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Consistent dynamics of the components of an elliptically polarised wave with zero mean amplitudes in a nonlinear isotropic gyrotropic medium in the adiabatic approximation

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Abstract. **The adiabatic approximation is used to obtain an analytical solution to a nonintegrable problem of propagation of a plane elliptically polarised light wave with zero mean amplitudes of orthogonal circularly polarised field components through an isotropic gyrotropic medium with local and nonlocal components of Kerr nonlinearity and second-order group velocity dispersion. We describe the aperiodic evolution of bound (attributable to the medium nonlinearity) paired states, which are responsible for the propagation of two orthogonal polarisation components – cnoidal waves with significantly different periods.**

Keywords: *cubic nonlinearity, spatial and frequency dispersion, linear and nonlinear gyrotropy, nonlinear Schrödinger equation, elliptical polarisation, adiabatic approximation, bound states, aperiodic dynamics.*

1. Introduction

Propagation of a plane elliptically polarised wave through an isotropic medium with Kerr nonlinearity and second-order group velocity dispersion is described by a system of two coupled nonlinear Schrödinger equations. In the general case, this system is not integrable [1 – 4] and, taking into account the terms responsible for linear and nonlinear gyrotropy, has the form [5 – 7]

$$\frac{\partial A_\pm}{\partial z} - \frac{\mathrm{i}k_2}{2}\frac{\partial^2 A_\pm}{\partial t^2} + \mathrm{i}[\mp\rho_0 + (\sigma_1/2 \mp \rho_1)|A_\pm|^2 + (\sigma_1/2 + \sigma_2)|A_\mp|^2]A_\pm = 0. \quad (1)$$

Here, $A_\pm(z,t)$ are the truncated amplitudes of the field components with the right- and left-hand circular polarisations and frequency ω propagating along the z axis; t is the time in the running coordinate system; the constant $k_2 = \partial^2 k/\partial\omega^2$ characterises the dispersion; and k is the wavenumber. Parameters $\sigma_1 = 4\pi\omega^2\chi^{(3)}_{xyxy}/(kc^2)$ and $\sigma_2 = 2\pi\omega^2\chi^{(3)}_{xxyy}/(kc^2)$ are determined by the independent components of the local cubic nonlinearity tensor $\chi^{(3)}(\omega;-\omega,\omega,\omega)$, and $\rho_{0,1} = 2\pi\omega^2\gamma_{0,1}/c^2$ are defined through the pseudoscalar constants $\gamma_{0,1}$ of linear and nonlinear gyration. The latter terms take into account the spatial nonlocality of these processes when the amplitudes of the field components slowly vary in the direction of z propagation on the wavelength scales.

A number of numerical [5, 6] and particular analytical [7 – 10] solutions to system (1) are known. To construct its approximate solutions, the linearization method was used [11, 12]. It was shown that excitation of one of the normal (in-phase or anti-phase) nonlinear modes results in a periodic regime of changes in the polarisation state, i.e., in propagation of elliptically polarised cnoidal waves. Otherwise, the Stokes parameters [13] change irregularly due to the beatings, because the frequencies of the two modes in the general case are incommensurable. In paper [14] system (1) was solved by using the adiabatic approximation [15 – 17]. Within the framework of the latter there were constructed bound (attributable to the nonlinearity) paired states of the field components corresponding to the consistent propagation of two waves – orthogonally polarised components with sign-alternating and sign-constant amplitudes and significantly different periods.

Below we also use the adiabatic approximation but consider a scenario that is slightly different with respect to [14]. In this scenario the amplitudes of both field components are sign alternating and their mean values are equal to zero. As is known, the signal, in which the zero-frequency harmonic in the Fourier spectrum is absent, is optimal for transmission along long paths. We will show that in this case there are similar (to those described in [14]) bound complexes with significantly differently scaled but consistent evolution of orthogonally polarised components of the wave in time, because the contributions of the latter to the total energy of the system are determined by the second-order moments (i.e., their intensities) and are not zeroed. Another important feature of the resulting approximate solution is the absence of exact particular solutions of (1) for the chosen values of the nonlinearity and gyrotropy parameters.

