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Dependence of the compensation error on the error of a sensor and corrector in an adaptive optics phase-conjugating system

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Abstract. **In the framework of a statistical model of an adaptive optics system (AOS) of phase conjugation, three algorithms based on an integrated mathematical approach are considered, each of them intended for minimisation of one of the following characteristics: the sensor error (in the case of an ideal corrector), the corrector error (in the case of ideal measurements) and the compensation error (with regard to discreteness and measurement noises and to incompleteness of a system of response functions of the corrector actuators). Functional and statistical relationships between the algorithms are studied and a relation is derived to ensure calculation of the mean-square compensation error as a function of the errors of the sensor and corrector with an accuracy better than 10 %. Because in adjusting the AOS parameters, it is reasonable to proceed from the equality of the sensor and corrector errors, in the case the Hartmann sensor is used as a wavefront sensor, the required number of actuators in the absence of the noise component in the sensor error turns out 1.5–2.5 times less than the number of counts, and that difference grows with increasing measurement noise.**

Keywords: *adaptive optics system, phase conjugation, deformable mirror, control algorithm, sensor error, corrector error, compensation error, matching of the number of counts to the number of actuators.*

1. Introduction

Operational efficiency of an adaptive optics system (AOS) of phase conjugation essentially depends on the control algorithm that is being developed at the stage of theoretical research with regard to the system quality indicator. Based on that algorithm, information coming from the wavefront (WF) sensor is converted into control actions that are fed to the corrector actuators, thereby forming the correcting phase distribution. In accordance with AOS purposes, the AOS quality indicators can be divided into two groups: indicators for information optical systems and indicators for energy optical systems. In information systems, the quality indicator is commonly considered as a probability of a desired event (see, for example, [1–7]). In energy systems, WF approximation errors or WF gradient errors (see, for example, [1, 2, 4, 7]), which are directly related to a light intensity at an optical system focus and a beam divergence angle, are commonly used as a quality indicator.

This paper is a continuation and development of work [8]. In [8], in the framework of a statistical model of a phase-conjugating AOS, three algorithms are described, each of them intended for minimisation of one of the following (energy) quality indicators: error I of a WF sensor, error II of a WF corrector, and error III of WF compensation. Error I of the WF sensor is conditioned by discreteness and measurement noises; it is determined by taking into account the sensor parameters under the assumption of an ideal corrector capable of exactly reproducing any given distribution function. Error II of the WF corrector is caused by the fact that the basis of the response functions of actuators, in which the measured phase distribution is presented, generally represents an incomplete system of functions. This error is determined with regard to the corrector parameters under the assumption of an ideal WF sensor. Approximation errors I and II allow evaluation of the limiting possibilities of an AOS in the problem of the required phase distribution formation. Optimisation of parameters of the AOS designated for compensating the phase distortions of the field (the number of sensor and corrector channels, operation speed, dynamic range of correction, etc.) is based on the analysis of error III that simultaneously takes into account all the aforementioned sources of errors.

In this paper, we introduce a coupling matrix between the WF sensor and corrector into the mathematical model of the AOS, which allows us, in the framework of an integrated mathematical approach to description of algorithms I, II and III, to study the functional and statistical relationships between these algorithms and corresponding approximation errors. The main objective of this research is to study the functional dependence of the compensation error on the sensor and corrector errors and to justify the possibility of designing a parameter-adjusted WF sensor and a corrector.

2. Basic calculation relations

As a rule, the target characteristics of an energy AOS do not depend on the phase averaged over the output aperture. Therefore, mathematical AOS models often contain expressions in which the mean value of a function is subtracted from the function itself. Let us denote by angle brackets the operation of spatial averaging of the function $f(\boldsymbol{\rho})$ over the variable $\boldsymbol{\rho}$ [$\boldsymbol{\rho} = (x, y)$ is the radius vector of a point in the plane]:

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$$\langle f(\boldsymbol{\rho})\rangle = \frac{\int P(\boldsymbol{\rho}) f(\boldsymbol{\rho})\,\mathrm{d}\boldsymbol{\rho}}{\int P(\boldsymbol{\rho})\,\mathrm{d}\boldsymbol{\rho}},$$

where $P(\boldsymbol{\rho})$ is the aperture function [$P(\boldsymbol{\rho}) = 1$ within the output light aperture of the AOS and $P(\boldsymbol{\rho}) = 0$ outside the aperture]. Two functions differing by the average value of one of them are denoted by pairwise lowercase and capital letters of the same alphabet and phonation: for example, if the function $f(\boldsymbol{\rho})$ is introduced into the text, then $F(\boldsymbol{\rho}) = f(\boldsymbol{\rho}) - \langle f(\boldsymbol{\rho})\rangle$; herewith $\langle F(\boldsymbol{\rho})\rangle = 0$.

