Quantum Electronics **44** (12) 1119–1128 (2014)

PACS numbers: 42.65.Wi; 81.05.Xj
DOI: 10.1070/QE2014v044n12ABEH015507

Modulation instability in a zigzag array of nonlinear waveguides with alternating positive and negative refractive indices*

A.A. Dovgiy

Abstract. **The modulation instability is analytically investigated in a zigzag array of tunnel-coupled optical waveguides with alternating refractive indices and Kerr nonlinearity. Particular solutions to a system of coupled nonlinear equations are found. They describe the propagation of electromagnetic waves that are uniform along the waveguide and their instability is studied. It is shown that the coupling coefficient between the waveguides, which are non-nearest neighbours, has a significant effect on the instability of the waves in question. When the coupling coefficient exceeds a certain threshold, the modulation instability disappears regardless of the radiation power. The influence of the ratio of the wave amplitudes in adjacent waveguides to the instability of the particular solutions is studied. Different variants of the nonlinear response in waveguides are considered. The studies performed present a new unusual type of the modulation instability in nonlinear periodic systems.**

Keywords: *modulation instability, tunnel-coupled waveguides, optical lattice, discrete diffraction, negative refraction, forward and backward waves, reciprocal lattice vector, instability increment.*

1. Introduction

The last 30–40 years are marked by intense theoretical and experimental studies of nonlinear discrete systems (see [1] and references therein). These structures are found in many material systems and are of undoubted scientific interest. Thus, in studying the wave propagation in nonlinear discrete structures, new types of solitons are found, which are not observed in continuous systems. The principal feature of discrete systems is that their linear properties are very different from the latter in continuous systems, and as a result, the nonlinear response demonstrates new effects, which have no analogues in continuous systems.

One of the important effects of nonlinear dynamics is the modulation instability (MI), which occurs in many material systems [2–16]. The modulation instability precedes the splitting of a spatially homogeneous wave into individual wave packets at high intensities. A propagating wave becomes unstable with respect to amplitude and phase modulation, the values of which begin to grow exponentially. Often this process of the wave destruction leads to the formation of a chain of solitons with gaps between them, inversely proportional to the spatial frequency corresponding to the maximum instability growth rate. Therefore, the MI is known in physics as a forerunner of soliton formation [17, 18].

An interesting and fruitful area of research, which examines new examples of discrete systems, is discrete optics of coupled waveguides [19]. Optical waveguides arranged relative to each other at a distance of wavelength can be coupled due to tunnel penetration of light from one waveguide to the other [20]. Systems made of a large number of tunnel-coupled waveguides form an optical lattice. Discrete optical system can be considered as a one-dimensional photonic crystal [21], and such structures can provide total reflection of radiation over a specified frequency range for any polarisation directions and angles of incidence [22, 23], so that they can be used as a frequency filter of electromagnetic radiation. Reflection of radiation in a particular frequency range is explained by the presence of a gap in the energy spectrum. Nonlinear interaction in nonlinear discrete systems can also lead to the MI [24, 25].

In periodic media, different sampling types may exert an unequal effect on the MI and lead to various conditions for forming localised modes (i.e., discrete solitons) [26, 27]. In particular, much attention is paid to experimental and analytical study of linear and nonlinear properties of arrays of identical waveguides [28–34]. Darmanyan et al. [35] considered such a system of coupled waveguides with a self-focusing Kerr nonlinearity. The MI process was studied and it was shown that, depending on the value of the reciprocal lattice vector, the system in question allows for two different scenarios of evolution of modulationally unstable solitary waves. The authors of papers [36, 37] studied a system of two coupled waveguides with a positive refractive index (PI) and a negative refractive index (NI), called an oppositely directed coupler (ODC). This system allows one to study the interaction of forward and backward waves; in addition, a nonlinear stationary solitary wave that propagates in both waveguides as a whole is formed in such a system. Litchinister et al. [38] showed that such a pair of waveguides is bistable, which manifests itself in an ambiguous dependence of the output pulse power on the input pulse power. Maimistov and Kazantseva [39] considered an ODC-based linear amplifier, determined the gain and found that compensation of losses is possible in the NI waveguide; they also studied the evolution of continuous waves in a dissipative ODC with a nonlinear PI waveguide. Xiang et al. [40] showed that the MI effect in an ODC with a Kerr nonlinearity occurs only at certain ratios of the amplitudes of forward and backward waves in the waveguides and specified nonlinearity coefficients. The modulation

*Presented at the III International Conference on Photonics and Information Optics, Moscow, Russia, January 2014.

A.A. Dovgiy National Research Nuclear University MEPhI, Kashirskoe shosse 31, 115409 Moscow, Russia;
e-mail: dovgiyalexandr@gmail.com

Received 2 April 2014; revision received 28 April 2014
Kvantovaya Elektronika **44** (12) 1119–1128 (2014)
Translated by I.A. Ulitkin

instability appears only in a limited range of power in the waveguides if NI waveguide nonlinearity is defocusing and PI waveguide nonlinearity is focusing. With increasing power the MI is suppressed and increases in the case of a conventional directional coupler. This unusual property of the MI effect stems from the interaction of forward and backward waves because in a conventional directional coupler the interacting waves are forward ones, and nothing like this is ever observed. Tatsing et al. [41] studied the effect of saturating nonlinearity on the MI in an ODC. Maimistov et al. [42, 43] studied an array of coupled waveguides with alternating positive and negative indices and showed that the spectrum of linear waves propagating in such a system has a band gap. Zezyulin et al. [44] found the existence of new discrete gap solitons in such a system of coupled waveguides and studied their properties. In addition, the system allows the formation of a nonlinear stationary solitary wave that propagates along the entire array of coupled waveguides as a whole [45].

Theoretical models describing the evolution of the waves in all the above variants of the arrays of tunnel-coupled waveguides take into account only the interaction of the nearest neighbours. The electromagnetic field is strongly localised in the channels of the array, and interaction with the rest of the waveguides is insignificant. Efremidis and Christodoulides [46] proposed a zigzag configuration for the coupled waveguide array, in which the interaction with the waveguides that are not the nearest neighbours can be just as significant as the interaction between the nearest neighbours. This leads to the appearance of new types of discrete solitons. Kazantseva and Maimistov [47] considered the modification of a zigzag array in which waveguides with even and odd numbers have positive and negative indices, respectively. They showed that the spectrum of linear waves has a band gap. In addition, they described the propagation of a nonlinear stationary solitary wave and the interaction between these waves, which, depending on the coupling coefficients, can behave as either gap solitons, or unstable solitary waves.

In this paper we study analytically the instability of electromagnetic waves (uniform along the waveguide) propagating in a zigzag array of coupled waveguides with alternating positive and negative indices (Fig. 1) in the presence of Kerr nonlinearity in the waveguides. The MI growth rate, an analytical expression of which is found by the method of linear stability analysis, is studied numerically. It is shown that the MI growth rate essentially depends on the parameters of the system in question: the coupling coefficient with waveguides following the nearest neighbours, the nonlinearity parameters, the ratio of the amplitudes of forward and backward waves in neighbouring waveguides. The MI is found to decrease significantly with increasing interaction force with non-adjacent waveguides independently of the field strength in the waveguide, which is a novelty for nonlinear periodic structures, because typically the MI only increases with increasing wave amplitude in such systems [48–50].

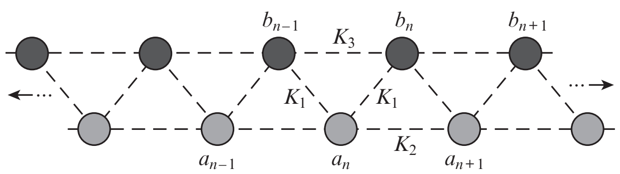

Figure 1. Zigzag array of coupled waveguides (cross section). Open circles show PI waveguides, dark ones – NI waveguides (see [45]).

