shown that equation (1) has an exact particular solution:

$$\phi(s,\xi)=\phi_0\mathrm{Ai}(\varepsilon\eta+4\gamma^2)\exp(-2\gamma\eta)\exp[\mathrm{i}(\varepsilon\eta+\xi^2/12)\xi/2], \quad (2)$$

where $\eta = s - (\varepsilon\xi^2/4)$; ϕ_0 is an arbitrary constant; Ai(x) is the Airy function [3]; and $\varepsilon = \pm1$ at positive and negative γ, respectively. The emergence of ε in the expression for γ is related to the requirement of exponential factor decay in (2) at the oscillating tail of the Airy function.

It is easy to make certain that

1) unlike self-consistent solutions for the cases of Kerr and most other types of nonlinearity, the amplitude of the solution of (2) is arbitrary;

2) as for other soliton-like solutions, in the intensity distribution the diffraction manifestations are absent and $|\phi(s,\xi)|^2=|\phi(\eta)|^2=|\phi_0|^2\mathrm{Ai}^2(\varepsilon\eta+4\gamma^2)\exp(-4\gamma\eta)$;

3) the profile of the distribution $|\phi(s,\xi)|^2$ experiencing no diffraction scattering is moving in the PRC along a parabolic trajectory $s = (\varepsilon\xi^2/4)$ + const;

4) the wavefront in this movement changes the slope (cross term in the phase) and is always (at fixed values of ξ) flat (the phase linearly depends on s and η);

5) forced removal of the nonlinearity (transition to limit $\gamma\to0$) does not eliminate the solution of (2) and transforms it to the form known from Berry's work [4]

$$\phi(s,\xi)=\phi_0\mathrm{Ai}(\varepsilon\eta)\exp[\mathrm{i}(\varepsilon\eta+\xi^2/12)\xi/2]. \quad (3)$$

However, in contrast to [4], ε in (3) can take either of two values $\varepsilon = \pm1$, which reflects the symmetry (1) with respect to the transformation $s \leftrightarrow -s$ at $\gamma = 0$ and the resulting existence of two mirror-symmetrical solutions in the form of Airy beams. In this limit the wave equation transforms into the standard form:

$$\mathrm{i}\frac{\partial\phi(s,\xi)}{\partial\xi}+\frac{1}{2}\frac{\partial^2\phi(s,\xi)}{\partial s^2}=0. \quad (4)$$

Note that the study of the peculiarities of self-similar solutions in the form of Airy beams, including some of the less idealised and more realistic (in terms of the practice) situations has attracted in recent years more and more attention of researchers [5–11].

With the above-mentioned taken into account, three main questions arise. First of all, why is the diffusion nonlinearity

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so remarkable that its emergence, in fact, only slightly modifies some class of particular solutions of the corresponding linear problem (4)? Second, why does this modification occur? And finally, what general properties characterise this class of particular solutions? We will show that, despite the presence of the nonlinear term in (1), the linear superposition principle holds true for partial solutions of a certain class and that this class includes not only solutions (2) and (3), but also some other well-known private solutions of (1) and (4) [12].

2. Linear medium

We begin our analysis with a linear problem (4) and rewrite the solution in the form

$$\phi(s,\xi) = \phi(\eta,\xi) = F(\eta,\xi)\exp[\mathrm{i}\varphi(\eta,\xi)], \tag{5}$$

where $\eta = \eta(s,\xi)$ and ξ play the role of the transverse and longitudinal coordinates, respectively; and $F(\eta,\xi)$, $\eta(s,\xi)$ and $\varphi(\eta,\xi)$ are some unknown functions. Then on the trajectory $s = s(\xi)$, described by the equation $\eta(s,\xi) = \eta_0 = \text{const}$, $F^2(\eta,\xi) = F^2(\eta_0,\xi)$ and $\varphi(\eta,\xi) = \varphi(\eta_0,\xi)$ in (5) will depend only on ξ, which describe the 'soliton-like' (independent of ξ) and divergent/convergent (dependent on ξ) beams that not necessarily propagate along a straight line.