Note that from the practical point of view, the resulting periodic and aperiodic solutions may be relevant in the design of the fibre optic communication lines, the development of Faraday decoupling elements for high-power lasers with high-Q ring resonators, etc.

2. Potential energy and adiabatic approximation

As in [14], we first separate the variables, assuming that

$$A_\pm(z,t) = r_\pm(t)\exp(\mathrm{i}\kappa_\pm z), \quad (2)$$

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where $\kappa_\pm$ are the separation constants, and $r_\pm(t)$ are the unknown real functions. Substituting expression (2) into (1), for $k_2 \neq 0$, we obtain the same (as in [14]) system of ordinary differential equations

$$\frac{d^2 r_\pm}{dt^2} = \frac{1}{k_2}[2\Delta\kappa_\pm + (\sigma_1 \mp 2\rho_1)r_\pm^2 + (\sigma_1 + 2\sigma_2)r_\mp^2]r_\pm. \quad (3)$$

Here, $\Delta\kappa_\pm = \kappa_\pm \mp \rho_0$. Below, the variables and parameters in (3) are considered dimensionless and normalised because of the choice of unit values for the constants describing the linear gyrotropy ($\rho_0 = 1$), dispersion ($k_2 = 1$) and one of the components of the local cubic nonlinearity ($\sigma_1 = 1$), which is equivalent to the following substitution of variables: $z\rho_0 \to z$, $t\sqrt{\rho_0/k_2} \to t$, $r_\pm\sqrt{\sigma_1/\rho_0} \to r_\pm$, $\sigma_2/\sigma_1 \to \sigma_2$, $\rho_1/\sigma_1 \to \rho_1$ and $\kappa_\pm/\rho_0 \to \kappa_\pm$.

Considering now (3) as a system describing the motion (evolution of the radius vector $\boldsymbol{r} = \{r_-, r_+\}$) of a unit-mass point, we introduce its potential energy $U(r_-, r_+)$:

$$U(r_-, r_+) = -\frac{1}{4k_2}[4\Delta\kappa_+ r_+^2 + 4\Delta\kappa_- r_-^2 + (\sigma_1 - 2\rho_1)r_+^4$$

$$+ (\sigma_1 + 2\rho_1)r_-^4 + 2(\sigma_1 + 2\sigma_2)r_+^2 r_-^2]. \quad (4)$$

In contrast to [14], we are interested in the equilibrium point $\boldsymbol{r}_{eq} = \{\tilde{r}_-, \tilde{r}_+\} = \{0, 0\}$, corresponding to the minimum of the potential energy U due to certain choice of the parameters. To analyse the behaviour of the solutions in the vicinity of this point, we will use the same (as in [14]) method of search for approximate solutions – adiabatic approximation [15–17].

Small oscillations in orthogonal directions $r_\pm$ in the vicinity of point $\boldsymbol{r}_{eq} = 0$ are the normal modes with frequencies $\omega_\pm$, given by the expressions

$$\omega_\pm^2 = \left.\frac{\partial^2 U(r_-, r_+)}{\partial r_\pm^2}\right|_{\boldsymbol{r}_{eq}} = -\frac{2}{k_2}\Delta\kappa_\pm. \quad (5)$$

Therefore, in view of the chosen normalisation ($k_2 = 1$) we are interested in the $\Delta\kappa_\pm < 0$ values corresponding to a local minimum of U and the oscillations, the frequency of which in one of the directions (for example, r_-) is much smaller than in the other (r_+), i.e.,

$$\Delta\kappa_+ \ll \Delta\kappa_-. \quad (6)$$

In this case, the adiabatic approximation should be applicable.

An alternative to approximate solutions of (3) with respect to the position of point $\boldsymbol{r}_{eq} = 0$ could be the exact particular solutions of sc and cs types [10, 18, 19], which may exist at $\rho_1 > 0$, $k_2(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2) < 0$ and $-\rho_1 < \sigma_2 < \rho_1$ and at $\rho_1 < 0$, $k_2(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2) > 0$ and $\rho_1 < \sigma_2 < -\rho_1$. However, as we have seen, in contrast to [14], in the case we consider below for $k_2 = 1$, $\sigma_1 = 1$, $\rho_0 = 1$, $\sigma_2 = 0.4$, $\rho_1 = -0.45$, $\Delta\kappa_+ = -10$ and $\Delta\kappa_- = -1$ these particular solutions do not exist.

