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Synchronisation in the phase model of three coupled lasers

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Abstract. **The problem of synchronisation of three lasers is considered within the phase approximation. The domains of complete synchronisation, partial synchronisation, two-frequency resonant regimes, and three-frequency quasi-periodicity have been found using bifurcation analysis, the method of Lyapunov exponent maps, and construction of phase portraits. The differences in the properties of a three-element chain and ring, as well as the influence of the coupling type, are discussed.**

Keywords: *laser synchronisation, quasi-periodic oscillations, bifurcation.*

1. Introduction

The problems related to synchronisation of sets of few lasers on the one hand and large laser arrays on the other hand are popular in laser physics [1–11]. Synchronisation makes it possible to increase significantly the intensity and quality of laser radiation. There are several ways to obtain synchronous lasing: injection of the external field of a single-frequency laser synchronising a laser ensemble, use of a spatial filter or Talbot cavity, synchronisation by Fourier coupling, etc. [1–11]. For example, optical coupling between two waveguide lasers was implemented in [2, 3] using a spatial filter. To this end, a diffraction grating was introduced into the lens system as a spatial filter. Optical coupling was provided by the radiation diffracted from this grating. The problem of synchronising even three lasers [2, 3, 6] (as well as the classical problem of synchronising three oscillators [12–15]) is complex. The following regimes may occur in this case: complete laser synchronisation, partial synchronisation of laser pairs (which corresponds to two-frequency quasi-periodicity), and more complex regimes of three-frequency quasi-periodicity. Note that an individual laser is in fact a self-oscillating system, where the presence of negative friction is provided by the active medium. Therefore, the problems of laser dynamics are fundamental and can be solved using the methods of nonlinear dynamics and bifurcation theory (see, for example, [8, 9, 11]). Recently, some new approaches and methods for studying these systems have been developed in nonlinear dynamics [13–19]. They allow one to analyse the structure of the parameter space and correctly reveal regimes of different types and the conditions for their occurrence (disappearance). We will apply this approach to the problems considered in [2, 3, 6].

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2. Adler–Khokhlov equation and its generalisation

The essence of the synchronisation phenomenon in systems of any physical nature is in mutual phase tuning of interacting subsystems [12, 20, 21]. Therefore, the factor of fundamental importance for describing synchronisation is the formulation of the corresponding phase equation

$$\dot{\theta} = \Delta - \mu \sin\theta. \qquad (1)$$

Here, θ is the relative phase of the subsystems; μ is the coupling coefficient; and Δ is the frequency detuning of the subsystems. Equation (1) describes also the problems of forced synchronisation (in this case, the variables and parameters are determined with respect to an external signal). Equation (1) was derived for the first time by Adler in 1946 [22]. He considered the radio-engineering problem of a triode generator excited by an external signal. Nevertheless, Adler understood the universal character of Eqn (1) and proposed also a mechanical model in the form of a pendulum in a highly viscous medium.

A similar equation was derived by Khokhlov in 1954 [23], who also formulated an important method of nonlinear theory, which consists in passing from the initial system to truncated equations for slow amplitudes with subsequent transition to phase equations*. Within this approach, a number of fundamental problems have been solved and some practical problems have been considered (in particular, synchronisation of reflex klystrons and molecular generators). Afterwards, the Adler–Khokhlov equation and its generalisations were also obtained for the problems of synchronising lasers of different types, coupled in different ways (semiconductor lasers, CO_2 lasers, etc.) [25, 5, 2, 3, 7–11].

A conventional generalisation of the Adler–Khokhlov equation is the transition to a system of several interacting elements. The generalised Adler–Khokhlov equation for three optically coupled lasers, combined into a chain, was reported in [6]. The initial equations have the form

* Akhmanov's review [24] (1986) even includes a special section: 'Method of step-by-step simplification of truncated equations (Khokhlov method)'.

$$E_i = [G\delta_i - \Gamma_i - \mathrm{i}(\omega_0 - \omega_i)]E_i + \sum_j m_{ij}E_j,$$
$$\dot{\delta}_i = -\gamma(\delta_i - \delta_{0i}) - 4G\delta_i|E_i|^2. \quad (2a)$$

Here, E_i is the complex field generated by the ith laser in the ensemble; δ_i is the population difference in this laser; δ_{0i} is the corresponding unsaturated inverse population; G is the coupling constant; γ and Γ_i are, respectively, the relaxation rates of the population and field in the cavity of the ith laser; ω_i is the fundamental frequency of the ith-laser cavity; ω_0 is the lasing-transition frequency; and m_{ij} is the coupling-coefficient matrix.

