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Entropy squeezing for qubit–field system under decoherence effect

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Abstract. **We study in detail the dynamics of field entropy squeezing (FES) for a qubit–field system whose dynamics is described by the phase-damped model. The results of calculations show that the initial state and decoherence play a crucial role in the evolution of FES. During the temporal evolution of the system under decoherence effect, an interesting monotonic relation between FES, Wehrl entropy (WE) and negativity is observed.**

Keywords: *qubit–field system, decoherence, field entropy squeezing.*

1. Introduction

In recent years, the properties of the dissipative variants of the Jaynes–Cummings model (JCM) have received renewed interest due to their use in implementation of quantum computation across the description of the atom–field interaction under decoherence effect. Unlike theoretical schemes relaying on dynamic effects, the JCM has found its real application through experimental progress in cavity quantum electrodynamics. For the JCM to be used in quantum information processing, it is important to understand the effect of decoherence on the field–matter interaction arising due to unavoidable influence of the environment. In this context, several papers have investigated the JCM with analytical approximations [1, 2] and numerical calculations [3–6], taking into account dissipation and phase damping. Remarkably, it has been shown that decoherence under dissipation (system loses energy by creating a bath quantum) has a significant impact on implementation of the experimental scheme in real physical situations. On the other hand, the phase damping effect describes the variation of coherence in the system state during the temporal evolution, where the interaction Hamiltonian between the external environment and the system does not commute with the system Hamiltonian during this process.

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Entanglement is a type of nonlocal correlation that has been playing an important role in the field of quantum information and processing. Precisely engineered entangled states of interest can indeed be both fragile and difficult to realise [7]. It is often viewed as a fragile and exotic feature of quantum mechanics, and its investigation is of practical and theoretical significance. From the point of scientific philosophy, this feature has been the focus of fundamental discussions and propositions of quantum mechanics since the works of Schrödinger and the celebrated Einstein–Podolsky–Rosen (EPR) paper [8]. Interestingly, entanglement is a property of nonlocal correlations between two or more quantum systems, which cannot be increased under local operations and classical communications [7]. Due to its properties, quantum entanglement is used as a essential resource for information processing tasks such as quantum computation, quantum teleportation [9], superdense coding [10], quantum cryptography [11, 12] and more recently, one-way quantum computation [13] and quantum metrology [14, 15]. Thus, different quantum problems cannot be solved by using classical approaches, which leads to an intensive search for new mathematical tools that would enable a proper measure of this phenomenon [16]. In particular, it is relevant to test whether a given quantum state is separable or may exhibit a certain quantum character. For this quantification, various entanglement characteristics (measures) have been demonstrated and provided such as concurrence [17–19], entanglement of formation [20–22], negativity [23–26], etc.

Investigating the entanglement phenomenon of the quantum system in the presence of decoherence has become recently an active field of endeavour, although many related aspects still need further efforts. There are a number of recent papers which examine the effect of decoherence on quantum entanglement and classical correlation. In this regard, the effect of phase damping on the classical correlation measured by Wehrl entropy (WE) and Wehrl phase distribution has been investigated [6]. It has been found that phase damping leads to long lived correlation of the system. Another work in this direction studies the effect of intrinsic decoherence on the entropy squeezing of coupled field-superconducting charge qubit [27]. It is reported there that the appearance and disappearance of entropy squeezing depend on the intrinsic decoherence.

The aim of our paper is to investigate and discuss in detail the time evolution of the FES, WE and entanglement measure in the presence of phase damping effect. Furthermore, we present the relationship between them in terms of the parameters involved in the system under consideration. This leads to the following question: can the FES be used as a parameter of entanglement and dynamical properties of the system in the presence of decoherence?