2. Theoretical model

In normalised variables the system of equations describing the evolution of optical waves in a zigzag coupled waveguide array under study (Fig. 1) has the form [47]:

$$\begin{aligned} &\mathrm{i}(\partial_\zeta + \partial_\tau)a_n + K_1(b_{n-1} + b_n) + K_2(a_{n-1} + a_{n+1}) \\ &\quad + r_1|a_n|^2 a_n = 0, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\mathrm{i}(-\partial_\zeta + \partial_\tau)b_n + K_1(a_n + a_{n+1}) + K_3(b_{n-1} + b_{n+1}) \\ &\quad + r_2|b_n|^2 b_n = 0, \end{aligned} \tag{1.2}$$

where a_n (b_n) is the normalised envelope of the electric field of a quasi-harmonic wave in a PI (NI) waveguide with number n; r_1 and r_2 are the nonlinearity parameters in PI and NI waveguides, respectively; K_1 ($K_1 > 0$) is the coupling coefficient between PI and NI waveguides; K_2 ($K_2 > 0$) is the coupling coefficient between PI waveguides; and K_3 ($K_3 > 0$) is the coupling coefficient between NI waveguides. Dissipation and second-order group velocity dispersion effects are neglected.

The coefficient K_1 is responsible for the interaction of the nearest neighbours, and the coefficients K_2 and K_3 – for interaction with waveguides following the nearest neighbours. Details of the derivation of equations (1) can be found in [42, 51]. Note that in equations (1) the coupling coefficients K_2 and K_3 may be arbitrary, but for simplicity we assume $K_2 = K_3$.

Unlike equation (1.1), in equation (1.2) the spatial derivative ∂_ζ has a minus sign in front, which is due to the fact that a backward wave with the vectors of the phase and group velocities having opposite directions propagates in the NI waveguide, whereas in the PI waveguide conventional forward waves propagate.

Equations (1.1) and (1.2) describe the propagation of optical waves in the absence of losses, but NI metamaterials, as a rule, have fairly large losses. However, Xiao et al. [52] demonstrated the possibility of obtaining a NI metamaterial, in which optical waves propagate without losses. This leaves a lot of hope that in the near future it will be possible to produce transparent NI materials.

3. Uniform waves. Dispersion relation

Let us consider the simplest solution to equations (1), which describes the propagation of plane waves in this system of coupled waveguides:

$$a_n = a\exp(\mathrm{i}\kappa\zeta + \mathrm{i}qn - \mathrm{i}\omega\tau) + \text{c.c.}, \tag{2.1}$$

$$b_n = b\exp(\mathrm{i}\kappa\zeta + \mathrm{i}qn - \mathrm{i}\omega\tau) + \text{c.c.}, \tag{2.2}$$

where κ is the wave propagation constant; $a^2 + b^2 = P$ is the total radiation power in a pair of PI and NI waveguides; q is the wave vector of the reciprocal lattice; and ω is the deviation from the frequency ω_0 of the carrier wave of a quasi-harmonic wave packet. Kazantseva and Maimistov [47] found the spectrum of linear waves (2) of system (1) at $r_1 = r_2 = 0$:

$$\begin{aligned} \omega^{(+),(-)} = &-(K_2 + K_3)\cos q \\ &\pm \sqrt{[\kappa - (K_2 - K_3)\cos q]^2 + 4K_1^2\cos^2(q/2)}, \end{aligned}$$

which has a band gap (Fig. 2). Radiation with frequencies from the band gap region cannot propagate in the waveguide array in question and therefore is reflected from it. In this regard, this waveguide array acts like a distributed Bragg mirror. However, when the values of the reciprocal lattice vector are equal to $q = \pi \pm 2\pi n$ ($n = 0, 1, \ldots$) the spectrum is gapless (Fig. 2). The points in Fig. 2 correspond to the one-dimensional resonance Bragg grating. It should be noted that the interaction with the waveguides following the adjacent ones influences the spectrum of propagating plane waves. Based on Fig. 2, we can make a conclusion that when the coupling coefficients K_2 and K_3 are increased, the branches of the spectrum corresponding to $q = 0$ descend, the branches of the spectrum with $q = \pi/2$ remain stationary and the branches of the spectrum with $q = \pi$ ascend, thereby reducing the gap. Thus, an increase in the coupling between nonadjacent waveguides can lead to degeneration of the band gap in the spectrum of propagating plane waves, thereby making it gapless.

With increasing field power in the waveguide, the nonlinear effects start exerting a significant influence, and the spectrum of the plane waves becomes distorted. Substituting solutions (2) into the system of nonlinear equations (1), we obtain the following nonlinear dispersion relation:

$$\begin{aligned} \omega &= -K_1 \frac{1+f^2}{f}\cos\frac{q}{2} - (K_2 + K_3)\cos q \\ &\quad - \frac{P}{2(1+f^2)}(r_1 + r_2 f^2), \\ \kappa &= -K_1 \frac{1-f^2}{f}\cos\frac{q}{2} + (K_2 - K_3)\cos q \\ &\quad + \frac{P}{2(1+f^2)}(r_1 - r_2 f^2), \end{aligned} \tag{3}$$

where the parameter $f = b/a$ (the ratio of the amplitude of the backward wave in the NI waveguide to the amplitude of the forward wave in the PI waveguide) describes how the power is distributed between the PI and NI waveguides. Figure 2 shows that the linear and nonlinear spectra coincide with good accuracy at low values of the radiation power in the waveguides. The value $f = 1$ corresponds to the bottom and $f = -1$ – to the top of the band gap. Positive and negative values of f lie on the low and upper branches of the spectrum (Fig. 2), respectively. In order to find a more general value of f, we consider the group velocity $v_{gr} = d\omega/d\kappa$:

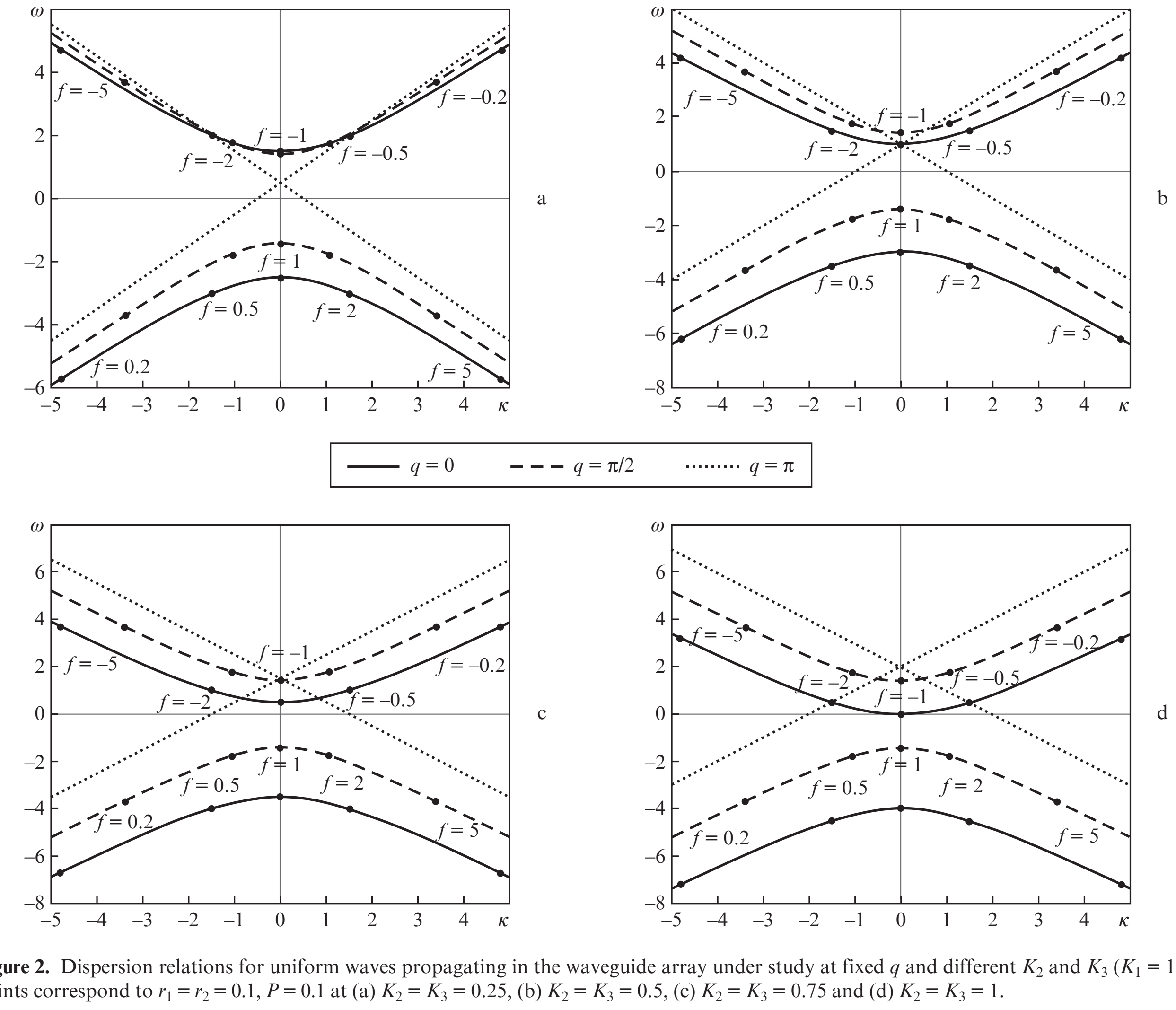

Figure 2. Dispersion relations for uniform waves propagating in the waveguide array under study at fixed q and different K_2 and K_3 ($K_1 = 1$). The points correspond to $r_1 = r_2 = 0.1$, $P = 0.1$ at (a) $K_2 = K_3 = 0.25$, (b) $K_2 = K_3 = 0.5$, (c) $K_2 = K_3 = 0.75$ and (d) $K_2 = K_3 = 1$.

$$v_{gr} = \frac{1-f^2}{1+f^2}.$$

The values of $\pm f$ correspond to the same group velocity but are located at different branches of the spectrum, while the value of f and f^{-1} correspond to the opposite group velocities and are on the same branch of the spectrum (Fig. 2). The case $|f| < 1$ corresponds to the forward propagation of the waves in the system, and $|f| > 1$ – to the backward propagation. When $|f| = 1$, which corresponds to the boundaries of the band gap in the spectrum, the group velocity is equal to zero. In this case, the energy is transferred in neither direct nor reverse directions.

4. Analysis of the stability of uniform waves

The procedure for investigating the stability of solutions (2) describing the propagation of uniform waves along the waveguides in the zigzag array under study is a well known method of linear stability analysis [18]. We introduce perturbations in the amplitudes of solutions (2) as follows:

$$a_n = (a + \tilde{a}_n)\exp(\mathrm{i}\kappa\zeta + \mathrm{i}qn - \mathrm{i}\omega\tau), \tag{4.1}$$

$$b_n = (b + \tilde{b}_n)\exp(\mathrm{i}\kappa\zeta + \mathrm{i}qn - \mathrm{i}\omega\tau), \tag{4.2}$$

where $\tilde{a}_n$ and $\tilde{b}_n$ are small quantities ($|\tilde{a}_n| \ll a, |\tilde{b}_n| \ll b$). Substituting (4) into the system of nonlinear equations (1) and linearizing it with respect to small perturbations, we can easily obtain the system of linear differential equations:

$$\begin{aligned}
&\mathrm{i}(\partial_\zeta + \partial_\tau)\tilde{a}_n - f\mu_1\tilde{a}_n + K_1(\tilde{b}_{n-1} + \tilde{b}_n) - \mu_2\tilde{a}_n \\
&\quad + K_2(\tilde{a}_{n-1} + \tilde{a}_{n+1}) + \rho_1(\tilde{a}_n + \tilde{a}_n^*) = 0, \\
&\mathrm{i}(-\partial_\zeta + \partial_\tau)\tilde{b}_n - (\mu_1/f)\tilde{b}_n + K_1(\tilde{a}_n + \tilde{a}_{n+1}) - \mu_3\tilde{b}_n \\
&\quad + K_3(\tilde{b}_{n-1} + \tilde{b}_{n+1}) + \rho_2(\tilde{b}_n + \tilde{b}_n^*) = 0,
\end{aligned} \tag{5}$$

where $\mu_1 = 2K_1\cos(q/2)$; $\mu_2 = 2K_2\cos q$; $\mu_3 = 2K_3\cos q$; $\rho_1 = r_1P/(1+f^2)$; and $\rho_2 = r_2Pf^2/(1+f^2)$.

The system of equations (5) will be sought for in the form

$$\tilde{a}_n = c_1\exp(\mathrm{i}k\zeta + \mathrm{i}Qn - \mathrm{i}v\tau) + d_1\exp(-\mathrm{i}k\zeta - \mathrm{i}Qn + \mathrm{i}v\tau), \tag{6.1}$$

$$\tilde{b}_n = c_2\exp(\mathrm{i}k\zeta + \mathrm{i}Qn - \mathrm{i}v\tau) + d_2\exp(-\mathrm{i}k\zeta - \mathrm{i}Qn + \mathrm{i}v\tau), \tag{6.2}$$

which is a more general case of propagating plane waves than solution (2). Here, the constants c_i and d_i ($i = 1, 2$) are the real values, k is the propagation constant, Q is the wave vector of the reciprocal lattice and v is the frequency of small perturbations. Substituting (6) into equation (5) leads to a system of four linear homogeneous algebraic equations of the first order with respect to the four unknowns

$$\begin{pmatrix} L_1 & M_1 & \rho_1 & 0 \\ M_1 & L_2 & 0 & \rho_2 \\ \rho_1 & 0 & L_3 & M_1 \\ 0 & \rho_2 & M_1 & L_4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ d_1 \\ d_2 \end{pmatrix} = 0, \tag{7}$$

$$L_1 = v - k - \mu_1 f - \mu_2 + M_2 + \rho_1,$$

$$L_2 = v + k - \mu_1/f - \mu_3 + M_3 + \rho_2,$$

$$L_3 = -v + k - \mu_1 f - \mu_2 + M_2 + \rho_1,$$

$$L_4 = -v - k - \mu_1/f - \mu_3 + M_3 + \rho_2,$$

where $M_1 = 2K_1\cos(Q/2)$, $M_2 = 2K_2\cos Q$ and $M_3 = 2K_3\cos Q$. The resulting system of algebraic equations (7) has a non-trivial solution when its determinant is zero. From this equation follows an algebraic equation of the fourth order with respect to the frequency of small perturbations v, which determines the dispersion relation for the waves generated by small perturbations:

$$v^4 - Av^2 + Bv + C = 0, \tag{8}$$

where

$$A = 2k^2 + 2M_1^2 + N_2^2 + N_3^2 - 2r_1\frac{P}{1+f^2}N_2 - 2r_2\frac{Pf^2}{1+f^2}N_3;$$

$$B = 2k\left(N_3^2 - N_2^2 + 2r_1\frac{P}{1+f^2}N_2 - 2r_2\frac{Pf^2}{1+f^2}N_3\right);$$

$$\begin{aligned}
C = k^4 &+ \left(2M_1 - N_2^2 - N_3^2 + 2r_1\frac{P}{1+f^2}N_2 + 2r_2\frac{Pf^2}{1+f^2}N_3\right)k^2 \\
&+ (M_1^2 - N_2N_3)\left[(M_1^2 - N_2N_3) - 4r_1r_2\left(\frac{Pf}{1+f^2}\right)^2\right. \\
&\left.+ 2r_2\frac{Pf^2}{1+f^2}N_2 + 2r_1\frac{P}{1+f^2}N_3\right];
\end{aligned}$$

$N_2 = \mu_1 f + \mu_2 - M_2$; and $N_3 = \mu_1/f + \mu_3 - M_3$.

The matrix of system (7) is called the stability matrix, which is used to investigate the stability of solutions (2) to equations (1). The modulation instability appears when at least one eigenvalue of the stability matrix has a non-zero imaginary part, which leads to an exponential increase in the amplitude of propagating plane waves (2) subjected to small perturbations (4). Thus, it is necessary to find the roots of equation (8), which is an algebraic equation of the fourth order and, according to the fundamental theorem of algebra, has four roots. The roots of equation (8), expressed in radicals, are determined by the Ferrari method and have the form

$$v_{1,2} = \frac{\sqrt{A+2s}}{2} \pm \sqrt{\frac{A-2s}{4} - \frac{B}{2\sqrt{A+2s}}},$$

$$v_{3,4} = -\frac{\sqrt{A+2s}}{2} \pm \sqrt{\frac{A-2s}{4} + \frac{B}{2\sqrt{A+2s}}},$$

where s is the cubic resolvent of equation (8), expressed in radicals by using Cardano's formula:

$$s = \sqrt[3]{-\frac{g}{2} + \sqrt{\left(\frac{g}{2}\right)^2 - \left(\frac{d}{3}\right)^3}} + \sqrt[3]{-\frac{g}{2} - \sqrt{\left(\frac{g}{2}\right)^2 - \left(\frac{d}{3}\right)^3}} - \frac{A}{6}$$

(here $d = A^2/12 + C$ and $g = A^3/108 - AC/3 - B^2/8$).