The equations for the real amplitudes A_i and phases ψ_i can be written as [6]

$$\dot{A}_i = -\Gamma_i A_i + G\delta_i A_i + \sum_j m_{ij}A_j\cos(\psi_i - \psi_j),$$
$$\dot{\psi}_i = (\omega_i - \omega_0) + \sum_j m_{ij}\frac{A_i}{A_j}\sin(\psi_i - \psi_j). \quad (2b)$$

The following phase equations can conventionally be obtained for three lasers connected into a chain [6]:

$$\dot{\theta} = \Delta_{21} - \mu_1\sin\theta + \mu_2\sin\phi,$$
$$\dot{\phi} = -\Delta_{23} - \mu_1\sin\phi + \mu_2\sin\theta. \quad (3)$$

Here, $\theta = \psi_1 - \psi_2$ is the relative phase of the first and second lasers with the coupling coefficient μ_1 and $\phi = \psi_2 - \psi_3$ is the relative phase of the second and third lasers with the coupling coefficient μ_2. The parameters Δ_{21} and Δ_{23} are the frequency detunings of the pairs of first and second and second and third lasers, respectively. The derivation of system (3) was described in detail in [6].

As was shown in [6], system (3) can be considered analytically to find the condition for complete synchronisation of all three lasers. In this case, all lasers are locked so that their phases ψ_i are constants. Then the derivatives of the relative phases are zero and Eqns (3) describe the equilibrium state:

$$\Delta_{21} - \mu_1\sin\theta + \mu_2\sin\phi = 0,$$
$$-\Delta_{23} - \mu_1\sin\phi + \mu_2\sin\theta = 0. \quad (4)$$

Equation (4) can be solved with respect to the sines of the phases:

$$\sin\theta = \frac{\mu_1\Delta_{21} - \mu_2\Delta_{23}}{\mu_1^2 - \mu_2^2},$$
$$\sin\phi = \frac{\mu_2\Delta_{21} - \mu_1\Delta_{23}}{\mu_1^2 - \mu_2^2}. \quad (5)$$

Equations (5) can be solved analytically; the solutions are presented in the form of pairs θ_1 and θ_2 (for the first equation) and ϕ_1 and ϕ_2 (for the second equation):

$$\theta_1 = \arcsin\left(\frac{\mu_1\Delta_{21} - \mu_2\Delta_{23}}{\mu_1^2 - \mu_2^2}\right),\ \theta_2 = \pi - \arcsin\left(\frac{\mu_1\Delta_{21} - \mu_2\Delta_{23}}{\mu_1^2 - \mu_2^2}\right),$$
$$\phi_1 = \arcsin\left(\frac{\mu_2\Delta_{21} - \mu_1\Delta_{23}}{\mu_1^2 - \mu_2^2}\right),\ \phi_2 = \pi - \arcsin\left(\frac{\mu_2\Delta_{21} - \mu_1\Delta_{23}}{\mu_1^2 - \mu_2^2}\right). \quad (6)$$

Thus, the equilibrium states in the phase plane correspond to the vertices of a rectangle with the coordinates (θ_1, ϕ_1), (θ_1, ϕ_2), (θ_2, ϕ_1), and (θ_2, ϕ_2) (Fig. 2a).