Substituting (5) into (4) and dividing the real and imaginary parts, we find

$$F(\eta,\xi)\left\{2\frac{\partial\varphi(\eta,\xi)}{\partial\eta}\frac{\partial\eta(s,\xi)}{\partial\xi}+2\frac{\partial\varphi(\eta,\xi)}{\partial\xi}+\left[\frac{\partial\varphi(\eta,\xi)}{\partial\eta}\right]^2\right.$$
$$\left.\times\left[\frac{\partial\eta(s,\xi)}{\partial s}\right]^2\right\}-\frac{\partial F(\eta,\xi)}{\partial\eta}\frac{\partial^2\eta(s,\xi)}{\partial s^2}$$
$$-\frac{\partial^2F(\eta,\xi)}{\partial\eta^2}\left[\frac{\partial\eta(s,\xi)}{\partial s}\right]^2 = 0, \tag{6a}$$

$$F(\eta,\xi)\left\{\frac{\partial\varphi(\eta,\xi)}{\partial\eta}\frac{\partial^2\eta(s,\xi)}{\partial s^2}+\frac{\partial^2\varphi(\eta,\xi)}{\partial\eta^2}\left[\frac{\partial\eta(s,\xi)}{\partial s}\right]^2\right\}$$
$$+2\frac{\partial F(\eta,\xi)}{\partial\eta}\left\{\frac{\partial\eta(s,\xi)}{\partial\xi}+\frac{\partial\varphi(\eta,\xi)}{\partial\eta}\left[\frac{\partial\eta(s,\xi)}{\partial s}\right]^2\right\}$$
$$+2\frac{\partial F(\eta,\xi)}{\partial\xi}= 0. \tag{6b}$$

Note that system (6) is exact and completely equivalent to (4). Next we need to make some assumptions and single out a certain class of solutions of (6).

We restrict ourselves to self-similar solutions, which, propagating (change in ξ), do not change the functional nature of the intensity distribution $F^2(\eta,\xi)$ in the transverse coordinate. Note that this requirement does not limit the analysis to soliton-like solutions only [13, 14], because the replacement $s \to \eta(s,\xi)$ admits ξ-dependent changes in the scale s. Hence, $F(\eta,\xi) \to F(\eta)$ and $\partial F(\eta,\xi)/\partial\xi \to \partial F(\eta)/\partial\xi \equiv 0$, given the fact that (6) can be rewritten in the form

$$F(\eta)\left\{2\frac{\partial\varphi(\eta,\xi)}{\partial\eta}\frac{\partial\eta(s,\xi)}{\partial\xi}+2\frac{\partial\varphi(\eta,\xi)}{\partial\xi}+\left[\frac{\partial\varphi(\eta,\xi)}{\partial\eta}\right]^2\right.$$
$$\left.\times\left[\frac{\partial\eta(s,\xi)}{\partial s}\right]^2\right\}-\frac{\mathrm{d}F(\eta)}{\mathrm{d}\eta}\frac{\partial^2\eta(s,\xi)}{\partial s^2}-\frac{\mathrm{d}^2F(\eta)}{\mathrm{d}\eta^2}\left[\frac{\partial\eta(s,\xi)}{\partial s}\right]^2 = 0, \tag{7a}$$

$$F(\eta)\left\{\frac{\partial\varphi(\eta,\xi)}{\partial\eta}\frac{\partial^2\eta(s,\xi)}{\partial s^2}+\frac{\partial^2\varphi(\eta,\xi)}{\partial\eta^2}\left[\frac{\partial\eta(s,\xi)}{\partial s}\right]^2\right\}$$
$$+2\frac{\mathrm{d}F(\eta)}{\mathrm{d}\eta}\left\{\frac{\partial\eta(s,\xi)}{\partial\xi}+\frac{\partial\varphi(\eta,\xi)}{\partial\eta}\left[\frac{\partial\eta(s,\xi)}{\partial s}\right]^2\right\} = 0. \tag{7b}$$

We now note that soliton-like solutions should have a plane wavefront, which, however, can be inclined relative to the normal to the transverse coordinate η. This inclination may depend on ξ, changing with propagation. Consequently,

$$\varphi(\eta,\xi) = \alpha(\xi) + \beta(\xi)\eta(s,\xi) \tag{8}$$

and

$$\frac{\partial\varphi(\eta,\xi)}{\partial\eta} = \beta(\xi),\ \frac{\partial\varphi(\eta,\xi)}{\partial\xi} = \frac{\mathrm{d}\alpha(\xi)}{\mathrm{d}\xi}+\frac{\mathrm{d}\beta(\xi)}{\mathrm{d}\xi}\eta(s,\xi),$$

where $\alpha(\xi)$ and $\beta(\xi)$ are some unknown functions. By substituting the corresponding expressions into (7), we find