Now we can investigate the instability of solutions of (2) by analysing the MI growth rate, which is found according to the known procedure [49] as

$$G = |\mathrm{Im}(v)_{max}|, \tag{9}$$

where $\mathrm{Im}(v)_{max}$ is the imaginary part of that root v_i of equation (8), which has a maximal imaginary part. If the growth rate (9) takes nonzero values, then solutions (2) are unstable.

5. Modulation instability growth rate

The MI growth rate (9) was studied by computer simulation because explicit analytic expressions are very difficult to derive since the roots of equation (8) are expressed in radicals using Cardano's formulas. Different variants of the nonlinear response in the waveguides of the system are considered: 1) all the waveguides are nonlinear, focusing or defocusing ($r_1 = r_2 > 0$ or $r_1 = r_2 < 0$); 2) some waveguides are focusing and others are defocusing ($r_1 = -r_2 \neq 0$); 3) some waveguides are linear and others are nonlinear ($r_1 = 0$, $r_2 \neq 0$ or $r_2 = 0$, $r_1 \neq 0$). We study how the coefficient of interaction with the waveguides located behind the nearest neighbours, the parameter f and the power P affect the MI growth rate.

It should be noted that there are particular cases when it is possible to obtain an explicit analytical expression of the MI growth rate. For example, when $r_1 = r_2$, $K_2 = K_3$ and $|f| = 1$. In this case, the expression for the MI growth rate (9) takes the form:

$$G = [\sqrt{(k^2 + M_1^2)(2N_2 - r_1 P)^2 - k^2 r_1^2 P^2} - (k^2 + M_1^2 + N_2^2 - r_1 P N_2)]^{1/2}. \qquad (10)$$

Figure 3 shows the effect of the power P on the MI growth rate (10) for a particular case when the vector of the reciprocal

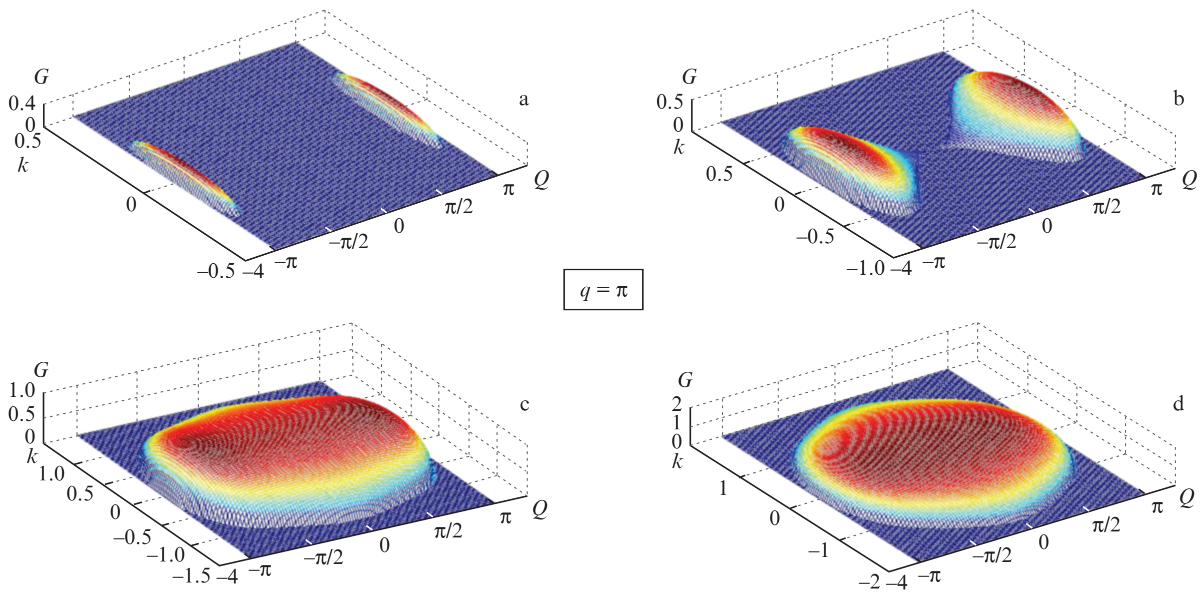

Figure 3. Dependences of the MI growth rate G on the reciprocal lattice vector Q and propagation constant k of small perturbations at P = (a) 5, (b) 10, (c) 15 and (d) 20 ($K_1 = 1$, $K_2 = K_3 = 0.25$, $r_1 = r_2 = 0.1$, $f = 1$, $q = \pi$).

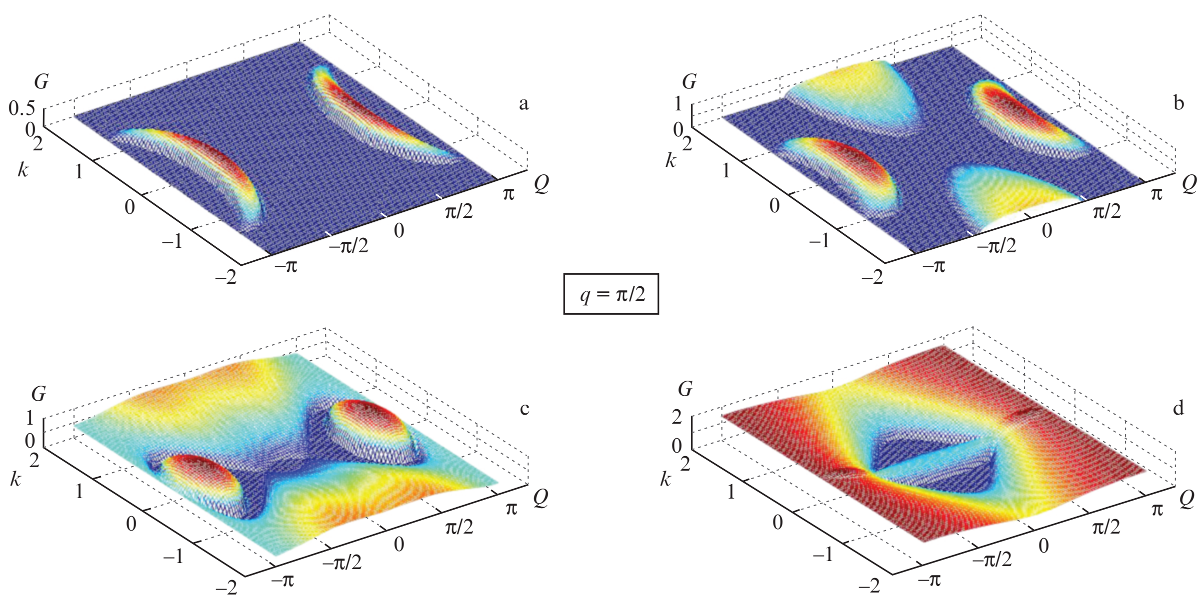

Figure 4. Dependences of the MI growth rate G on the reciprocal lattice vector Q and propagation constant k of small perturbations at P = (a) 10, (b) 15, (c) 20 and (d) 30 ($K_1 = 1$, $K_2 = K_3 = 0.25$, $r_1 = r_2 = 0.1$, $f = 1$, $q = \pi/2$).

lattice of plane waves is $q = \pi$ and the spectrum is gapless (Fig. 2). It can be concluded that in this case, when the power of the field in the waveguides is increased, solutions (2) become unstable. The influence of power on the stability of solutions (2) is of threshold character (Fig. 3b). Above a threshold power solutions (2) become completely unstable (Figs 3c and 3d). The same case was investigated in Figure 4, but for $q = \pi/2$, when the spectrum of plane waves has a gap. Similarly to the case $q = \pi$, the influence of power on the stability of solutions (2) with $q = \pi/2$ is of threshold character, but the presence of a gap in the spectrum increases the power threshold, which is easily seen by comparing Figs 3 and 4.