Let us now vary the frequency detunings so as to increase the magnitude of the right-hand side of the first equation in system (5). When this value becomes unity, the solutions in the pair merge, and all four equilibrium states disappear simultaneously. The situation for the second equation in (5) is similar. Thus, the condition for disappearance of the solutions is the equality of the right-hand sides in (5) to ± 1:

$$\mu_1\Delta_{21} - \mu_2\Delta_{23} = \pm(\mu_1^2 - \mu_2^2),$$
$$\mu_2\Delta_{21} - \mu_1\Delta_{23} = \pm(\mu_1^2 - \mu_2^2). \quad (7)$$

Relations (7) set two pairs of mutually parallel lines in the frequency-detuning plane; the intersections of these lines form a parallelogram determining the complete-synchronisation domain. This result was obtained in [6] and discussed in [2, 3]. However, we should note that a similar result was described within the oscillation theory by Landa in 1980 [12]*. Therefore, the complete-synchronisation domain can be referred to as Landa's parallelogram (Fig. 1a).

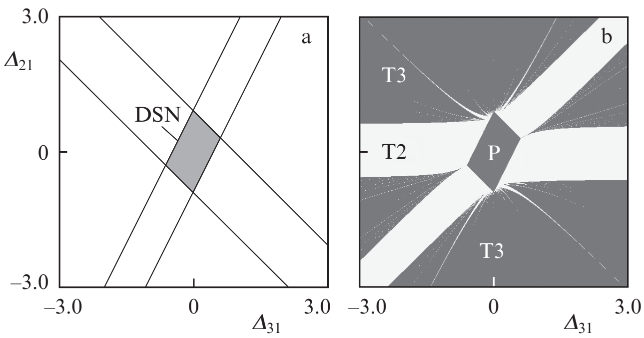

Figure 1. a) Lines (7) and the complete-synchronisation domain of system (3) (DSN is one of four segments of the degenerate saddle–node bifurcation line). (b) Corresponding Lyapunov map (P is the complete-synchronisation domain and T2 and T3 are the domains of two- and three-frequency quasi-periodicity). The frequency detuning of the third and first lasers $\Delta_{31} = \Delta_{21} - \Delta_{23}$ is plotted on the abscissa axis; $\mu_1 = 0.6$, $\mu_2 = 0.3$.

However, it was found that even a slight complication in the transition from Eqn (1) to Eqns (3) highly enriches the pattern of possible oscillation regimes. An analysis of these regimes requires a numerical investigation applying the methods of the theory of dynamic systems and nonlinear dynamics.

First of all, we present several phase portraits of system (3) in the plane of relative phases θ and ϕ to illustrate the main behavioural types. Due to the phase property of 2π periodicity, the phase dynamics can be considered in the ranges $0 < \theta < 2\pi$ and $0 < \phi < 2\pi$. If a phase trajectory leaves this square, for example, through the right side, it appears at the corresponding point on the left side. The same holds true for the upper and lower boundaries.

Figure 2 shows the phase portraits corresponding to the main types of regimes provided by system (3). The system in Fig. 2a has four equilibrium states. This situation is consistent

* This result was published in an English monograph in 1996 [26].

with the analytical consideration. One can see a stable site (1), two saddles (2, 3), and an unstable site (4). The relative oscillator phases tend to the stable equilibrium point in the course of time. The rates of change in the phases of each oscillator become constant and equal. This is the regime of complete synchronisation of all three lasers.