3. Solution of the problem in the adiabatic approximation

In cases when inequality (6) is fulfilled, the approximate solutions are constructed according to the scheme described in [14]. First, for the selected values of $\Delta\kappa_\pm$ we fix an arbitrary (valid) value r_- and seek for a solution for $r_+(t)$ in the selected class of elliptic functions. In this case, we solve the equation

$$\frac{d^2 r_+}{dt^2} = \frac{1}{k_2}[2\Delta\kappa_+ + (\sigma_1 - 2\rho_1)r_+^2 + (\sigma_1 + 2\sigma_2)r_-^2]r_+, \quad (7)$$

assuming that

$$r_+(t) = B_+ \mathrm{sn}(\nu_+ t, \mu_+). \quad (8)$$

Here, $\mathrm{sn}(\nu t, \mu)$ and $\mathrm{cn}(\nu t, \mu)$ (see below) are the Jacobi elliptic functions with the modulus $0 \leqslant \mu \leqslant 1$ [20]. Substituting (8) into (7), we find

$$B_+^2 = -\frac{2\mu_+^2[2\Delta\kappa_+ + (\sigma_1 + 2\sigma_2)r_-^2]}{(1 + \mu_+^2)(\sigma_1 - 2\rho_1)}, \quad (9a)$$

$$\nu_+^2 = -\frac{2\Delta\kappa_+ + (\sigma_1 + 2\sigma_2)r_-^2}{k_2(1 + \mu_+^2)}. \quad (9b)$$

One of the parameters in (8) and (9) (we assume that it is μ_+) remains free, and therefore, we define a whole family of solutions of the corresponding type.

Next, we average the second equation of (3) over the fast oscillations $r_+(t)$:

$$\frac{d^2 r_-}{dt^2} = \frac{1}{k_2}[2\Delta\kappa_- + (\sigma_1 + 2\rho_1)r_-^2 + (\sigma_1 + 2\sigma_2)\langle r_+^2\rangle_t]r_-. \quad (10)$$

Here $\langle ... \rangle_t$ denotes averaging over time. Because

$$\langle \mathrm{sn}^2(\nu_+ t, \mu_+)\rangle_t = \frac{1}{\mu_+^2}\left[1 - \frac{E(\mu_+)}{K(\mu_+)}\right], \quad (11)$$

where $K(\mu_+)$ and $E(\mu_+)$ are the complete elliptic integrals of the first and second kind [20], and taking into account (9a), we obtain

$$\langle r_+^2\rangle_t = -\frac{2[2\Delta\kappa_+ + (\sigma_1 + 2\sigma_2)r_-^2]}{(1 + \mu_+^2)(\sigma_1 - 2\rho_1)}\left[1 - \frac{E(\mu_+)}{K(\mu_+)}\right], \quad (12)$$

and equation (10) takes the form

$$\frac{d^2 r_-}{dt^2} = \frac{1}{k_2}\left\{2\Delta\kappa_- - \frac{4\Delta\kappa_+(\sigma_1 + 2\sigma_2)}{(1 + \mu_+^2)(\sigma_1 - 2\rho_1)}\left[1 - \frac{E(\mu_+)}{K(\mu_+)}\right]\right.$$

$$\left. + \left[(\sigma_1 + 2\rho_1) - \frac{2(\sigma_1 + 2\sigma_2)^2}{(1 + \mu_+^2)(\sigma_1 - 2\rho_1)}\right]\left[1 - \frac{E(\mu_+)}{K(\mu_+)}\right]r_-^2\right\}r_-. \quad (13)$$