$$F(\eta)\left\{2\beta(\xi)\frac{\partial\eta(s,\xi)}{\partial\xi}+2\left[\frac{\mathrm{d}\alpha(\xi)}{\mathrm{d}\xi}+\frac{\mathrm{d}\beta(\xi)}{\mathrm{d}\xi}\eta(s,\xi)\right]+\beta^2(\xi)\right.$$
$$\left.\times\left[\frac{\partial\eta(s,\xi)}{\partial s}\right]^2\right\}-\frac{\mathrm{d}F(\eta)}{\mathrm{d}\eta}\frac{\partial^2\eta(s,\xi)}{\partial s^2}-\frac{\mathrm{d}^2F(\eta)}{\mathrm{d}\eta^2}\left[\frac{\partial\eta(s,\xi)}{\partial s}\right]^2 = 0, \tag{9a}$$

$$F(\eta)\beta(\xi)\frac{\partial^2\eta(s,\xi)}{\partial s^2}+2\frac{\mathrm{d}F(\eta)}{d\eta}$$
$$\times\left\{\frac{\partial\eta(s,\xi)}{\partial\xi}+\beta(\xi)\left[\frac{\partial\eta(s,\xi)}{\partial s}\right]^2\right\}= 0. \tag{9b}$$

We now require that for the desired class of solutions $\partial^2\eta(s,\xi)/\partial s^2 = 0$, whence

$$\eta(s,\xi) = \mu(\xi) + \nu(\xi)s, \tag{10}$$

where $\mu(\xi)$ and $\nu(\xi)$ are some unknown functions. Condition (10) reflects the self-similar nature of the desired solutions, allowing for them a ξ-dependent transverse shift and an s uniform (for a given value of ξ) scaling factor. Note that restriction (10) is very efficient because it removes the term with the first derivative $\mathrm{d}F(\eta)/\mathrm{d}\eta$ in (9a) and the term proportional to $F(\eta)$ in (9b). The family of motion trajectories of points of equal intensity $F^2(\eta_0)$ in this case is given by the expression $s = [\eta_0 - \mu(\xi)]/\nu(\xi)$, where η_0 is a constant. Substituting (10) into (9b), we find that

$$\frac{\mathrm{d}F(\eta)}{\mathrm{d}\eta}\left[\frac{\mathrm{d}\mu(\xi)}{\mathrm{d}\xi}+\frac{\mathrm{d}\nu(\xi)}{\mathrm{d}\xi}s+\beta(\xi)\nu^2(\xi)\right]= 0, \tag{11}$$

whence

$$\nu(\xi) = \nu_0 = \text{const}, \tag{12a}$$

$$\frac{\mathrm{d}\mu(\xi)}{\mathrm{d}\xi}+\nu_0^2\beta(\xi) = 0. \tag{12b}$$

It is easy to see that the motion trajectories of points of equal intensity for this class of solutions are similar and simply shifted by s relative to each other. It follows from (9a) that

$$\frac{\mathrm{d}^2F(\eta)}{\mathrm{d}\eta^2} - F(\eta)\left\{-\beta^2(\xi) + \frac{2}{v_0^2} \times\left[\frac{\mathrm{d}\alpha(\xi)}{\mathrm{d}\xi} + \mu(\xi)\frac{\mathrm{d}\beta(\xi)}{\mathrm{d}\xi}\right] + \frac{2}{v_0}\frac{\mathrm{d}\beta(\xi)}{\mathrm{d}\xi}s\right\} = 0, \tag{13}$$

and we can draw an important conclusion. Because according to the initial assumption that the function $F(\eta)$ cannot explicitly depend on the longitudinal coordinate ξ (the required solution is a self-similar solution), the coefficient in curly brackets at $F(\eta)$ in this equation must be reduced to a certain function of the transverse coordinate η, i.e., the condition

$$-\beta^2(\xi) + \frac{2}{v_0^2}\left[\frac{\mathrm{d}\alpha(\xi)}{\mathrm{d}\xi} + \mu(\xi)\frac{\mathrm{d}\beta(\xi)}{\mathrm{d}\xi}\right] + \frac{2}{v_0}\frac{\mathrm{d}\beta(\xi)}{\mathrm{d}\xi}s = H(\eta)$$

must be met, where $H(\eta)$ is some unknown function. Taking into account condition (10) and the linearity of the left-hand side of the relationship derived with respect to s, this requirement can be satisfied only in two cases: $H(\eta) \propto (\eta + \mathrm{const})$ and (13) is reduced to the equation for the Airy function [3], or $H(\eta) = \mathrm{const}$ and (13) transforms into the equation of undamped harmonic oscillations. We emphasise again that these two possibilities exhaust the class of self-similar solutions in question.