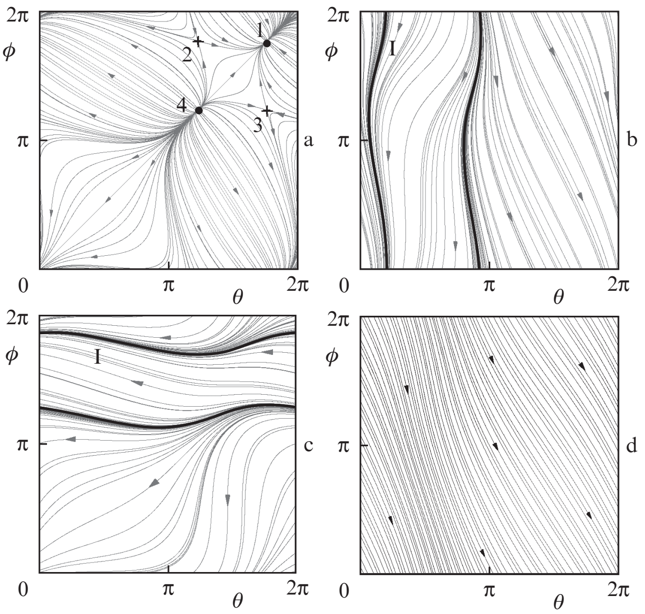

Figure 2. Phase portraits of system (3): (a) regime of complete synchronisation of all oscillators ($\Delta_{21} = -0.25$, $\mu_1 = 0.75$, and $\mu_2 = 0.375$), (b) regime of partial locking of the first and second oscillators ($\Delta_{21} = 0.125$, $\mu_1 = 0.25$, and $\mu_2 = 0.125$), (c) regime of partial locking of the second and third oscillators ($\Delta_{21} = -0.375$, $\mu_1 = 0.25$, $\mu_2 = 0.125$), and (d) three-frequency quasi-periodic regime ($\Delta_1 = -1$ and $\mu = 0.25$). The detuning is $\Delta_{31} = -0.5$.

While approaching the boundaries of the parallelogram in Fig. 1a, the equilibrium points become closer pairwise (for example, 1 and 3, as well as 2 and 4) and simultaneously merge and disappear. It is a kind of degenerate saddle–node bifurcation, when the stable equilibrium point and a saddle, as well as the unstable equilibrium point and a saddle, simultaneously merge.

Let us now consider Fig. 2b. It reveals two new objects: attractive and repulsive invariant curves. The stable curve is indicated by the letter I. Here, the phase θ in the regime corresponding to the stable invariant curve is not steady-state; it oscillates around some averaged value. Since $\theta = \psi_1 - \psi_2$ by definition, this oscillation indicates that the phases of the first and second oscillators are close and their difference does not increase. This regime can be characterised as locking of the first and second lasers. However, this locking is partial rather than complete: the relative phase changes with time. Note that the relative phase θ in Fig. 2b oscillates near zero; therefore, lasers are almost in-phase synchronised.

Figure 2c shows another version of arrangement of the attractive invariant curve. In this case, the phase $\phi = \psi_2 - \psi_3$ oscillates around a stationary value, whereas the phase θ changes in the entire possible range. Thus, we have the regime of partial locking of the second and third lasers in this case.

Figure 2d shows one more version of the oscillation regime of the system. Here, one can see a 'flow' of phase trajectories, which closely fill the phase square. Each phase changes in the entire possible range: $0 < \theta < 2\pi$, $0 < \phi < 2\pi$.

How do the above-described regimes manifest themselves in the dynamics of the initial system? In this case, the number of fundamental frequencies is the sum of the number of fundamental frequencies for phase equations (3) and the fundamental optical frequency ω_0. Therefore, the state of stable equilibrium corresponds to the limiting cycle. The invariant curve corresponds to the regime of two-frequency quasi-periodicity and two-frequency invariant torus, whereas the trajectory flow in Fig. 2d corresponds to the regime of three-frequency quasi-periodicity and invariant torus of higher dimension.

Now it is of interest to find out how the regimes of these types are represented in the frequency-detuning plane. To solve this problem, we will use a numerical method of nonlinear dynamics: analysis of Lyapunov exponent maps [13, 14, 18, 19]. Lyapunov exponents characterise compression (expansion) of the phase volume of a dynamic system [27]. System (3) is of second order; therefore, it is characterised by two Lyapunov exponents, Λ_1 and Λ_2. Using the standard technique [27], we will calculate both Lyapunov exponents at each point in the plane of parameters Δ_{31} and Δ_{21}. Afterwards, we will colour this plane according to the exponent values to visualise the following regimes:

(i) $\Lambda_1 < 0$, $\Lambda_2 < 0$ (stable equilibrium state P);

(ii) $\Lambda_1 = 0$, $\Lambda_2 < 0$ (stable invariant curve, two-frequency quasi-periodic regime T2);

(iii) $\Lambda_1 = 0$, $\Lambda_2 = 0$ (phase-trajectory flow, three-frequency quasi-periodic regime T3).