2. Master equation and its dynamics

The model to be treated is an intensity-dependent JCM of a qubit interacting resonantly with a single mode of the radiation field in a cavity where the coupling is intensity dependent and preserves the energy of the system. We assume that the environment is at zero temperature and dissipation is affected through a phase-damping reservoir. Under the rotating-wave approximation, the interaction Hamiltonian of the system reservoir is given by

$$\hat{H}_I = \lambda(\hat{A}\,|\uparrow\rangle\langle\downarrow| + \hat{A}^\dagger|\downarrow\rangle\langle\uparrow|). \tag{1}$$

Here, λ is the coupling constant; $\hat{A} = \hat{a}\sqrt{\hat{n}}$; $\hat{A}^\dagger = \sqrt{\hat{n}}\,\hat{a}^\dagger$; $\hat{n} = \hat{a}^\dagger\hat{a}$; and the factor $\sqrt{\hat{n}}$ is the interaction term, which is no longer linear in the field variables and represents an intensity dependent coupling.

In this case, the master equation for the density matrix operator $\hat{\rho}$ under phase damping of the cavity field and at a zero temperature bath can be written as

$$\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[\hat{H}_I,\hat{\rho}(t)] + \gamma[2\hat{n}\hat{\rho}(t)\hat{n} - \hat{n}^2\hat{\rho}(t) - \hat{\rho}(t)\hat{n}^2], \tag{2}$$

where γ is the phase-damping constant. The dressed-state representation can be used to obtain an exact solution to Eqn (2) in the case of a high-Q cavity [2]. To derive the master equation in the high-Q limit, Eqn (2) can be written as

$$\frac{\mathrm{d}\hat{\Lambda}(t)}{\mathrm{d}t} = \gamma\exp(\mathrm{i}\hat{H}_I t)$$

$$\times[2\hat{n}\hat{\rho}(t)\hat{n} - \hat{n}^2\hat{\rho}(t) - \hat{\rho}(t)\hat{n}^2]\exp(-\mathrm{i}\hat{H}_I t), \tag{3}$$

where $\hat{\Lambda}(t) = \exp(\mathrm{i}\hat{H}_I t)\hat{\rho}(t)\exp(-\mathrm{i}\hat{H}_I t)$ and the initial state of the whole system can be expressed through the density matrix. Then, adding these inputs and applying secular approximation [2], in which we neglect the oscillatory terms, to the master equation (3), we obtain the density matrix

$$\frac{\hat{\Lambda}(t)}{\mathrm{d}t} = \frac{\gamma}{2}\sum_{n,m=0}^{\infty}\{(2n+1)(2m+1)[\Gamma_{nn}^{++}\hat{\Lambda}(t)\Gamma_{mm}^{++}$$

$$+\Gamma_{nn}^{--}\hat{\Lambda}(t)\Gamma_{mm}^{--} + \Gamma_{nn}^{++}\hat{\Lambda}(t)\Gamma_{mm}^{--} + \Gamma_{nn}^{--}\hat{\Lambda}(t)\Gamma_{mm}^{++}]$$

$$+\Gamma_{nn}^{+-}\hat{\Lambda}(t)\Gamma_{mm}^{-+}\exp(2\mathrm{i}t\mu_{nm}) + \Gamma_{nn}^{-+}\hat{\Lambda}(t)\Gamma_{mm}^{+-}\exp(-2\mathrm{i}t\mu_{nm})\}$$

$$-\frac{\gamma}{4}\sum_{n=0}^{\infty}[(2n+1)^2+1][(\Gamma_{nn}^{++} + \Gamma_{nn}^{--})\hat{\Lambda}(t)$$

$$+\hat{\Lambda}(t)(\Gamma_{nn}^{++} + \Gamma_{nn}^{--})], \tag{4}$$

where $\Gamma_{nm}^{(ji)} = |\phi_n^{(j)}\rangle\langle\phi_m^{(i)}|$; $j,\ i$ = '+', '–'; $|\phi_n^{\pm}\rangle$ are two eigenstates of the total Hamiltonian (1) for a lossless cavity; $\mu_{nm} = \mu_n - \mu_m$; $\pm\mu_n = \pm\lambda(n+1)$ are the eigenvalues of the operator. The expression for the eigenvectors in the rotating wave approximation (RWA) can be written as

$$\begin{pmatrix}|\phi_n^+\rangle\\|\phi_n^-\rangle\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}|n,\uparrow\rangle\\|n+1,\downarrow\rangle\end{pmatrix}. \tag{5}$$