Let the function $\Phi(\boldsymbol{\rho})$ [$\phi(\boldsymbol{\rho})$] describe the measured (subject to reconstruction and/or correction) random phase distribution with a known statistics. The phase distribution function $\Psi(\boldsymbol{\rho})$ [$\psi(\boldsymbol{\rho})$] that is actually formed by the AOS is generally different from the function $\Phi(\boldsymbol{\rho})$ and has, similar to $\Phi(\boldsymbol{\rho})$, a stochastic nature. The AOS control algorithm should bring the difference

$$\Delta(\boldsymbol{\rho}) = \Phi(\boldsymbol{\rho}) - \Psi(\boldsymbol{\rho}) \quad (1)$$

closer to zero. As an AOS quality indicator, we consider the mean square (variance) of the phase error (1):

$$\Delta^2 = \overline{\langle \Delta^2(\boldsymbol{\rho})\rangle}, \quad (2)$$

where the bar denotes statistical averaging over the ensemble. In the problem of WF compensation, quantity (2) is directly associated (at $\Delta^2 < 1$) with the Strehl parameter $I_S \approx \exp(-\Delta^2)$ equal to relative reduction in radiation intensity at the optical system focus [9].

The problem of calculating errors I, II and III are similar to each other. In all these problems, the current and/or *a priori* information about the phase distribution $\Phi(\boldsymbol{\rho})$ is present. In all these problems, the distribution function $\Psi(\boldsymbol{\rho})$ formed by the AOS is represented in the basis of the corrector response functions. In solving problem I, we may assume that the conditionally used WF corrector is capable of reproducing exactly any given response functions. Therefore, the results of the analytical solution of problems I, II and III can be reduced to an integrated form. Below, we present the main general relations obtained by optimising the system according to the minimum condition of indicator (2), and then we specify the content of the control algorithms with the peculiarities of each of these problems taken into account.

The expression for the optimal function $\Psi(\boldsymbol{\rho})$ in all these problems can be reduced to the form

$$\Psi(\boldsymbol{\rho}) = MHR(\boldsymbol{\rho}). \quad (3)$$

Here the matrix $M = \|M_m\|$ represents a row (the matrix dimension is $1\times\mu$) composed of the WF sensor counts; μ is the number of counts; the quantities $\{M_m\}$ are of stochastic nature because they are functionally associated with the random distribution $\Phi(\boldsymbol{\rho})$ under reconstruction and, in addition, generally comprise a noise component; the matrix function $R(\boldsymbol{\rho}) = \|R_n(\boldsymbol{\rho})\|$ represents a column ($\nu\times 1$) of linearly independent corrector response functions $R_n(\boldsymbol{\rho})$ [$r_n(\boldsymbol{\rho})$]; ν is the number of actuators of the WF corrector; $H = \|H_{mn}\|$ is the deterministic coupling matrix ($\mu\times\nu$) between the WF sensor and corrector; and the quantities $\{H_{mn}\}$ depend on the matrix function $R(\boldsymbol{\rho})$, the autocorrelation characteristics for $\Phi(\boldsymbol{\rho})$ and M and the correlation matrix $\overline{M\Phi(\boldsymbol{\rho})}$. The expressions for the matrix H as applied to each of the problems (I, II, III) are given in Appendix 1.

Expression (3) takes into account that the AOS control system contains three major devices: a sensor (counts M), an actuating device [corrector, the response function $R(\boldsymbol{\rho})$] and a device for generating the control actions (matrix H) to adjust the sensor data to the corrector capabilities.

Equation (3) assumes linearity of the response function $\Psi(\boldsymbol{\rho})$ of the WF corrector relative to the control actions (the superposition property [1] of the actuator response functions $\{R_n(\boldsymbol{\rho})\}$). Therefore,

$$\Psi(\boldsymbol{\rho}) = CR(\boldsymbol{\rho}), \quad C = MH. \quad (4)$$

where $C = \|C_n\|$ is the row matrix ($1\times\nu$) of control actions; and C_n is the weighting factor of the response function of the nth actuator in the phase distribution.