Now we can find an analytical solution for the second component of the field $r_-(t)$ in the corresponding class of elliptic functions. Substituting

$$r_-(t) = B_- \mathrm{cn}(\nu_- t, \mu_-) \quad (14)$$

into (13), we obtain

$$B_-^2 = -\{4\mu_-^2[\Delta\kappa_-(1 + \mu_+^2)(\sigma_1 - 2\rho_1)K(\mu_+)$$

$$-2\Delta\kappa_+(\sigma_1 + 2\sigma_2)(K(\mu_+) - E(\mu_+))]\}\{(2\mu_-^2 - 1)[(1 + \mu_+^2) \times$$

$$\times(\sigma_1^2 - 4\rho_1^2)K(\mu_+) - 2(\sigma_1 + 2\sigma_2)^2(K(\mu_+) - E(\mu_+))]\}^{-1}, \quad (15a)$$

$$v_-^2 = \frac{2}{k_2(2\mu_-^2 - 1)}$$

$$\times\left\{\Delta\kappa_- - \frac{2\Delta\kappa_+(\sigma_1 + 2\sigma_2)}{(1 + \mu_+^2)(\sigma_1 - 2\rho_1)}\left[1 - \frac{E(\mu_+)}{K(\mu_+)}\right]\right\}. \quad (15b)$$

Note that, although point $\mu_- = 2^{-1/2}$ in (15) is singular, the parameter μ_- is also independent.

Substituting (14) into (8) and (9), we derive the final expression for the desired consistent evolution of the fast circularly polarised component of the light field in the form

$$r_+(t) = B_+\mathrm{sn}(v_+t, \mu_+), \quad (16a)$$

$$B_+^2 = -\frac{2\mu_+^2[2\Delta\kappa_+ + (\sigma_1 + 2\sigma_2)B_-^2\mathrm{cn}^2(v_-t, \mu_-)]}{(1 + \mu_+^2)(\sigma_1 - 2\rho_1)}, \quad (16b)$$

$$v_+^2 = -\frac{2\Delta\kappa_+ + (\sigma_1 + 2\sigma_2)B_-^2 cn^2(v_-t, \mu_-)}{k_2(1 + \mu_+^2)}, \quad (16c)$$

where B_-^2 and v_-^2 are given in (15). Equations (14)–(16) define a desired consistent solution $r_\pm(t)$ of problem (3) in the first iteration [correction of the dependence $r_+(t)$ to (16) by substituting expression (14) into (8) and (9)] of the calculation in the adiabatic approximation. In principle, the iterative series can be continued [21, 22] by refining the expression for $\langle r_+^2 \rangle_t$ followed by substitution of (16) into (10), and so on. However, we will not do this here.

Recall again that the parameters $\mu_\pm$ in (14)–(16) are free and their variations define a family of approximate solutions of the corresponding class. Restrictions on the values of these parameters are only due to the fact that at any time moments t, the condition of applicability of the adiabatic approximation must be valid. However, because the spectrum of nonlinear oscillations is continuous, we will compare the squares of their periods, which describe the position of the maxima of the spectral density for the corresponding oscillations. Since according to [20] for the solutions (14) and (16), these periods are defined by the expressions $T_+ = 4K(\mu_+)/v_+$ and $T_- = 4K(\mu_-)/v_-$, instead of (6) we obtain

$$\frac{K^2(\mu_-)v_+^2}{K^2(\mu_+)v_-^2} \ll 1. \quad (17)$$

Taking into account above relations (15) and (16), condition (17) allows us now to choose such values of $0 < \mu_\pm < 1$ ($\mu_- \neq 2^{-1/2}$), which provide the possibility of using the adiabatic approximation.

Note, however, that the correct assessment of the applicability limits of the adiabatic approximation is an independent, complex and still unsolved problem (see, e.g., [32]).

4. Behaviour of the solutions

The behaviour of approximate solutions (14)–(16) is illustrated in Fig. 1, which shows the greyscale map of the $U(r_-, r_+)$ distribution in the vicinity of the equilibrium point $\boldsymbol{r}_{\mathrm{eq}} = 0$. The values of the parameters ($k_2 = 1, \sigma_1 = 1, \rho_0 = 1, \sigma_2 = 0.4, \rho_1 = -0.45$, $\Delta\kappa_+ = -10$ and $\Delta\kappa_- = -1$) are chosen such that the potential well is elongated in the r_- direction and forms a long 'channel'. The thick solid line shows the trajectory $r_+(r_-)$, corresponding to the solution of (14)–(16) for $\mu_+ = 0.3584$ and $\mu_- = 0.95$ at $t \in [0, 43]$. Note that an alternative periodic exact particular solution of sc type [8, 10] in the case under consideration (for the parameters used above) is absent.