5.1. Influence of the coupling coefficient K_2 on the MI growth rate

As was already noted, in the coupled zigzag waveguide array in question, interaction with nonadjacent waveguides can be as significant as with adjacent ones (note that in conventional arrays of coupled waveguides, essential only is the interaction of the nearest neighbours). It is therefore of interest to investigate the effect of this additional interaction on the instability of plane waves propagating in the zigzag array under study and compare it with the MI in the conventional arrays of coupled waveguides or Bragg gratings. Usually, plane waves are unstable in nonlinear Bragg gratings, and instability increases with increasing wave amplitude [48–50].

Figure 5 shows the effect of the coupling coefficient K_2 on the MI growth rate (10) at $q = \pi$, when the spectrum is gapless. Interaction with the waveguides following the nearest neighbours significantly affects the instability of plane waves. By increasing K_2, the instability of plane waves (Figs 5a–c) is significantly reduced, and there is quite a large area of stability, which does not change with increasing field strength (Fig. 5d). Thus, we can conclude that the influence of the coupling coefficient K_2 on the instability of plane waves has a threshold character. To find the threshold value of K_2, we study the expression for the MI growth rate (10) at arbitrary q. For simplicity, we consider expression (10) at $k = 0$:

$$G = [(r_1P/2)^2 - (|r_1P/2 - N_2| - 2K_1|\cos(Q/2)|)^2]^{1/2},$$

where $N_2 = 2K_1\cos(q/2) + 2K_2\cos q - 2K_2\cos Q$. It follows that the instability disappears regardless of the value of P, if the inequality

$$|\cos(Q/2)| \geqslant K_1/2K_2 + \cos(q/2) \tag{11}$$

is fulfilled.

The wave vector of the reciprocal lattice of small perturbations Q is a real value, and therefore the absolute value of the cosine cannot be greater than unity. Thus, the latter inequality holds only when $K_2 \geqslant K_1/4\sin^2(q/4)$ and $q \neq 0$. As a consequence, for the threshold value of the coupling coefficient K_2 we have

$$K_2^{\text{th}} = K_1/4\sin^2(q/4). \tag{12}$$

When $q = 0$, inequality (11) is not met at any Q, and the instability persists regardless of the K_2 value. The minimum threshold value K_2^{th} is reached at $q = \pi$: $K_2^{\text{th}}(\pi) = K_1/2$; in this case, only $Q = 0$ satisfies inequality (11), so that the instability disappears only at the centre of the Brillouin zone (Fig. 5a). When the threshold is exceeded, the stability region increases (Figs 5b–d). Inequality (11) allows one to find easily the boundaries of the stability region:

$$-2\arccos[K_1/2K_2 + \cos(q/2)] \leqslant Q$$
$$\leqslant 2\arccos[K_1/2K_2 + \cos(q/2)]. \tag{13}$$

At $K_2 < K_1/4\sin^2(q/4)$ the stability of plane waves is not observed, and the threshold power at which solutions (2) in this case become completely unstable, is given by the expression

$$P^{\text{th}}(q) = 4\frac{K_1\cos^2(q/4) - K_2\sin^2(q/2)}{r_1}$$

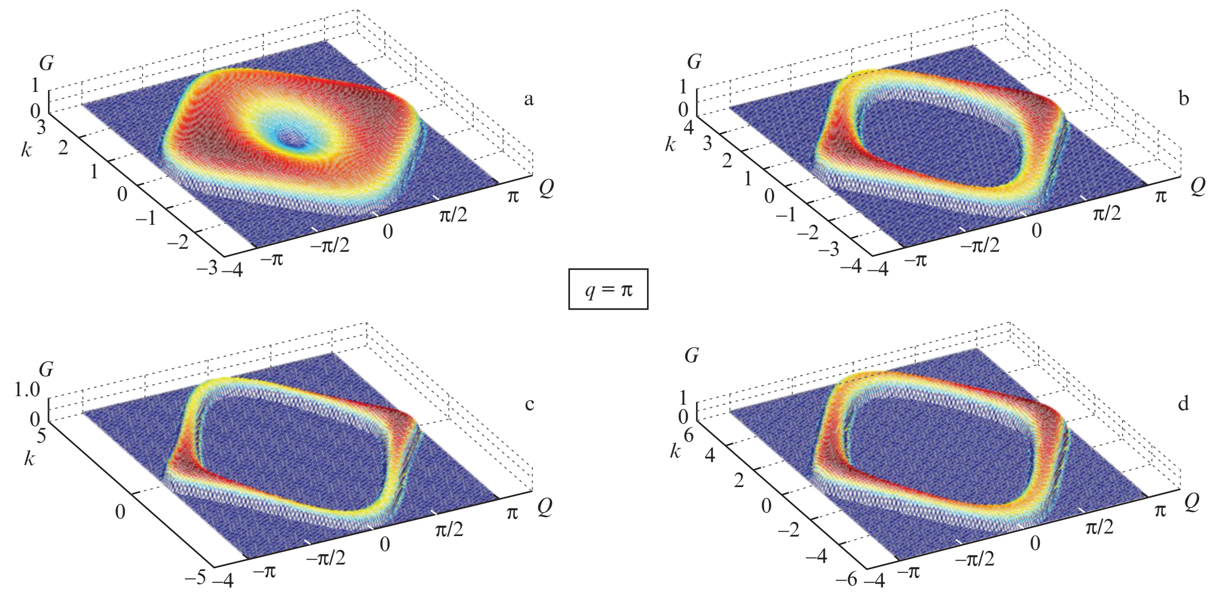

Figure 5. Dependences of the MI growth rate G on the reciprocal lattice vector Q and propagation constant k of small perturbations at (a) $P = 20$, $K_2 = K_3 = 0.5$, (b) $P = 20$, $K_2 = K_3 = 0.75$, (c) $P = 20$, $K_1 = 1$ and (d) $P = 40$, $K_2 = K_3 = 1$ ($K_1 = 1$, $r_1 = r_2 = 0.1$, $f = 1$, $q = \pi$).

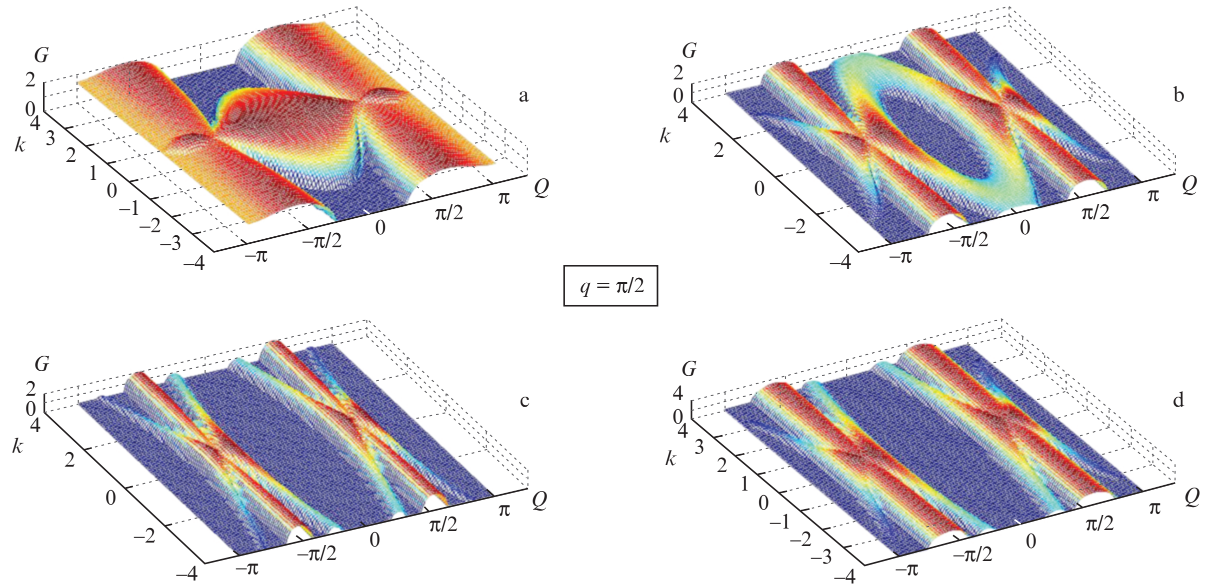

Figure 6. Dependences of the MI growth rate G on the reciprocal lattice vector Q and propagation constant k of small perturbations at (a) $P = 40$, $K_2 = K_3 = 1$, (b) $P = 40$, $K_2 = K_3 = 2$, (c) $P = 40$, $K_2 = K_3 = 3$ and (d) $P = 60$, $K_2 = K_3 = 3$ ($K_1 = 1$, $r_1 = r_2 = 0.1$, $f = 1$, $q = \pi/2$).