A Lyapunov exponent map obtained in this way is shown in Fig. 1b. One can see a complete-synchronisation domain, which corresponds to analytical consideration. In addition, there are wide domains (bands) of two-frequency regimes, determined by two possible resonant conditions for the system:

$$\Delta_{21} = 0, \ \ \Delta_{21} = \Delta_{31}. \tag{8}$$

The fundamental frequencies of the first and second (second and third) lasers coincide in the first (second) case. Accordingly, there are two bands in Fig. 1b that correspond to the two-frequency regimes and lie in a finite range in the vicinity of values (8). These bands adjoin the boundaries of Landa's parallelogram. Thus, the exit beyond the parallelogram boundaries leads to destruction of the complete-synchronisation regime and transition to partial synchronisation of a particular laser pair.

In turn, the stable and unstable invariant curves merge at the boundaries of two-frequency regime bands. In terms of the initial system, one can say that a saddle–node bifurcation of the invariant tori arises. Afterwards, the tori disappear and the regime of three-frequency quasi-periodicity occurs. Note that the destruction of the regime of laser-pair synchronisation due to this bifurcation does not correspond to lines (7), as one would expect at the first glance; i.e., the occurrence of a pair of solutions for one relative phase is not related to the destruction of the synchronisation regime for the corresponding oscillator pair.

Another specific feature of the pattern in Fig. 1b is the presence of many thin tongues, corresponding to the two-frequency regime; their shape is most complex in the vicinity of the parallelogram vertices. These domains reflect the presence of resonant two-frequency regimes of different types.

Their invariant curves have a more complex configuration and various (including large) numbers of intersections with the phase-square sides in comparison with the curves of simplest types shown in Figs 2b and 2c (see [14, 18] for details).

Let us briefly consider the coupling type. Positive coupling coefficients in (1) and (3) correspond (in terms of the oscillation theory) to dissipative coupling [12, 20]. However, the coupling coefficient may be negative, a situation that is widespread in problems of laser physics [2, 3]. This case can be referred to as active coupling. It can easily be seen that the change in the coupling coefficient sign in Eqn (3) is equivalent to the phase replacement

$$\theta \to \theta + \pi, \ \phi \to \phi + \pi. \tag{9}$$

Thus, the negative coupling coefficient corresponds to the antiphase laser synchronisation. At the same time, the regime patterns for the laser chain in the parameter plane for dissipative and active couplings exactly coincide due to the replacement (9).

3. Phase dynamics of three globally coupled lasers

The phase equations for a more general case, where coupling of the third and first lasers is also possible (i.e., where all three lasers are optically coupled), were reported in [2, 3]. Under these conditions, the generalised Adler–Khokhlov equation has the form

$$\dot\theta = \Delta_{21} - \mu_1 \sin\theta + \mu_2 \sin\phi - m\sin(\theta+\phi),$$
$$\dot\phi = -\Delta_{23} - \mu_1 \sin\phi + \mu_2 \sin\theta - m\sin(\theta+\phi). \tag{10}$$

The coefficient m takes into account the possibility of coupling the third and first lasers. Note that the parameters μ_1 and μ_2 are expressed in terms of the parameters M and c (used in [3]) through the relations $\mu_1 = (c + 1/c)M$ and $\mu_2 = M/c$. In turn, the coefficients M and c can be calculated by optical methods for the initial system. The calculation details and the relationship with the parameters of the initial laser system were reported in [2, 3].

First, we will discuss the influence of the coupling type on the structure of the equations. Let us change the signs of all coefficients in (10). Now replacement (9) is insufficient: the sign of the constant m, which determines the coupling between the boundary lasers, must be changed additionally:

$$\theta \to \theta + \pi, \ \phi \to \phi + \pi, \ m \to -m. \tag{11}$$

Thus, the structure of the parameter plane in the case of global coupling of all lasers depends on the sign of the coupling parameter. At the same time, the case where all coefficients in system (10) are negative is equivalent to the situation where the coefficients μ_1 and μ_2 are positive, but the coefficient m is negative*.