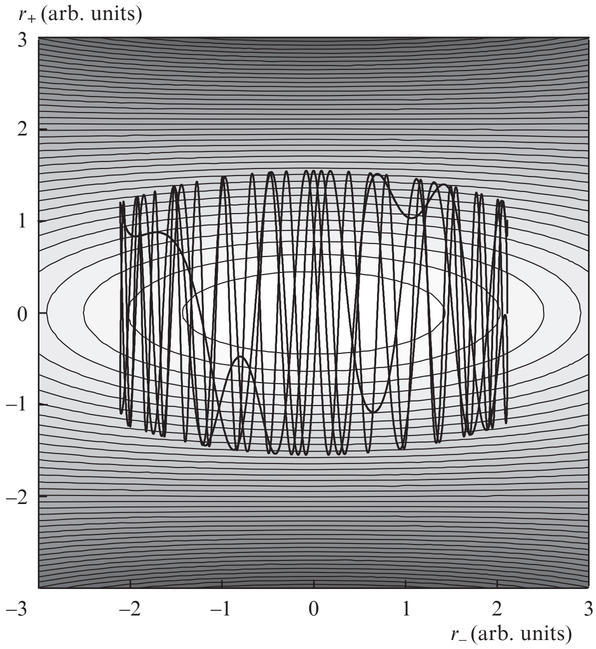

Figure 1. Greyscale map of $U(r_-, r_+)$ distribution in the vicinity of point $\boldsymbol{r}_{\mathrm{eq}} = 0$ at $k_2 = 1, \sigma_1 = 1, \rho_0 = 1, \sigma_2 = 0.4, \rho_1 = -0.45, \Delta\kappa_+ = -10, \Delta\kappa_- = -1$. The potential well is elongated in the r_-direction. The thick solid line shows the $r_+(r_-)$ trajectory corresponding to the solution of (14)–(16) at $\mu_+ = 0.3584$ and $\mu_- = 0.95$ for $t \in [0, 43]$.

Here, we should make two important remarks. First, we consider the case of excitation of two normal nonlinear modes (Fig. 1). In this case, the possibility of using the adiabatic approximation is determined by the significant difference of their frequencies. Therefore, the solutions with multiple periods of changes in $r_\pm(t)$ (the case of regular evolution) can be considered almost impossible. Second, the $r_+(r_-)$ trajectory of the solution (Fig. 1) is a classical Lissajous figure [23]. However, in the case of independent harmonic oscillations, the phase space of the system [part of the $\{r_-, r_+\}$ plane filled in during the $r_\pm(t)$ evolution] would represent a rectangle [23]. With the consistent evolution of the $r_\pm(t)$ field components we deal with nonlinear Lissajous figures [24], and the phase space of the system is determined by the geometry of the potential well in the vicinity of point $\boldsymbol{r}_{\mathrm{eq}} = 0$ (Fig. 1). In principle, this allows the nonlinear coupling behaviour to be investigated. Note that a similar approach (method of trajectories) is widely used in nonlinear dynamics [24, 25], chemical physics [26–28], nonlinear acoustics [29] and rheology [30, 31].

Dependences $r_+(t)$ (16) and $r_-(t)$ (14)–(15) for the same values of the parameters ($k_2 = 1, \sigma_1 = 1, \rho_0 = 1, \sigma_2 = 0.4, \rho_1 = -0.45, \Delta\kappa_+ = -10, \Delta\kappa_- = -1$ at $\mu_+ = 0.3584$ and $\mu_- = 0.95$) and the behaviour of the evolution of the intensity $I(t) = r_+^2(t) + r_-^2(t)$ for the obtained approximate solution are shown in Fig. 2. In the initial parts of the dependences one can clearly see the slow modulation of high-frequency nonlinear oscillations of a rapidly varying field component $r_+(t)$ (Fig. 2a) and