(see Fig. 3b). The minimum threshold power is reached at $q = \pi$ and the maximum – at $q = 0$.

Let us calculate numerically these powers. To do this, we will use the data of Eisenberg et al. [33], who investigated experimentally the formation of discrete solitons in a system of coupled AlGaAs waveguides: $K_1 = 1$ mm^{-1}, $r_1 = 5$ m^{-1} W^{-1}. In calculations we assume $K_2 = 0.25$ mm^{-1}. Then, $P^{\text{th}}(\pi) = 200$ W and $P^{\text{th}}(0) = 800$ W. The resulting thresholds are in good agreement with the results of Meier et al. [53], who investigated experimentally the MI in a system of coupled waveguides made of AlGaAs. An increase in the coupling coefficient K_2 reduces to a certain point the power threshold and when the threshold value K_2^{th} (12) is reached, the instability begins to disappear (Fig. 5a) independently of the power quantity (Figs 5b and 5c).

Figure 6 shows the effect of the coupling coefficient K_2 on the MI growth rate (10) at $q = \pi/2$, when the spectrum has a band gap (Fig. 2). Similarly to $q = \pi$, an increase in K_2 leads to a substantial decrease in the instability of plane waves with $q = \pi/2$; however, a gap in the spectrum increases the threshold value of K_2 as compared with the previous case, which can be easily seen if we compare Figs 5 and 6 [this also follows from relation (12)]. Moreover, it follows from (13) that the area of stability at $q = \pi/2$ is narrower than at $q = \pi$, and when $q = 0$, it entirely degenerates.

Thus, interaction with the waveguides located behind the nearest neighbours in the zigzag coupled waveguide array in question has a significant effect on the instability of the propagating uniform waves. Besides, the MI effect has some features that have not been observed previously in conventional arrays of coupled waveguides and Bragg gratings. However, in a zigzag array considered in [46], a similar effect of the MI absence was observed at small values of the coefficient of interaction with the waveguides that are not the nearest neighbours.

5.2. Influence of the parameter *f* on the MI growth rate

The dependence of the MI growth rate (9) on f is investigated by computer simulation. We have considered the nonlinear response of the system of coupled waveguides (variants 1 to 3, Section 5). Figures 7 and 8 show the effect of the parameter f on the instability of plane waves (2) at $q = \pi$ and $\pi/2$, respectively. The most stable region is observed in the case of variant 3 (Figs 7c, d and 8c, d). If PI waveguides are nonlinear and NI waveguides are linear, the instability occurs when the amplitude of the forward wave is greater than that of the backward wave (Figs 7c and 8c). In an opposite case, when NI waveguide are nonlinear and PI waveguides are linear, the instability increases with increasing $|f|$, if the amplitude of the backward wave is greater than that of the forward wave (Figs 7d and 8d). Variants 1 and 2 proved unstable, but the greatest instability occurs when NI waveguides are defocusing (Figs 7b, e and 8b, e). It should be noted that the calculations are made for $Q = 0$. For other values of Q there appear different peculiarities in the MI behaviour, which are discussed below. From a comparison of Figs 7 and 8 it can be concluded that the fundamental difference of the dependence of the MI growth rate (9) on the parameter f for plane waves with $q = \pi$ and $q = \pi/2$ is that in the former case the MI growth rate is symmetrical with respect to f, and in the second – not.

5.3. Influence of the NI waveguide nonlinearity r_2 and power P on the MI growth rate

Figure 9 shows the dependence of the MI growth rate (9) on the NI waveguide nonlinearity r_2 and power P at $q = \pi/2$. At $Q = \pi/2$ and positive values of f we observe an uncharacteristic dependence of the MI growth rate (9) on P for a defocusing NI waveguide and a focusing PI waveguide (Figs 9a and 9b). The modulation instability occurs only in a limited range of P values and is suppressed with increasing power. A similar effect is observed in an oppositely directed coupler [40]. At $Q = \pi$ and negative values of f the instability is completely absent (irrespective of power) when all the waveguides are focusing (Figs 9c and 9d). Similar features of the MI effect have not been observed previously in conventional Bragg gratings.

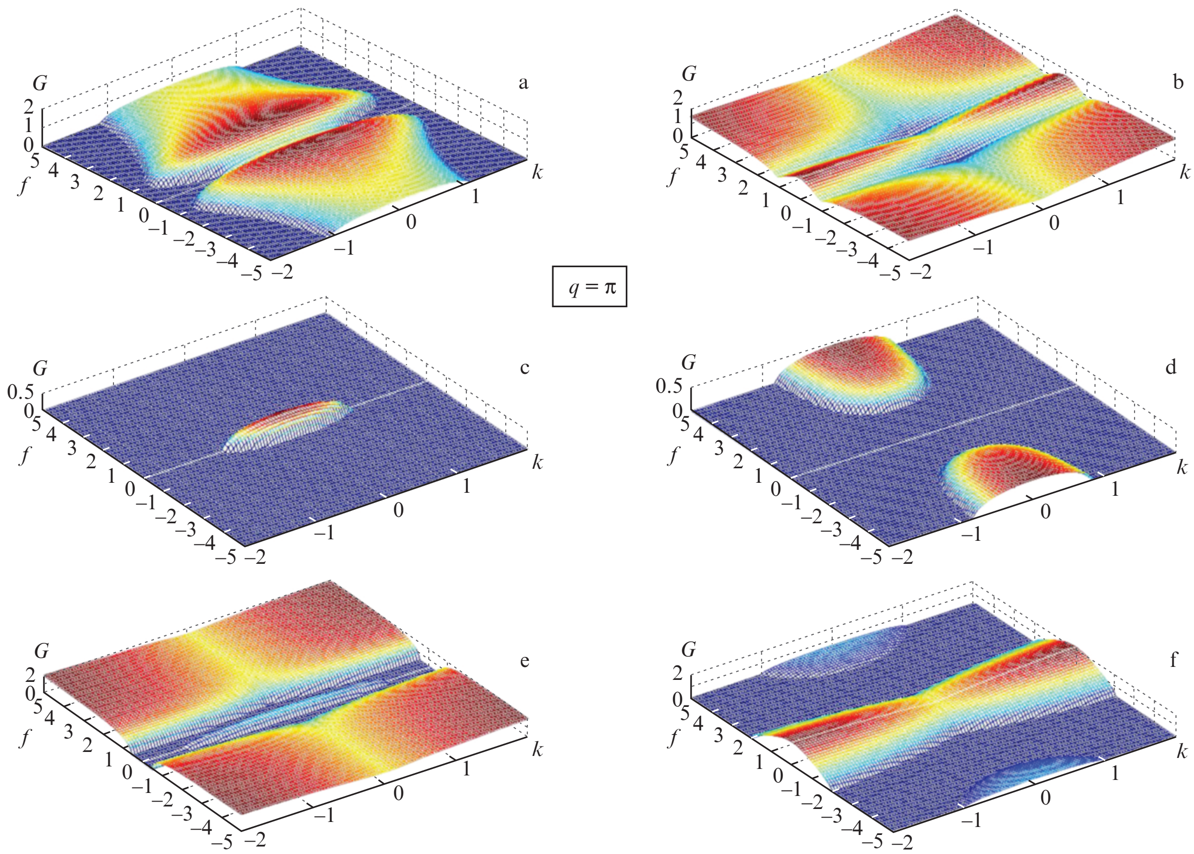

Figure 7. Dependences of the MI growth rate G on the parameter f and propagation constant k of small perturbations at (a) $r_1 = r_2 = 0.1$, (b) $r_1 = r_2 = -0.1$, (c) $r_1 = 0.1$, $r_2 = 0$, (d) $r_1 = 0$, $r_2 = 0.1$, (e) $r_1 = 0.1$, $r_2 = -0.1$ and (f) $r_1 = -0.1$, $r_2 = 0.1$ ($K_1 = 1$, $K_2 = K_3 = 0.25$, $P = 20$, $q = \pi$, $Q = 0$).