Now we must determine the condition for complete synchronisation of all three lasers. System (10) may have equilibrium states, which will be sought for by equating the rates of change in the relative phases to zero:

$$\Delta_{21} - \mu_1 \sin\theta + \mu_2 \sin\phi - m\sin(\theta+\phi) = 0,$$
$$-\Delta_{23} - \mu_1 \sin\phi + \mu_2 \sin\theta - m\sin(\theta+\phi) = 0. \tag{12}$$

Unfortunately, the approach [6] is invalid in this case: the condition of equality of the sines to unity does not yield a correct solution. The conditions for bifurcation occurrence must be sought for more correctly. If $m \neq 0$, this condition is as follows: the Jacobian of the perturbation matrix of system (10) turns to zero [21, 28]; this leads to rather cumbersome analytical expressions. Therefore, we will present below the results of numerical determination of the complete-synchronisation domain**.

The pattern of bifurcation lines for system (10) in the plane of laser frequency detunings Δ_{23} and Δ_{21} is shown in Fig. 3. This pattern was obtained numerically using the standard MatCont package. Examples of the phase portraits at the points indicated by lowercase letters a–f in Fig. 3 are shown in Figs 4a–4f, respectively. The coefficients were chosen to be $\mu_1 = 0.351$, $\mu_2 = 0.108$, and $m = -0.139$; these values correspond to the parameters $M = 0.162$, $c = 1.5$, and $m = -0.139$ used in [2, 3]. According to the above remark, this case is equivalent to the situation where all coupling coefficients are negative (case of active coupling).

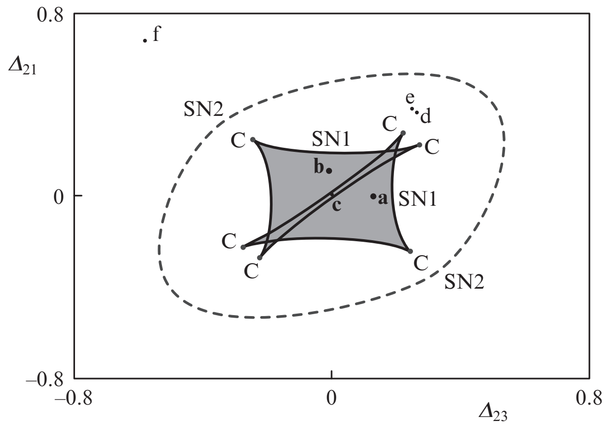

Figure 3. Bifurcation lines of the equilibrium states of system (10): SN1 and SN2 are the saddle–node bifurcation lines for the stable and unstable sites, respectively, and C are the cusp points. The region where at least one stable equilibrium state may exist is shown in gray.

The three phase portraits in Figs 4a–4c illustrate stable equilibrium states of the system, which correspond to the complete synchronisation of all three lasers. In the cases shown in Figs 4a and 4b, the system has four equilibrium points, one of which is a stable site, another is an unstable site, and the two others are saddles. There are six equilibrium points in Fig. 4c (two stable sites, three saddles, and one unstable site). Thus, in contrast to the laser chain, this system is characterised by bistability (i.e., coexistence of two stable equilibrium points).

* Specifically this situation was considered in [2, 3].

** This point should be emphasised, because the conditions $\theta = \pm\pi/2$ and $\phi = \pm\pi/2$ (i.e. $\sin\theta = \pm 1$ and $\sin\phi = \pm 1$) were interpreted in [3] as the conditions determining the synchronisation-domain boundary in the general case, which does not hold true. One can make sure of it by comparing the pattern of bifurcation lines obtained below from Fig. 2 in [3], which was also reported in [2].