From the above formulas, we can find an analytical solution to the system density matrix in the general initial state

$$\hat{\rho}(t) = \frac{1}{2}\sum_{n,m=0}^{\infty}\ \sum_{\substack{r,s=\pm k\\k=0}}^{1}\ \sum_{l=0}^{1}\exp\left(-\frac{\gamma}{2}t\right)\exp[-\gamma t(n-m)^2]$$

$$\times\exp[-\mathrm{i}t(r\mu_m - s\mu_n)]\,U_{nm}^{(rs)}\,|1-l,m+l\rangle\langle 1-k,n+k|, \tag{6}$$

where

$$U_{nm}^{(rs)} = \langle\phi_m^{(r)}|\rho^{\mathrm{AF}}(0)|\phi_n^{(s)}\rangle\left\{1+\delta_{rs}\left[\cosh\left(\frac{\gamma t\delta_{nm}}{2}\right)-1\right]\right\}$$

$$+\delta_{nm}\delta_{rs}\langle\phi_m^{(-r)}|\rho^{\mathrm{AF}}(0)|\phi_n^{(-s)}\rangle\sinh\left(\frac{\gamma t}{2}\right); \tag{7}$$

and δ_{nm} is the Kronecker delta. Let us assume that the initial state of the system is the product of the states: $\hat{\rho}^{\mathrm{AF}}(0) = \hat{\rho}^{\mathrm{A}}(0)\otimes\hat{\rho}^{\mathrm{F}}(0)$, with the atom initially in the excited state, i.e., $\hat{\rho}^{\mathrm{A}}(0) = |\uparrow\rangle\langle\uparrow|$, while the initial state of the field is given by

$$\hat{\rho}^{\mathrm{F}}(0) = \frac{|\alpha\rangle\langle\alpha| + r|-\alpha\rangle\langle-\alpha|}{1+r}. \tag{8}$$

At $r = 0$, the optical field arises from the coherent state, while at $r = 1$, the field exhibits the statistical mixture of coherent states $|\alpha\rangle$ and $|-\alpha\rangle$, where $|\alpha\rangle$ is a state containing $|\alpha|^2$ photons on average. For simplicity, the amplitude α is assumed real throughout the paper without loss of generality.

3. Different classical and quantum quantifiers

We begin our discussion by presenting the overall state of different kinds of classical and quantum quantifiers such as the FES, quantum entanglement and WE.

3.1. Field entropy squeezing

Important tools have been developed in recent years for the systematic investigation of the squeezing of quantum systems. In this regard, the relationship between squeezing and entangled state transformations has been discussed [28]. The entropy uncertainty relation, which introduces the concept of entropy squeezing, has described some highly sensitive effects of the field squeezing [29, 30]. This work has been extended to discuss the FES for another system [31]. The inequality $\Delta X\Delta Y \geqslant \hbar/2$ has been presented in [32] for the operators that satisfy the condition $[X, Y] = \mathrm{i}\hbar$, where ΔA is the standard deviation of the observable A. This equation was derived by Heisenberg as the true mathematical expression of the uncertainty principle for the coordinate–momentum pair. An alternative mathematical formulation of the uncertainty principle is provided by the inequality [33, 34]

$$\delta X\,\delta Y \geqslant \pi\mathrm{e}\hbar, \tag{9}$$

where δA is the exponential of the differential entropy corresponding to the observable A.