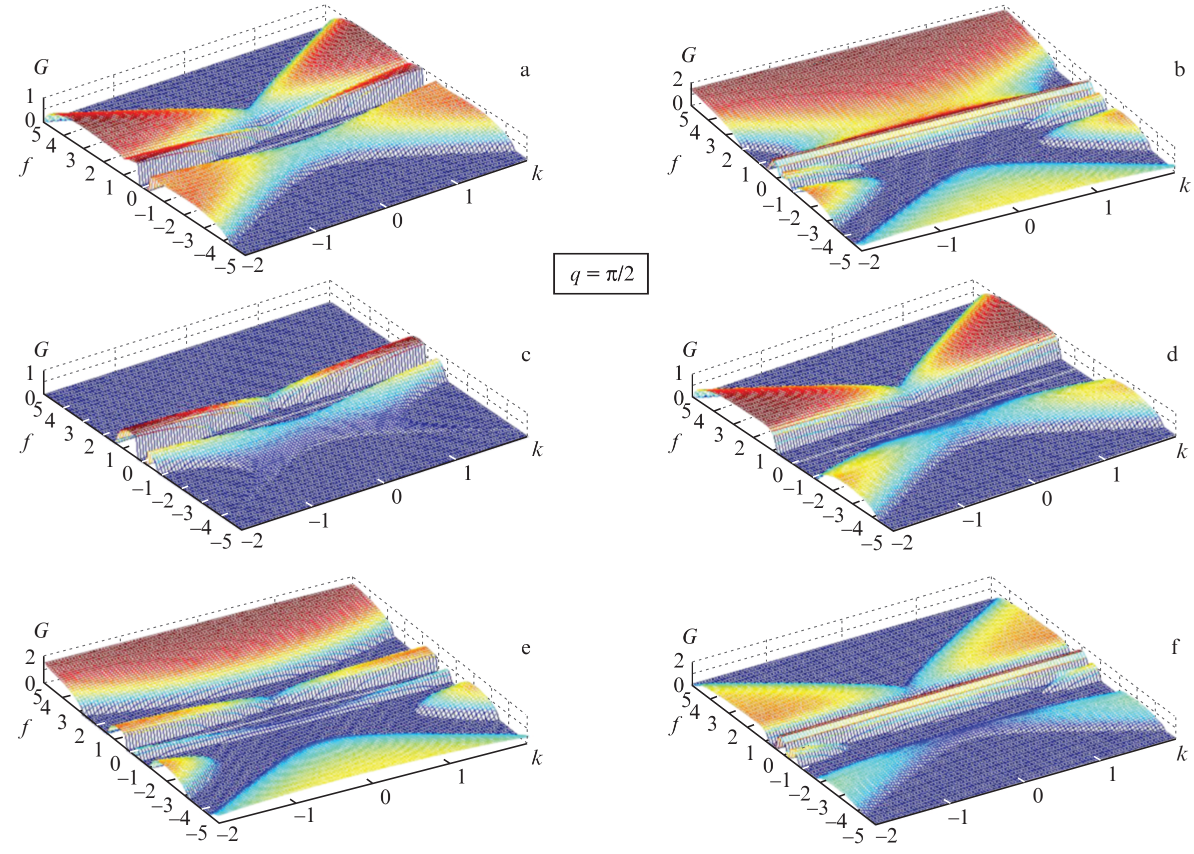

Figure 8. Dependences of the MI growth rate G on the parameter f and propagation constant k of small perturbations at (a) $r_1 = r_2 = 0.1$, (b) $r_1 = r_2 = -0.1$, (c) $r_1 = 0.1$, $r_2 = 0$, (d) $r_1 = 0$, $r_2 = 0.1$, (e) $r_1 = 0.1$, $r_2 = -0.1$ and (f) $r_1 = -0.1$, $r_2 = 0.1$ ($K_1 = 1$, $K_2 = K_3 = 0.25$, $P = 20$, $q = \pi/2$, $Q = 0$).

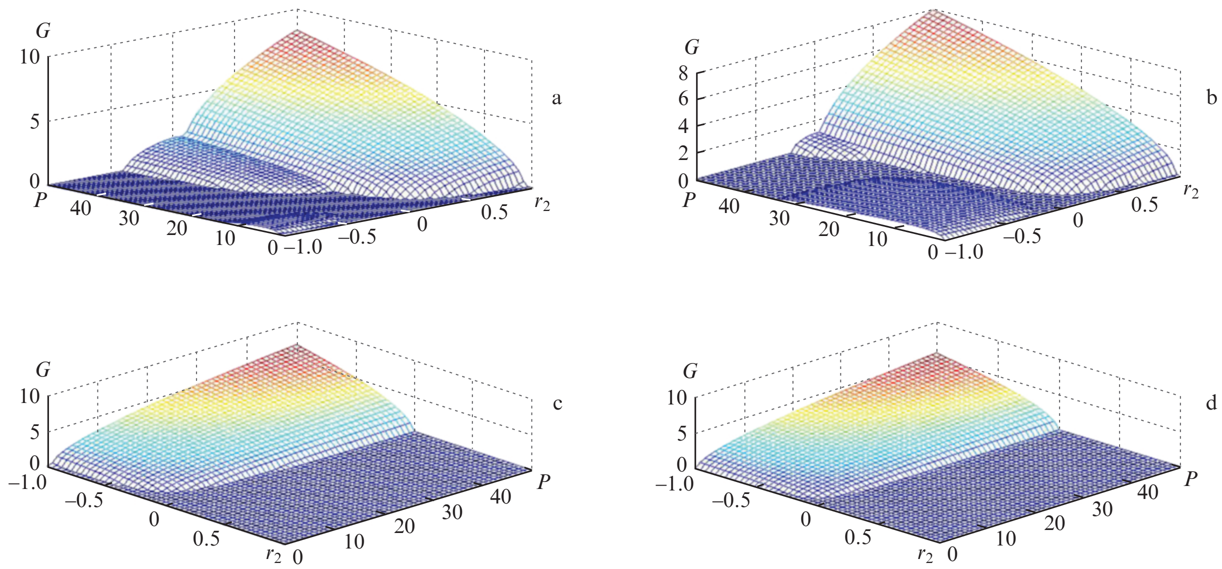

Figure 9. Dependences of the MI growth rate G on the power P and the NI waveguide nonlinearity r_2 at different values of f and q: (a) $f = 1$, $Q = \pi/2$, (b) $f = 1.5$, $Q = \pi/2$, (c) $f = -1$, $Q = \pi$ and (d) $f = -1.5$, $Q = \pi$ ($K_1 = 1$, $K_2 = K_3 = 0.25$, $r_1 = 0.1$, $q = \pi/2$, $k = 1$).

6. Conclusions

We have analytically investigated the effect of the modulation instability in a zigzag array of tunnel-coupled waveguides in which positive index waveguides alternate with negative index waveguides. All the waveguides are made of an optically nonlinear material with Kerr nonlinearity. The zigzag-like spatial configuration of a system of coupled waveguides allows one to take into account the interaction with the waveguides following the nearest neighbours.

The method of research is based on the theory of coupled modes. We have obtained particular solutions of coupled nonlinear equations describing the propagation of uniform waves in an array of coupled waveguides. The method of linear stability analysis is used to study the solutions for instability. We have found that the coupling between non-adjacent waveguides significantly affects the instability of the waves that are uniform along the waveguides. When the threshold is exceeded, the value of which depends on the wave vector of the reciprocal lattice of uniform waves, the instability begins to disappear regardless of the radiation power. Such a feature in the MI behaviour has not been observed previously in conventional Bragg gratings [48–50] and is a new result, because usually the instability only increases with increasing wave amplitude.

We have studied the influence of the ratio of the amplitudes of forward and backward waves on the stability of the particular solutions obtained. It is shown that if the spectrum has a gap, the instability growth rate is asymmetric with respect to f, and if the spectrum is gapless, the instability growth rate is symmetric. Different variants of the nonlinear response in waveguides are considered. If use is made of focusing positive index waveguides and defocusing negative index waveguides at certain values of the ratio of forward and backward waves and the reciprocal lattice vector, the instability is observed only in a limited range of power and suppressed with its further increase. A similar effect was observed in an oppositely directed coupler [40].