We now require, for example, that equation (13) transforms into the equation for the Airy function [3]. To this end, the condition

$$-\beta^2(\xi) + \frac{2}{v_0^2}\left[\frac{\mathrm{d}\alpha(\xi)}{\mathrm{d}\xi} + \mu(\xi)\frac{\mathrm{d}\beta(\xi)}{\mathrm{d}\xi}\right] + \frac{2}{v_0}\frac{\mathrm{d}\beta(\xi)}{\mathrm{d}\xi}s = \eta = \mu(\xi) + v_0 s$$

should be met, whence

$$\alpha(\xi) = \alpha_0 + \frac{1}{3}\left(\beta_0 + \frac{1}{2}v_0^2\xi\right)^3, \tag{14a}$$

$$\beta(\xi) = \beta_0 + \frac{1}{2}v_0^2\xi, \tag{14b}$$

$$\mu(\xi) = \mu_0 - \left(\beta_0 + \frac{1}{2}v_0^2\xi\right)^2, \tag{14c}$$

where α_0, β_0 and μ_0 are the integration constant. As a result, the found self-similar solution is described by the expressions

$$\eta(s,\xi) = \mu_0 + v_0 s - \left(\beta_0 + \frac{1}{2}v_0^2\xi\right)^2, \tag{15a}$$

$$\varphi(\eta,\xi) = \alpha_0 + \left(\beta_0 + \frac{1}{2}v_0^2\xi\right)\eta(s,\xi) + \frac{1}{3}\left(\beta_0 + \frac{1}{2}v_0^2\xi\right)^3, \tag{15b}$$

$$\phi(s,\xi) = \phi_0 \mathrm{Ai}(\eta)\exp[\mathrm{i}\varphi(\eta,\xi)]. \tag{15c}$$

Equations (15a) and (15b) can be significantly simplified. Thus, the requirement that in the plane $\xi = 0$ the wavefront coincides with the s axis, determines the value of $\beta_0 = 0$. The choice of the point location $s = 0$ suggests that $\mu_0 = 0$. And finally, the value of the constant α_0, renormalising the phase-shift velocity in ξ, also can be zeroed by an appropriate choice of the wave vector projection on the axis. Therefore, finally,

$$\eta(s,\xi) = v_0 s - \left(\frac{1}{2}v_0^2\xi\right)^2, \tag{16a}$$

$$\varphi(\eta,\xi) = \frac{1}{2}v_0^2\xi\left[\eta(s,\xi) + \frac{1}{3}\left(\frac{1}{2}v_0^2\xi\right)^2\right], \tag{16b}$$

which, in principle, coincides with the solution of Berry [4]. However, it follows from (16a) that, as in (3), the ability to change the sign $v_0 \to -v_0$ determines the existence of two mirror-symmetric (with respect to the ξ axis) families of motion trajectories of points of equal intensity specified by the expression $s = v_0^{-1}[\eta_0 + (v_0^2\xi/2)^2]$.

Other unconsidered self-similar solutions should correspond to the situation when (13) is reduced to the undamped oscillation equation. To this end, it is necessary to fulfil the condition

$$-\beta^2(\xi) + \frac{2}{v_0^2}\left[\frac{\mathrm{d}\alpha(\xi)}{\mathrm{d}\xi} + \mu(\xi)\frac{\mathrm{d}\beta(\xi)}{\mathrm{d}\xi}\right] + \frac{2}{v_0}\frac{\mathrm{d}\beta(\xi)}{\mathrm{d}\xi}s = -\omega_0^2 = \mathrm{const},$$

whence

$$\alpha(\xi) = \alpha_0 + \frac{v_0^2}{2}(\beta_0^2 - \omega_0^2)\xi, \tag{17a}$$

$$\beta(\xi) = \beta_0, \tag{17b}$$

$$\mu(\xi) = \mu_0 - \beta_0 v_0^2\xi. \tag{17c}$$

As a result, one more solution found is described by the expressions

$$\eta(s,\xi) = \mu_0 + v_0 s - \beta_0 v_0^2\xi, \tag{18a}$$

$$\varphi(\eta,\xi) = \alpha_0 + \beta_0\eta(s,\xi) + \frac{v_0^2}{2}(\beta_0^2 - \omega_0^2)\xi = \alpha_0 + \beta_0\mu_0 + \beta_0 v_0 s - \frac{v_0^2}{2}(\beta_0^2 + \omega_0^2)\xi, \tag{18b}$$

$$\phi(s,\xi) = \phi_0\cos(\omega_0\eta)\exp[\mathrm{i}\varphi(\eta,\xi)]. \tag{18c}$$