The coordinate and momentum entropy of the field are defined as [32]

$$S_\xi(t) = -\int\langle\xi|\hat{\rho}^{\mathrm{F}}(t)|\xi\rangle\ln\langle\xi|\hat{\rho}^{\mathrm{F}}(t)|\xi\rangle\,\mathrm{d}\xi, \tag{10}$$

where $\xi = x, p$ is the coordinate or momentum. The Fock state $|n\rangle$ of the field can be written in terms of the coordinate and the momentum representations as

$$(\langle x|n\rangle, \langle p|n\rangle) = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} \times \left(H_n(x)\exp\left(-\frac{x^2}{2}\right), \frac{H_n(p)}{\mathrm{i}^n}\exp\left(-\frac{p^2}{2}\right)\right), \quad (11)$$

where $H_n(\xi)$ are the Hermite polynomials. The entropy uncertainty relation of coordinate and momentum is given by [32]

$$\exp[S_x(t)]\exp[S_p(t)] \geqslant \pi\mathrm{e}. \quad (12)$$

In this considered case, the FES in terms of the variable $x(p)$ is given by the expression [35]

$$\delta_{x(p)} = \exp[S_{x(p)}(t)] - \sqrt{\pi}\,\mathrm{e}, \quad (13)$$

where for $\delta_{x(p)} < 0$, the coordinate (momentum) of the field is squeezed in entropy. Here, we compare the FES dynamics with the negativity. In other words, we use the concept of the negativity [25] to quantify the amount of entanglement of the final qubit–field state (6), which is defined by

$$N(\hat{\rho}) = \frac{\|\hat{\rho}^{\mathrm{T}_A}\| - 1}{2}, \quad (14)$$

where $\hat{\rho}^{\mathrm{T}_A}$ denotes the partial transpose of $\hat{\rho}$ with respect to subsystem A. The trace-normalised Hermitian operator $\|\hat{\rho}^{\mathrm{T}_A}\| \equiv \mathrm{Tr}\sqrt{(\hat{\rho}^{\mathrm{T}_A})^\dagger \hat{\rho}^{\mathrm{T}_A}}$ has the matrix elements

$$\langle i_A, j_B|\hat{\rho}^{\mathrm{T}_A}|k_A, l_B\rangle = \langle k_A, j_B|\hat{\rho}|i_A, l_B\rangle. \quad (15)$$

The negativity varies from $N(\hat{\rho}) = 0$ for an unentangled state to $N(\hat{\rho}) = 1$ for a maximally entangled state as for the well known case of EPR states.

3.2. Wehrl entropy

Wehrl entropy is used in treating the dynamics of quantum systems [36–38]. This measure has been successfully applied in description of different properties of the quantum optical fields such as phase-space uncertainty [39] and decoherence [40]. In addition, the problem of measuring quantum correlations (entanglement) in phase space with application of the WE has been discussed in [41]. It has been found that the degree of the intermode correlation strongly depends on the photon number difference in two-mode Fock states. On the other hand, the effect of phase damping of the classical correlation measured by WE and Wehrl phase distribution has been investigated in [6, 42].

Any quantum state, described by a density matrix $\hat{\rho}$, can be represented by the Husimi quasi-distribution function, $Q_\rho(v) = (1/\pi)\langle v|\hat{\rho}|v\rangle$, where $|v\rangle$ is the coherent state. This function in the β space is defined by the expression

$$Q_\beta(t) = \frac{1}{\pi}\langle\beta|\hat{\rho}^{\mathrm{F}}(t)|\beta\rangle, \quad (16)$$

and the WE of a quantum state $\hat{\rho}$ is defined by the relation [36]

$$S_\rho = -\int_\Omega Q_\rho(v)\ln Q_\rho(v)\,\mathrm{d}v. \quad (17)$$

According to Eqn (17), the field Wehrl space entropy is given by [43, 44]:

$$S_{\mathrm{W}}(t) = -\int_0^{2\pi}\left\{\int_0^\infty [Q_\beta(t)\ln Q_\beta(t)]\,|\beta|\,\mathrm{d}|\beta|\right\}\mathrm{d}\Omega. \quad (18)$$

To the best of our knowledge, there have been no previous studies on the dynamics of the FES for a qubit–field system whose dynamics is described by the phase-damped model.