The investigation of the instability of uniform-along-the-waveguide waves propagating in the zigzag coupled waveguide array in question has made it possible to observe a new feature in the behaviour of the modulation instability in nonlinear periodic systems. With an increase in the coupling strength between the waveguides following the nearest neighbours, the instability vanishes at any values of the field power in the considered waveguides of the array, whereas the instability only increases with increasing wave amplitude (both in a nonlinear continuous medium and in nonlinear discrete structures).

Acknowledgements. The author is pleased to thank his colleagues V.G. Ivanov, I.S. Besedin and D.A. Abin for useful and fruitful discussions.

References

1. Flach S., Willis C.R. *Phys. Rep.*, **295**, 182 (1998).
2. Marklund M., Shukla P.K., Betschart G., et al. *J. Exp. Theor. Phys.*, **99**, 9 (2004).
3. Gauthier P., Gobert O., Comte M., et al. *Phys. Rev. A*, **65**, 033804 (2002).
4. Machabeli G.Z., Luo Q., Vladimirov S.V., et al. *Phys. Rev. E*, **65**, 036408 (2002).
5. Begue M.L., Ghizzo A., Bertrand P., et al. *J. Plasma Phys.*, **62**, 367 (1999).
6. Lai R., Sievers A.J. *Phys. Rev. B*, **57**, 3433 (1998).
7. Carr L.D., Brand J. *Phys. Rev. Lett.*, **92**, 040401 (2004).
8. Strecker K.E., Partridge G.B., Truscott A.G., et al. *Nature*, **417**, 150 (2002).
9. Peyrard M., Bishop A.R. *Phys. Rev. Lett.*, **62**, 2755 (1989); Peyrard M., Boesch R., Kourakis I., in *Collective Proton Transport in Hydrogen Bonded Systems, Proc. NATO Advanced Research Workshop (Heraklion, Crete, 1991)*. Ed. by T. Bountis (New York: Plenum Press, 1991).
10. Hasegawa A. *Opt. Lett.*, **9**, 288 (1984).
11. Peccianti M., Conti C., Assanto G., et al. *Nature*, **432**, 733 (2004).
12. Carvalho M.I., Singh S.R., Christodoulides D.N. *Opt. Commun.*, **126**, 167 (1996).
13. Gomila D., Zambrini R., Oppo G.-L. *Phys. Rev. Lett.*, **92**, 253904 (2004).
14. Fang H., Malendevich R., Schiek R., et al. *Opt. Lett.*, **25**, 1786 (2000).

15. Akhmanov S.A., Khokhlov R.V., Sukhorukov A.P., in *Laser Handbook*. Ed. by F.T. Arecchi, E.O. Schulz-DuBois (Amsterdam: North-Holland, 1972) p. 1151.
16. Xianqiong Zhong, Tingting Tang, Anping Xiang, Ke Cheng. *Opt Commun*., **284** (19), 4727 (2011).
17. Stegeman G.I., Segev M. *Science*, **280**, 889 (1998).
18. Agrawal G.P. *Nonlinear Fiber Optics* (San Diego: Academic Press, 2001).
19. Lederer F., Stegeman G.I., Christodoulides D.N., et al. *Phys. Rep*., **463** (1–3), 1 (2008).
20. Marcuse D. *Bell. Syst. Tech. J*., **50** (6), 1791 (1971).
21. Usievich B.A., Nurligareev D.Kh., Sychugov V.A., et al. *Kvantovaya Elektron*., **37** (6), 580 (2007) [*Quantum Electron*., **37** (6), 580 (2007)].
22. Winn J.N., Fink Y., Fan S., et al. *Opt. Lett*., **23** (20), 1573 (1998).
23. Chigrin D.N., Lavrinenko A.V., Yarotsky D.A., et al. *Appl. Phys. A*, **68**, 25 (1999).
24. Christodoulides D.N., Lederer F., Silberberg Y. *Nature (London)*, **424**, 817 (2003).
25. Kivshar Yu.S., Peyrard M. *Phys. Rev. A*, **46**, 3198 (1992).
26. Kivshar Yu.S., Salerno M. *Phys. Rev. E*, **49** (4), 3543 (1994).
27. Christodoulides D.N., Joseph R.I. *Opt. Lett*., **13** (9), 794 (1988).
28. Aceves A.B., De Angelis C., Trillo S., et al. *Opt. Lett*., **19** (5), 332 (1994).
29. Eisenberg H.S., Morandotti R., Silberberg Y., et al. *J. Opt. Soc. Am. B*, **19** (12), 2938 (2002).
30. Sukhorukov A.A., Kivshar Y.S., *Phys. Rev. E*, **65** (3), 036609 (2002).
31. Aceves A.B., De Angelis C., Peschel T., et al. *Phys. Rev. E*, **53** (1), 1172 (1996).
32. Efremidis N.K., Sears S., Christodoulides D.N. *Phys. Rev. E*, **66** (4), 046602 (2002).
33. Eisenberg H.S., Silberberg Y., Morandotti R., et al. *Phys. Rev. Lett*., **81** (16), 3383 (1998).
34. Szameit A., Garanovich I.L., Heinrich M., et al. *Nature Phys*., **5**, 271 (2009).
35. Darmanyan S., Relke I., Lederer F. *Phys. Rev. E*, **55** (6), 7662 (1997).
36. Maimistov A.I., Gabitov I.R., Lichinitser N.M. *Opt. Spektrosk*., **104** (2), 292 (2008).
37. Kazantseva E.V., Maimistov A.I., Ozhenko S.S. *Phys. Rev. A*, **80**, 43833 (2009).
38. Litchinitser N.M., Gabitov I.R., Maimistov A.I. *Phys. Rev. Lett*., **99**, 1139002 (2007).
39. Maimistov A.I., Kazantsev E.V. *Opt. Spektrosk*., **112** (2), 291 (2012).
40. Xiang Y., Wen Sh., Dai X., et al. *Phys. Rev. E*, **82**, 056605 (2010).
41. Tatsing P.H., Mohamadou A., Bouri C., Tiofack C.G.L., Kofane T.C. *J. Opt. Soc. Am. B*, **29** (12), 3218 (2012).
42. Maimistov A.I., Gabitov I.R. *Eur. Phys. J. Special Topics*, **147** (1), 265 (2007).
43. Maimistov A.I., Kazantseva E.V., Desyatnikov A.S., in *Kogerentnaya optika i opticheskaya spektroskopiya: XVI Vserossiiskaya molodezhnaya nauchnaya shkola, sbornik lektsionnykh zametok* (Coherent Optics and Optical Spectroscopy: XVI All-Russian Youth Scientific School, a Collection of Lecture Notes) (Kazan: KFI, 2012) p. 21.
44. Zezyulin D.A., Konotop V.V., Abdullaev F.K. *Opt. Lett*., **37** (19) 3930 (2012).
45. Dovgiy A.A., Maimistov A.I. *Opt. Spektrosk*., **116** (4), 673 (2014).
46. Efremidis N.K., Christodoulides D.N. *Phys. Rev. E*, **65**, 056607 (2002).
47. Kazantseva E.V., Maimistov A.I. *Kvantovaya Elektron*., **43** (9), 807 (2013) [*Quantum Electron*., **43** (9), 807 (2013)].
48. Porsezian K., Senthilnathan K., Devipriya S. *IEEE J. Quantum Electron*., **41**, 789 (2005).
49. Ganapathy R., Senthilnathan K., Porsezian K. *J. Opt. B: Quantum Semiclas. Opt*., **6**, S436 (2004).
50. Litchinitser N.M., McKinstrie C.J., de Sterke C.M., Agrawal G.P. *J. Opt. Soc. Am. B*, **18**, 45 (2001).
51. Maimistov A.I., Basharov A.M. *Nonlinear Optical Waves* (Dordrecht – Boston – London: Kluwer Academic Publishers, 1999).
52. Xiao S., Drachev V.P., Kildishev A.V., Ni X., Chettiar U.K., Yuan H.-K., Shalaev V.M. *Nature*, **466**, 735 (2010).
53. Meier J., Stegeman G.I., Christodoulides D.N., Silberberg Y., Morandotti R., Yang H., Salamo G., Sorel M., Aitchinson J.S. *Phys. Rev. Lett*., **92**, 163902 (2004).