Expressions (18a) and (18b) can be also simplified by selecting the values of the integration constants. It is easy to see (18b) that now the wavefront retains its orientation in space and the case, when it coincides with the s axis, corresponds to two possibilities. In the first one (the most interesting) $\beta_0 = 0$, and at an appropriate choice of the point position $s = 0$ and the initial phase ($\alpha_0 = 0$ and $\mu_0 = 0$)

$$\eta(s,\xi) = v_0 s, \tag{19a}$$

$$\varphi(\eta,\xi) = -\frac{v_0^2}{2}\omega_0^2\xi, \tag{19b}$$

which corresponds to the interference pattern of two plane waves propagating symmetrically at some angle to the ξ axis.

In the second case we will have to assume that $v_0 = 0$, and the solution

$$\eta(s,\xi) = \mu_0, \tag{20a}$$

$$\varphi(\eta,\xi) = \alpha_0 + \beta_0\mu_0 \tag{20b}$$

corresponds to propagation of a plane wave with a complex amplitude $\tilde{\phi}_0 = \phi_0\sin(\mu_0\omega_0)\exp[\mathrm{i}(\alpha_0 + \beta_0\mu_0)] = \mathrm{const}$ strictly along the ξ axis.

3. Photorefractive crystal with diffusion nonlinearity

Let us turn to the case of slit beam propagation in the PRC. After the substitution of the same form of the self-similar solution $\phi(s,\xi) = F(\eta)\exp[i\varphi(\eta,\xi)]$, the term responsible for the diffusion nonlinearity in the wave equation (1) takes the form

$$\gamma\frac{\partial|\phi(s,\xi)|^2}{\partial s}\frac{\phi(s,\xi)}{|\phi(s,\xi)|^2} = 2\gamma\frac{dF(\eta)}{d\eta}\frac{\partial\eta(s,\xi)}{\partial s}\exp[i\varphi(\eta,\xi)]. \quad (21)$$

Therefore, the entire chain of preceding arguments is sound up to equation (13), and only in (21) there is an additional term proportional to $dF(\eta)/d\eta$ and responsible for the exponential decay/increase (depending on the sign of γ) in the transverse coordinate

$$\frac{d^2F(\eta)}{d\eta^2} + 4\frac{\gamma}{v_0}\frac{dF(\eta)}{d\eta} - F(\eta)\Big\{-\beta^2(\xi) + \frac{2}{v_0^2}\Big[\frac{d\alpha(\xi)}{d\xi} + \mu(\xi)\frac{d\beta(\xi)}{d\xi}\Big] + \frac{2}{v_0}\frac{d\beta(\xi)}{d\xi}s\Big\} = 0. \quad (22)$$

Because solution (15) found previously in the form of an Airy beam is localised in η and, depending on the sign of v_0 exponentially decreases either at $\eta \to +\infty$ or at $\eta \to -\infty$, we can expect that the exponentially increasing/decaying tail (at a reasonable rise/fall rate), which appears at the expense of an additional member, will not prevent the existence of a corresponding localised solution.

One can see that the account for a purely diffusion nonlinearity due to saturation retains the linearity of the problem for the considered class of self-similar solutions. Now, however, due to the appearance of the of term proportional to $dF(\eta)/d\eta$ in (22), the mirror symmetry of the problem of propagation along the transverse coordinate turns violated, as the replacement $s \leftrightarrow -s$ modifies the wave equation.

Equation (22) can easily be found by using the standard replacement of variables $F(\eta) \to G(\eta)\exp(\sigma\eta)$, where $\sigma = -2\gamma/v_0$, which determines the final form of the equation to be solved

$$\frac{d^2G(\eta)}{d\eta^2} - G(\eta)\Big\{-\beta^2(\xi) + \frac{2}{v_0^2}\Big[\frac{d\alpha(\xi)}{d\xi} + \mu(\xi)\frac{d\beta(\xi)}{d\xi}\Big] + \frac{2}{v_0}\frac{d\beta(\xi)}{d\xi}s + \Big(\frac{2\gamma}{v_0}\Big)^2\Big\} = 0. \quad (23)$$