4. Numerical results and discussion

In this section, we introduce the dynamics of the FES under phase-damping effect when the field arises from a mixed state and the atom – from the upper state. In this case we present the correlation of the FES to the degree of entanglement and WE.

Based on Eqns (13), (14) and (18), we present the main results for the evolution of the field entropy squeezing components $\delta_{x,p}$, negativity $N(\hat{\rho})$, Wehrl entropy S_{W} and Wehrl phase distribution. All the curves in Figs 1 and 2 are plotted for the mean photon number $\bar{n} = 10$. The time t is normalised to the inverse coupling constant λ.

Figure 1 presents the results of investigation of the influence of the initial field state at $r = 0$ and 1 on the dynamics of the different quantities in the absence of the phase-damping effect. One can see that δ_x, δ_p and S_{W} exhibit a periodic time dependence. Of interest is the fact that both δ_x and S_{W} are very sensitive to the change in the initial state of the coherent state. To this end, the field does not have any impact on the dynamics of δ_p, which means that the FES presents a rich structure, which can be explored in various physical branches. Moreover, one can see that there is large squeezing in δ_x for $r = 0$ and sharp squeezing for $r = 1$ at $\lambda t = (2m+1)\pi/2$, where $m = 0, 1, 2, \ldots$. At the same time, there is no squeezing at all for δ_p. On the other hand, the WE is greater at $r = 1$, indicating that the optical field becomes more quantum mechanical as it tends to the statistical mixture of the states.

Also, we can see that, in each periodicity, the entanglement is initially increased to a maximum and decreased to a zero at $\lambda t = (2m+1)\pi/2$. Such a system suffers a sudden death (i.e., the complete loss of the entanglement after a finite time). Furthermore, the system exhibits a sudden birth of entanglement thereafter: it can be increased to a maximum and decreased to minimum during the time $\lambda t = m\pi$. In the case of the coordinate entropy squeezing of the optical field (Fig. 1a), the lower curve shows that δ_x increases to a maximum, exhibiting local minima and maxima for short time, and then decreases to a minimum at $\lambda t = (2m+1)\pi/2$. The entropy S_{W} increases to a maximum followed by local minima and maxima; after that, it tends to a local minimum at $\lambda t = m\pi$.

Then, by investigating the dynamics of δ_x and δ_p of the qubit–field interaction, we have found an interesting correlation between both quantities during the evolution. We can observe a direct monotonic relationship between δ_p and $N(\hat{\rho})$ in the range $(2m+1)\pi/2 - \varepsilon \leqslant \lambda t \leqslant (2m+1)\pi/2 + \varepsilon$, where ε is a small value. In this case, we deal with the same behaviour in this time interval with minimal values at $\lambda t = (2m+1)\pi/2$, which means that both quantities are symmetric with respect to points $\lambda t = (2m+1)\pi/2$. On the other hand, we can see that the δ_p and $N(\hat{\rho})$ quantities provide an indirect (or direct) relation during the evolution, which manifests itself as an inverse monotonic change in δ_p from the maximum, accompanied by an increase in $N(\hat{\rho})$ from its minimal values. However, this is not a general result since δ_p and entanglement are