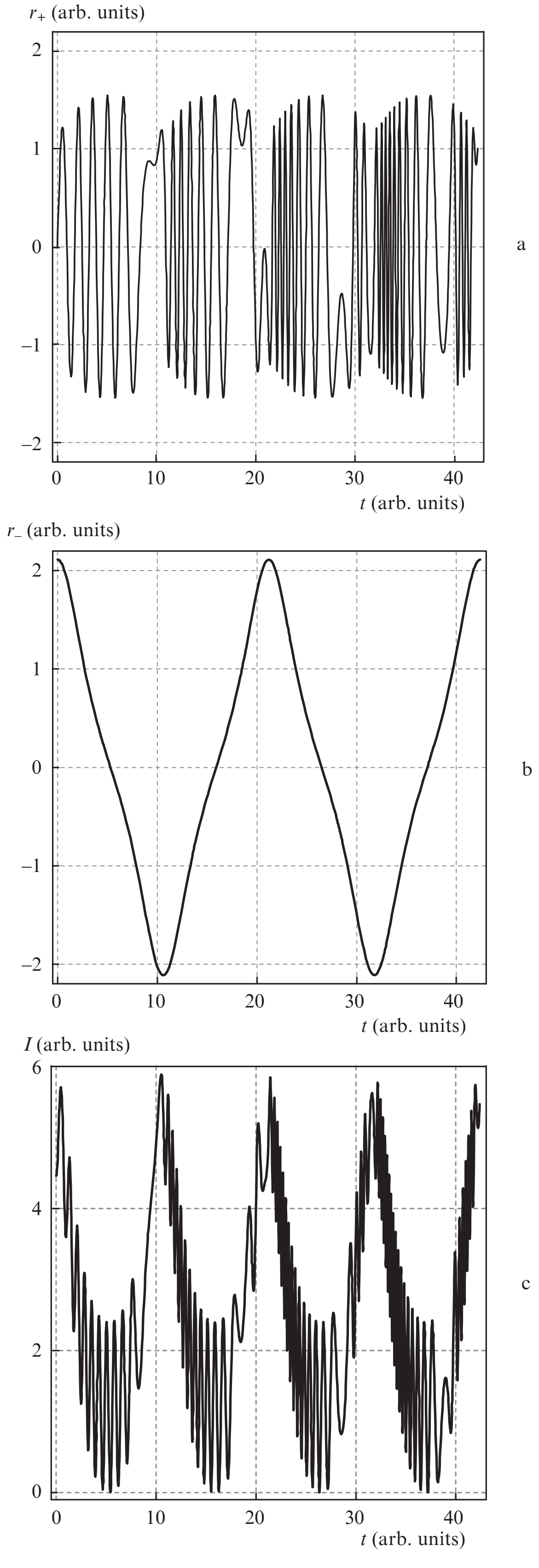

Figure 2. Dependences $r_+(t)$ (16) and $r_-(t)$ (14)–(15) (a and b) as well as $I(t) = r_+^2(t) + r_-^2(t)$ (c) at $k_2 = 1, \sigma_1 = 1, \rho_0 = 1, \sigma_2 = 0.4, \rho_1 = -0.45, \Delta\kappa_+ = -10, \Delta\kappa_- = -1, \mu_+ = 0.3584$ and $\mu_- = 0.95$ for $t \in [0, 43]$.

the total intensity $I(t)$ (Fig. 2c) by low-frequency nonlinear oscillations of a slowly varying component $r_-(t)$ (Fig. 2b). Then, the evolution of $r_+(t)$ and $I(t)$ clearly becomes more irregular and the brokenness of the corresponding dependences grows.

A gradual (with increasing t) change in the behaviour of dependences $r_+(t)$ and $I(t)$ (Fig. 2) can be explained as follows. A set of harmonics, which are present in the Fourier spectrum of the amplitude $r_+(t)$, as a result of slow modulation of its parameters, is phased in a certain way by a particular choice of initial conditions at the time moment $t = 0$. The initial phase relations can be modified, for example, due to changes in the initial phases of the functions sn and cn, which in this case are also the solutions of the corresponding equations. During the subsequent evolution (propagation), the phase shifts between different harmonics vary due to frequency dispersion, and we observe their gradual 'randomisation' (the system gradually 'forgets' the initial conditions). This fact distinguishes this stage from the initial stage of evolution, where the dependences $r_+(t)$ and $I(t)$ are almost regular. The calculation of the dynamics of the components for different initial phases of sn and cn functions confirmed the fact that for chosen values of the parameters the initial stage duration Δt is always ~10. The above-presented interpretation is very similar to the classical interpretation of multiphoton processes (including so-called non-Markovian relaxation processes) presented, for example, in [32].