As before, after the condition

$$-\beta^2(\xi) + \frac{2}{v_0^2}\Big[\frac{d\alpha(\xi)}{d\xi} + \mu(\xi)\frac{d\beta(\xi)}{d\xi}\Big] + \frac{2}{v_0}\frac{d\beta(\xi)}{d\xi}s + \Big(\frac{2\gamma}{v_0}\Big)^2 = \eta = \mu(\xi) + v_0 s$$

is met, expression (23) is again reduced to the equation for the Airy function, which implies that

$$\alpha(\xi) = \alpha_0 - 2\gamma^2\xi + \frac{1}{3}\Big(\beta_0 + \frac{1}{2}v_0^2\xi\Big)^3, \quad (24a)$$

$$\beta(\xi) = \beta_0 + \frac{1}{2}v_0^2\xi, \quad (24b)$$

$$\mu(\xi) = \mu_0 - \Big(\beta_0 + \frac{1}{2}v_0^2\xi\Big)^2, \quad (24c)$$

and the self-similar solution found is described by the expressions

$$\eta(s,\xi) = \mu_0 + v_0 s - \Big(\beta_0 + \frac{1}{2}v_0^2\xi\Big)^2, \quad (25a)$$

$$\varphi(\eta,\xi) = \alpha_0 - 2\gamma^2\xi + \Big(\beta_0 + \frac{1}{2}v_0^2\xi\Big)\eta(s,\xi) + \frac{1}{3}\Big(\beta_0 + \frac{1}{2}v_0^2\xi\Big)^3, \quad (25b)$$

$$\phi(s,\xi) = \phi_0 \mathrm{Ai}(\eta)\exp\Big(-\frac{2\gamma}{v_0}\eta\Big)\exp[i\varphi(\eta,\xi)]. \quad (25c)$$

Equations (25a) and (25b) can also be simplified by selecting the values of the integration constants. The case, where for $\xi = 0$ the wavefront coincides with the s axis, corresponds to the two possibilities. In the first one, $\beta_0 = 0$, and at an appropriate choice of the point position $s = 0$ and the initial phase ($\alpha_0 = 0$ and $\mu_0 = 0$)

$$\eta(s,\xi) = v_0 s - \Big(\frac{1}{2}v_0^2\xi\Big)^2, \quad (26a)$$

$$\varphi(\eta,\xi) = -2\gamma^2\xi + \frac{1}{2}v_0^2\xi\eta(s,\xi) + \frac{1}{3}\Big(\frac{1}{2}v_0^2\xi\Big)^3, \quad (26b)$$

which in view of the arbitrariness of the amplitude (due to the linearity of the problem) corresponds to the solution in the form of a modified Airy beam, given in [1].

In contrast to the case of propagation in a linear medium considered in the previous section, the mirror symmetry is now absent in the problem because requirement of the spatial localisation of the desired self-similar solutions over the transverse coordinate uniquely determines the sign of v_0 (through the sign of $\sigma = -2\gamma/v_0$). The same requirement of spatial localisation of the field leads to a ban on all other above-considered possibilities at which either $v_0 = 0$, or equation (23) is reduced to the equation of oscillations, because all such solutions will be infinitely growing in the field amplitude either at $s \to +\infty$, or at $s \to -\infty$.

4. Conclusions

We have shown that a slit Airy beam is localised over the transverse coordinate by a self-similar solution to the linear problem of propagation of a radiation beam with a plane wavefront. This class of solutions may also include the known non-localised solutions: a plane wave propagating along the longitudinal coordinate and interference structures formed by a pair of plane waves symmetrically propagating at an angle to the longitudinal axis. All other interference structures that can be constructed based on the linear superposition principle will not apply to this class of solutions due to different projections of the wave vector in the propagation direction. The deviation of the motion trajectories of points of equal intensity from the longitudinal axis is determined for the Airy beam in this case up to a sign, which reflects the mirror symmetry of the wave equation over the transverse coordinate.

The appearance of the diffusion nonlinearity retains the linearity of the propagation problem for the considered class of self-similar solutions. However, the presence of nonlinearity of this type violates the mirror symmetry of the propagation problem over the transverse coordinate (replacement $s \leftrightarrow -s$ changes the wave equation). At the same time, taking into account the requirements of the spatial localisation, of the two possible localised self-similar solutions with a plane wavefront the diffusion nonlinearity singles out only one and deforms (modulates by an additional exponential factor) the profile of the intensity distribution in the transverse coordinate, corresponding to the Airy beam. Solutions of this class in the form of plane waves and interference structures formed by plane waves symmetrically propagating at angles to the longitudinal axis are thus prohibited.

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