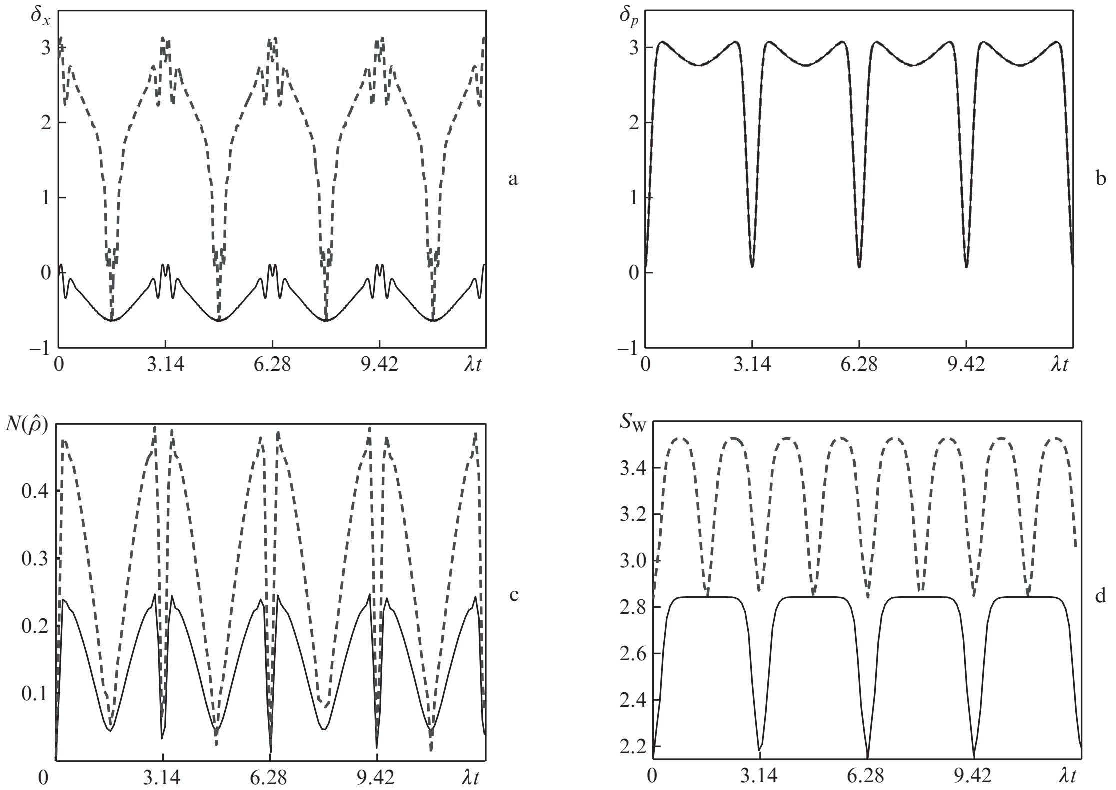

Figure 1. Time evolution of the FES for (a) the coordinate δ_x, (b) momentum δ_p, (c) negativity $N(\hat{\rho})$ and (d) Wehrl entropy S_W at $\alpha = \sqrt{10}$, $\gamma/\lambda = 0$, $r = 0$ (solid curves) and 1 (dashed curves).