Figure 3 illustrates the foregoing and shows the time dependence of $\arg = tv_+(t)$ (solid line) of the elliptic function sn in expression (16a). One can easily see that as a result of strictly periodic modulation of $v_+(t)$ (16c), the $\arg t$ dependence becomes very complex and aperiodic. The open circles show the points $\{t_i, \arg t_i\}$, at which $\arg t_i = 4K(\mu_+)n$, where i, $n = 0, 1, 2, ...$ are the integers, and the initial value of the elliptic function sn remains the same [20]. One can see from the presented dependence that the points $t = t_i$ are arranged nonuniformly on the time axis. Moreover, in the course of evolution (change in t_i) the $\arg t$ value varies nonmonotonically and at sufficiently large n the same arg value appears at different

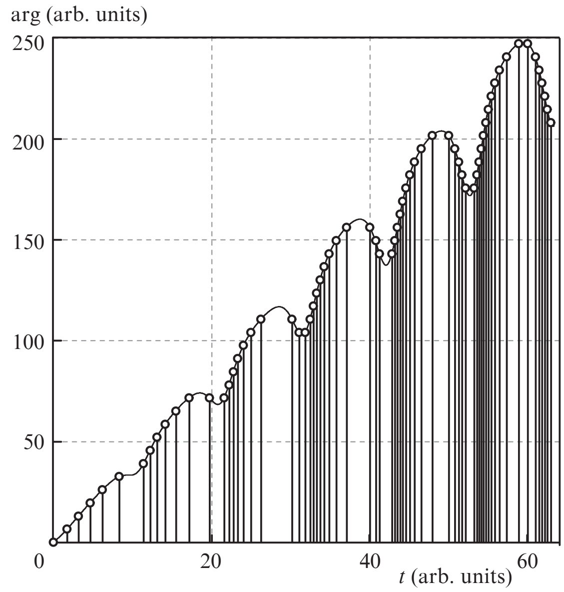

Figure 3. Time dependence of $\arg = tv_+(t)$ (solid curve) of the elliptic function sn (16a) at $k_2 = 1, \sigma_1 = 1, \rho_0 = 1, \sigma_2 = 0.4, \rho_1 = -0.45, \Delta\kappa_+ = -10, \Delta\kappa_- = -1, \mu_+ = 0.3584$ and $\mu_- = 0.95$ for $t \in [0, 63]$. The open circles show the points at which $\arg = 4K(\mu_+)n$, where $n = 0, 1, 2, ...$ is an integer.

time moments. This means that during the evolution, the change in the field components $r_+(t)$ over time in accordance with (16) becomes purely aperiodic, which is fully consistent with the behaviour of the dependences shown in Fig. 2.

'Chaotic' behaviour of the evolution of the polarisation state is illustrated in Fig. 4, showing the trajectory of the end of the Stokes vector $\boldsymbol{s}$ on the Poincaré sphere [13] at $k_2 = 1$, $\sigma_1 = 1$, $\rho_0 = 1$, $\sigma_2 = 0.4$, $\rho_1 = -0.45$, $\Delta\kappa_+ = -10$ and $\Delta\kappa_- = -1$ for the approximate solution of (14)–(16) at $\mu_+ = 0.3584$ and $\mu_- = 0.95$. For greater clarity we assume that the observation plane moves with constant velocity and $t = 0.54z$ ($t \in [0, 43]$). One can see a purely aperiodic behaviour of the evolution of the polarisation state. The reason behind the irregular changes, as in [8, 9, 11, 12], is incommensurable frequencies and dephasing of the Fourier spectrum harmonics of the $r_+(t)$ function.