In the statistical model of the AOS, one of the most important characteristics of the random function $\Psi(\boldsymbol{\rho})$ is represented by the autocorrelation function $K_{\Psi}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$. On the basis of (4) we have

$$K_{\Psi}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \overline{\Psi(\boldsymbol{\rho}_1)\Psi(\boldsymbol{\rho}_2)} = R^{\mathrm{T}}(\boldsymbol{\rho}_1)\overline{CC}R(\boldsymbol{\rho}_2), \quad (5)$$

where

$$\overline{CC} = \|\overline{C_n C_m}\| = H^{\mathrm{T}}\,\overline{MM}\,H \quad (6)$$

is the correlation matrix ($\nu\times\nu$) of control actions; $\overline{MM} = \|\overline{M_n M_m}\|$ is the correlation matrix ($\mu\times\mu$) of counts of the WF sensor; and the superscript T denotes transposition. The matrix $\overline{MM}$ is considered nondegenerate. If the rank of the original correlation matrix is smaller than the number of counts, it is replaced by a submatrix that forms the basis minor; the row of counts is accordingly changed [10].

In all the problems (I, II, III) the minimised error (variance) of approximation (2) is calculated by the formula

$$\Delta^2 = \sigma_{\Phi}^2 - \sigma_{\Psi}^2. \quad (7)$$

Here σ_{Φ}^2 and σ_{Ψ}^2 are the variances of the AOS reconstructed and formed phase distributions, respectively. Taking into account (5), we obtain

$$\sigma_{\Phi}^2 = \langle K_{\Phi}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\rangle, \quad \sigma_{\Psi}^2 = \langle K_{\Psi}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\rangle = \mathrm{Sp}(\langle RR\rangle\overline{CC}), \quad (8)$$

where $K_{\Phi}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \overline{\Phi(\boldsymbol{\rho}_1)\Phi(\boldsymbol{\rho}_2)}$ is the correlation function of the reconstructed phase distribution; $\langle RR\rangle = \|\langle R_n(\boldsymbol{\rho})R_m(\boldsymbol{\rho})\rangle\|$ is the coupling matrix ($\nu\times\nu$) of the corrector response functions; and Sp(...) denotes the matrix trace [10]. Equation (7) is obtained with regard to the fact that, as shown below [the second formula in (15)], the error Δ and the function Ψ are uncorrelated.

Expressions (4)–(8) are referred to the zonal method of the AOS control. However, in a search AOS with zonal control, the target function optimisation is complicated by the fact that the response functions of actuators are spatially interrelated (the coupling matrix $\langle RR\rangle$ is nondiagonal in the general case). The nonorthogonality of the control channels leads to systematic errors and reduction of the signal-to-noise ratio. These drawbacks turn out less significant in the search AOS with modal control [1,2]. Note that, in some cases, the use of modal algorithms in non-search regime [2] allows one to reduce the number of computational operations and thus increase the AOS operating speed.

Consider an AOS with a basis of the orthonormal modes that can be error-free reproduced by the WF corrector. In this case the mode response $F_n(\boldsymbol{\rho})$ $[f_n(\boldsymbol{\rho}),\ n = 1, 2, ..., \nu]$ is implemented by supplying the appropriate control actions into each of ν actuators simultaneously. Then $F(\boldsymbol{\rho}) = AR(\boldsymbol{\rho})$, where $A = \|A_{mn}\|$ is the $(\nu\times\nu)$ matrix of transition from the basis $\{R_n(\boldsymbol{\rho})\}$ to the basis $\{F_n(\boldsymbol{\rho})\}$. Given orthonormality of the modes, we have $A\langle RR\rangle A^{\mathrm{T}} = E$, where E is the identity matrix. Herewith, $\Psi(\boldsymbol{\rho}) = \hat{C}F(\boldsymbol{\rho})$, where $\hat{C} = CA^{-1}$ is the matrix of the control mode actions, and A^{-1} is the matrix inverse to the A matrix. Each mode $F_n(\boldsymbol{\rho})$ makes the contribution $S_n^2 = \langle\langle K_\Psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) F_n(\boldsymbol{\rho}_1) F_n(\boldsymbol{\rho}_2)\rangle\rangle \geqslant 0$ to the variance of the phase distribution:

$$\sigma_\Psi^2 = \mathrm{Sp}(\overline{\hat{C}\hat{C}}) = \sum_{n=1}^{\nu} S_n^2,$$

where $\overline{\hat{C}\hat{C}} = A^{-\mathrm{T}}\overline{CC}A^{-1}$ is the correlation matrix of the control mode actions; and $A^{-\mathrm{T}} = (A^{\mathrm{T}})^{-1}$.