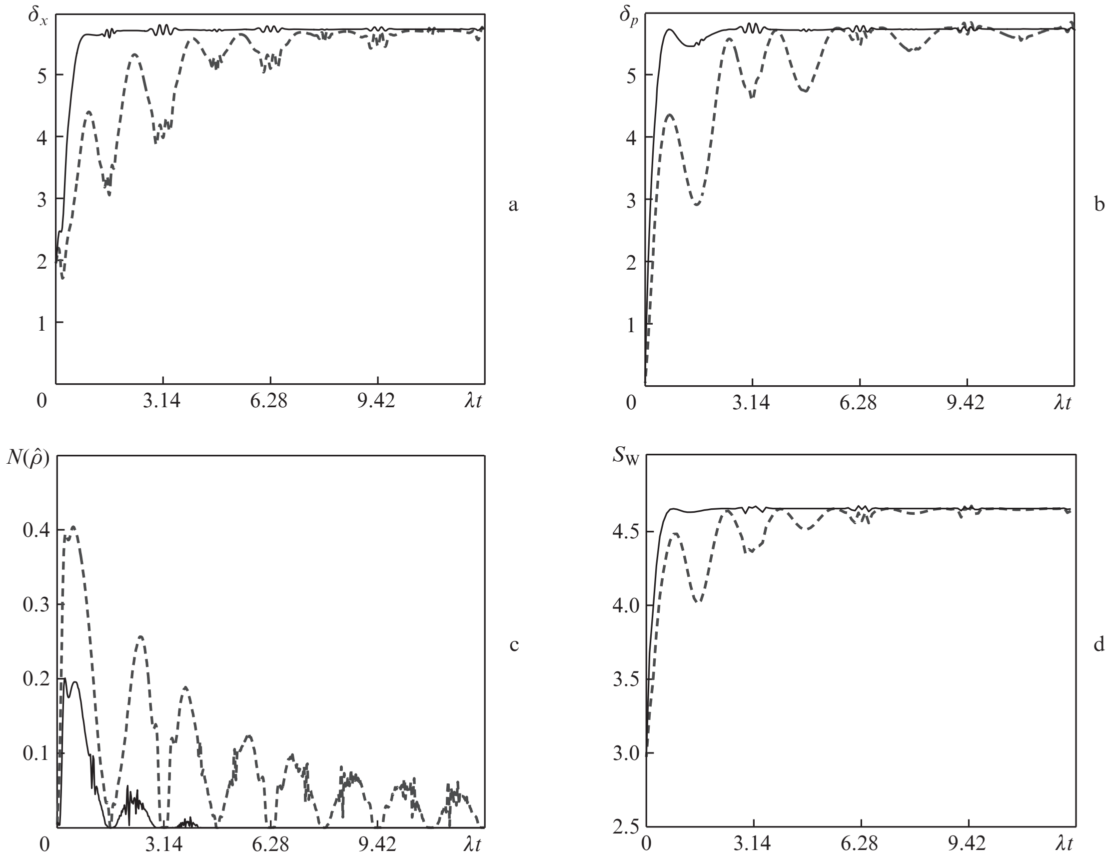

Figure 2. Time evolution of the FES for (a) the coordinate δ_x, (b) momentum δ_p, (c) negativity $N(\hat{\rho})$ and (d) Wehrl entropy S_W at $\alpha = \sqrt{10}$, $r = 1$, $\gamma/\lambda = 0.3$ (solid curves) and 0.05 (dashed curves).

two different and independent physical quantities with no simple relative ordering between them.

To visualise the effect of the phase damping on the dynamics of the above quantities, we have plotted in Fig. 2 the evolution of the different quantities in terms of various parameters γ/λ at $r = 1$. In this considered case, the behaviour of the system has completely changed under decoherence effect. It follows from Fig. 2 that the negativity decreases with increasing phase damping parameter and almost disappears as the time becomes significantly large, which transfers the system into a separable state when its nonlocal correlation is completely lost. On the other hand, the effect of decoherence on δ_x, δ_p and S_W is found to be similar where there is a monotonic relation between them exhibiting the same behaviour during the time evolution. Interestingly, δ_x and δ_p increase with increasing γ/λ and squeezing is not observed at all during the evolution. In addition, we have found that the WE tends to stabilise for different values of γ/λ, which indicates that the field is not dependent on the environment and becomes more quantum mechanical in this limit. The results obtained show that decoherence effect may destroy the entanglement and restrain the squeezing phenomenon during the evolution.

5. Conclusions

We have studied in detail the field entropy squeezing dynamics for a qubit–field system in the presence of phase damping effect including the squeezing of coordinate and momentum components in terms of the initial state setting and decoherence parameter. We have shown that the field entropy exhibits a rich structure by providing different physical phenomena via a proper choice of the involved system parameters under consideration. Interestingly, the field entropy has a different order depending on the optical field form and phase damping parameter exhibiting a monotonic relationship with the degree of entanglement and the statistical proprieties of the field for different ranges of dimensionless time. Particularly, we have found that the amount of different quantities is very sensitive to the degree of mixture of the optical field. In addition, decoherence effect may destroy the amount of entanglement and restrain the squeezing phenomenon during the evolution. Our observations may have important implications in exploiting this quantity in quantum information theory. In the future, we plan to investigate the evolution of the field entropy squeezing of multiqubit systems including the effects of finite-temperature environments and the interqubit distance. It will be important to study the non-Markovian dynamics of the field entropy squeezing which is useful for better understanding of the relationship between the field entropy squeezing, and classical-quantum quantifiers in the decoherence process.

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