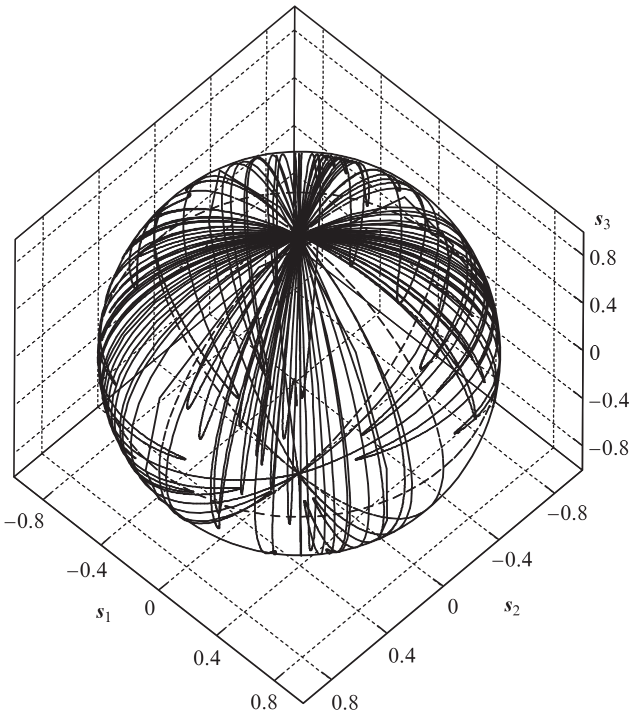

Figure 4. Trajectories of the motion of the end of the vector $\boldsymbol{s}$ on the Poincaré sphere at $k_2 = 1$, $\sigma_1 = 1$, $\rho_0 = 1$, $\sigma_2 = 0.4$, $\rho_1 = -0.45$, $\Delta\kappa_+ = -10$ and $\Delta\kappa_- = -1$ for the approximate solution of (14)–(16) at $\mu_+ = 0.3584$ and $\mu_- = 0.9$. The observation plane moves at a constant velocity so that $t = 0.54z$, $t \in [0, 43]$.

5. Conclusions

Thus, using the adiabatic approximation, we have obtained an analytical solution to a nonintegrable problem of propagation of a plane elliptically polarised light wave with zero mean amplitudes of orthogonal circularly polarised components of the light field through an isotropic gyrotropic medium with local and nonlocal components of Kerr nonlinearity and second-order group velocity dispersion. We have found that in this case, as in those situations that were previously considered in [14], there exist bound (attributable to the nonlinearity) states of orthogonally polarised components of the light field with significantly differently scaled but time consistent evolution. The reason behind this is that the contributions of the latter to the total energy of the system are not zeroed in this situation because they are defined by the second-order moments (intensity). It is shown that the solutions obtained in this approximation correspond to the case of simultaneous excitation of two normal nonlinear modes at substantially different frequencies, i.e., they are responsible for the propagation regimes (aperiodic evolution), resembling the polarisation 'chaos'. Note also that in the situation considered here (for the specific choice of the values of a set of parameters), exact particular periodic solutions, which would be an alternative to the class of approximate solutions we have described here, do not exist.

We should also note that a gradual transition from a situation with almost regular dynamics of the fast field component at the initial stage to an aperiodic ('chaotic') evolution in the subsequent stages is very similar to the classical transition from regular dynamics (inhomogeneous broadening of the transition) to irreversible relaxation processes due to the so-called multiphoton transitions described, for example, in [32].

In the vicinity of the considered minimum $\boldsymbol{r}_{\rm eq} = 0$, one can similarly construct various differently scaled solutions to problem (1) both for a potential well elongated along the r_+ axis, when the components r_+ and r_- will be slow and fast, respectively, and in the form of other combinations of elliptic functions: $\mathrm{cn}(\nu_+ t, \mu_+)$ and $\mathrm{sn}(\nu_- t, \mu_-)$, $\mathrm{sn}(\nu_+ t, \mu_+)$ and $\mathrm{sn}(\nu_- t, \mu_-)$, and $\mathrm{cn}(\nu_+ t, \mu_+)$ and $\mathrm{cn}(\nu_- t, \mu_-)$.

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