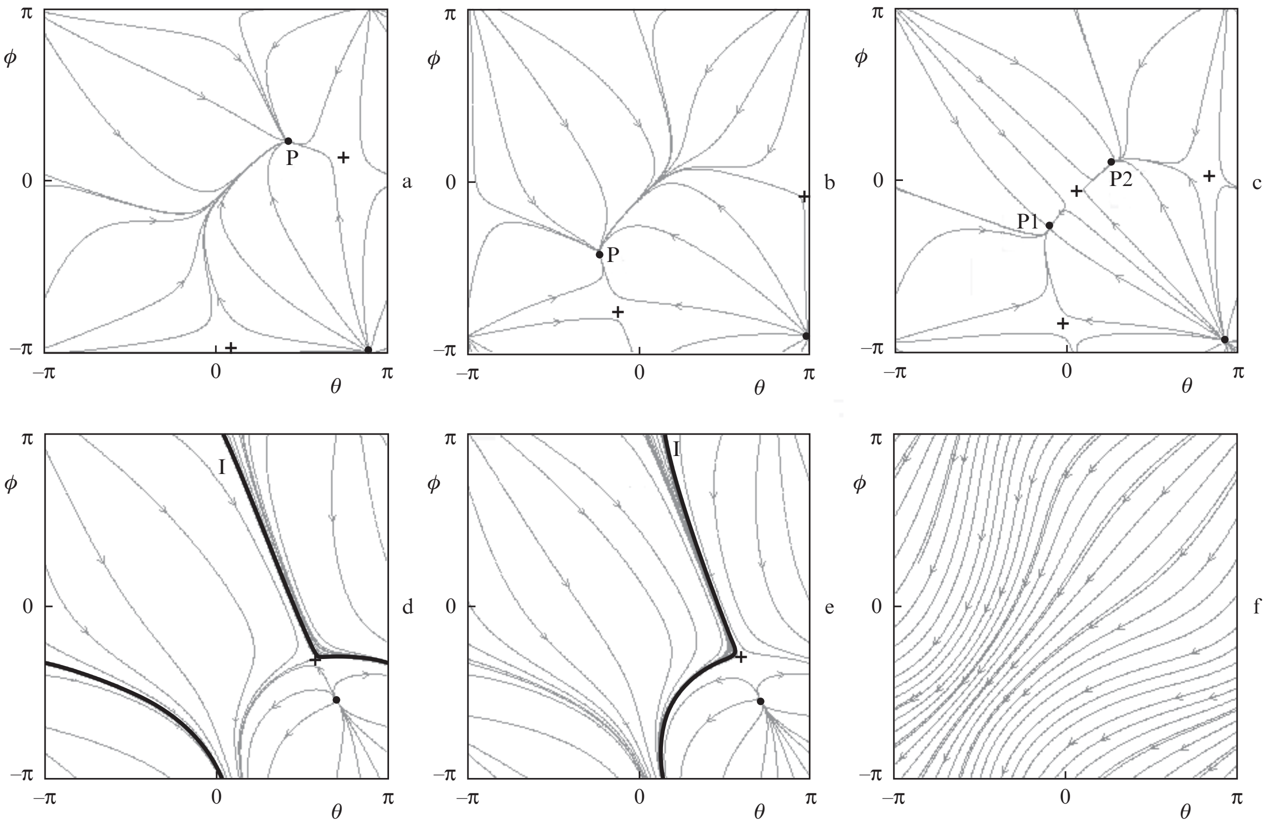

Figure 4. (a – f) Characteristic phase portraits constructed at the points indicated by the corresponding lowercase letters in Fig. 3. Points P are stable equilibrium points, saddles are indicated by crosses, and the stable invariant curves I are selected bold.

The lines of saddle – node bifurcations converge pairwise at the cusp points C [29], where characteristic spikes with a semicubical singularity exist. The complete-synchronisation domain in Fig. 3 lies within the dashed oval, which is a line of saddle – node bifurcation SN2, where the unstable site and saddle merge.

Thus, the radical difference between the system under consideration and a laser chain is that bifurcations of stable and unstable sites do not occur simultaneously if all lasers are intercoupled.

The phase portraits in Figs 4d and 4e demonstrate that two-frequency quasi-periodicity regimes may also exist. The phase trajectories in Fig. 4f fill closely the entire phase square, which corresponds to three-frequency quasi-periodicity.

To reveal the general pattern of the regimes for system (10), we will construct a Lyapunov exponent map (Fig. 5). One can see the complete-synchronisation domain, which corresponds to our bifurcation analysis. However, both overlapping sheets in Fig. 3 are projected onto the same parameter plane; therefore, they are visualised in Fig. 5 as a single domain. Figure 5 shows the entire system of two-frequency quasi-periodicity domains having a complex organisation. The three widest bands corresponding to these regimes lie near the lines describing the resonant conditions

$$\Delta_{21} = 0, \ \ \Delta_{23} = 0, \ \ \Delta_{21} = \Delta_{23}. \qquad (13)$$

Condition (13) corresponds to the three fundamental resonances in the system. The fundamental frequencies of the first and second, second and third, and first and third lasers coincide in the first, second, and third cases, respectively. For example, Fig. 4d illustrates the first case, and the relative phase of the first and second oscillators $\theta = \psi_1 - \psi_2$ oscillates around some equilibrium point. Accordingly, the regime in which the second and third oscillators are partially locked arises along the vertical band of two-frequency quasi-periodicity in Fig. 5. Finally, the regime of partial locking of the first and third oscillators can be observed along the diagonal. The phase portrait for this case is shown in Fig. 4d. The complete-synchronisation domain is formed by the intersection of the

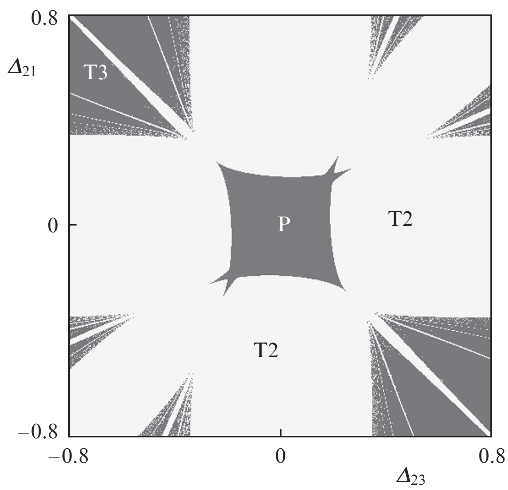

Figure 5. Lyapunov exponent map for system (10) at $\mu_1 = 0.351$, $\mu_2 = 0.108$, and $m = -0.139$.

three aforementioned bands, corresponding to three versions of partial synchronisation in the system.

Let us now discuss how the transition from the regime of partial synchronisation of the first and third lasers to the regime of locking the first and second lasers may occur in the case of global coupling. This transition can be implemented due to the saddle–node bifurcation of the invariant curves (as in the case of the laser chain). However, there may be another scenario. If the trajectory in the frequency-detuning plane lies near the complete-synchronisation domain, the following nonlocal bifurcation arises: the invariant curve passes through the saddle in the phase plane, merging with its separatrices (the tracking direction along the invariant curve is set by the unstable manifold of the saddle). This mechanism can be illustrated by the transition from Fig. 4d to Fig. 4e. The invariant curve, when rounding the saddle, turns to left (with respect to the phase trajectory direction) in Fig. 4d and to right in Fig. 4e.

4. Conclusions

The regime pattern for three interacting lasers is rather complicated even within the phase approximation. It can be revealed by applying jointly the bifurcation analysis and the method of Lyapunov maps and by constructing phase portraits. The main regimes are as follows: complete laser synchronisation, partial synchronisation of laser pairs, resonant two-frequency regimes of different types, and three-frequency regimes. The regime pattern includes bifurcations of the equilibrium states, as well as saddle–node and nonlocal bifurcations of the invariant curves. The dynamics for a three-element chain differs significantly from that for a ring (for example, there are different types of bifurcations corresponding to destruction of the complete-synchronisation regime). In addition, bistability in the form of coexistence of two stable equilibrium states, which correspond to the complete synchronisation of three lasers, is typical of global coupling. Another specific feature is that the regime pattern for a laser ring in the parameter space depends on the sign of the coupling coefficient, whereas the pattern for a laser chain is independent of this parameter.

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