Of particular interest are the Karhunen–Loeve modes [7, 10]. They possess a distinctive feature of extremality. Consider various bases with numbering of modes in each of them satisfying the condition $S_n^2 \geqslant S_{n+1}^2$ $(n = 1, 2, ..., \nu-1)$. In this case, the variance of WF approximation using the first k $(1 \leqslant k \leqslant \nu)$ Karhunen–Loeve modes does not exceed the error resulting from the use of other bases with the same ν and k. Therefore, in the AOS with modal control, it is advisable to use the Karhunen–Loeve basis, which allows obtaining the minimum approximation error as quick as possible. Given work [10] and the results of works [8, 11], the Karhunen–Loeve modes can be found as the solution of the matrix equation with respect to the A and S^2 values or from equivalent integral equations for $F(\boldsymbol{\rho})$ and S^2:

$$A\langle RR\rangle\overline{CC} = S^2A, \quad \langle F(\boldsymbol{\rho}_1)K_\Psi(\boldsymbol{\rho}_1, \boldsymbol{\rho})\rangle = S^2F(\boldsymbol{\rho}). \tag{9}$$

Here $S^2 = \|S_n^2\,\delta_{nm}\|$; δ_{nm} is the Kronecker symbol; averaging in the second relation is performed over the variable $\boldsymbol{\rho}_1$; and the K_Ψ and $\overline{CC}$ values are determined by expressions (5) and (6). For completeness of the statistical AOS model, let us note the following. If the WF sensor and corrector are ideal, the Karhunen–Loeve mode can be found from the second (integral) equation in (9) at $K_\Psi = K_\Phi$. In this case, in the matrix equation

$$\overline{CC} = \langle RR\rangle^{-1}\|\langle K_\Phi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) R_n(\boldsymbol{\rho}_1) R_m(\boldsymbol{\rho}_2)\rangle\|\langle RR\rangle^{-1}$$

and the original basis $\{R_n(\boldsymbol{\rho})\}$ must represent a complete set of functions (for example, Zernike polynomials [9]).

The Karhunen–Loeve modes allow approximation of a random function by a sum of spatially orthogonal functions with statistically independent coefficients. Relations (9) generalise this description method to the problems in which the Karhunen–Loeve basis is formed from a set (finite or infinite) of the given functions. Methods for solving equations (9) are known [10].

In the framework of the statistical AOS model, relations (1)–(9) give a general description of the algorithms for optimal phase formation. Three matrices are required to calculate the AOS characteristics: the matrix function $R(\boldsymbol{\rho})$ of the WF corrector responses, the stochastic matrix M of the WF sensor counts and the deterministic matrix H of the correlation between the sensor and corrector. Using these matrices, all other matrices, functions and values present in (2)–(9) can be found. The expressions for the generating matrices $R(\boldsymbol{\rho})$, M and H are given in Appendix 1. A detailed description of the algorithm for AOS control depends on its designation. According to formulas (A1.4), (A1.6) and (A1.7), we obtain for problems I, II and III, respectively,

$$R(\boldsymbol{\rho}) = \|\overline{M_m\Phi(\boldsymbol{\rho})}\|, \quad M = \|M_m\|, \quad H = \overline{MM}^{-1}, \tag{10}$$

$$R(\boldsymbol{\rho}) = \|R_n(\boldsymbol{\rho})\|, \quad M = \|\langle\Phi(\boldsymbol{\rho})R_n(\boldsymbol{\rho})\rangle\|, \quad H = \langle RR\rangle^{-1}, \tag{11}$$

$$\begin{gathered} R(\boldsymbol{\rho}) = \|R_n(\boldsymbol{\rho})\|, \quad M = \|M_m\|, \\ H = \overline{MM}^{-1}\|\langle\overline{M_m\Phi(\boldsymbol{\rho})}R_n(\boldsymbol{\rho})\rangle\|\langle RR\rangle^{-1}. \end{gathered} \tag{12}$$

To implement algorithm (3), one must also know the statistical properties of the function $\Phi(\boldsymbol{\rho})$ and the counts $\{M_m\}$, as well as their joint correlation function. These characteristics can be determined experimentally by direct measurements and/or found from *a priori* information.

Now we proceed to examination of the relationships between the algorithms. To distinguish between the algorithms I, II and III, let us rename $\Psi(\boldsymbol{\rho})$, $\Delta(\boldsymbol{\rho})$, σ_Ψ^2 and Δ^2 by means of a subscript. We introduce the functions $\Psi_\xi(\boldsymbol{\rho})$, $\Delta_\xi(\boldsymbol{\rho}) = \Phi(\boldsymbol{\rho}) - \Psi_\xi(\boldsymbol{\rho})$, $\sigma_\xi^2 = \langle\overline{\Psi_\xi^2(\boldsymbol{\rho})}\rangle$ and $\Delta_\xi^2 = \sigma_\Phi^2 - \sigma_\xi^2$, where the subscript ξ corresponds to μ, ν and $\mu\nu$ for problems I, II and III, respectively. We investigate $\Psi_\xi(\boldsymbol{\rho})$ as an operator $\Psi_\xi(\Phi)$ being linear with respect to Φ [neglecting the functional dependence of the counts M on $\Phi(\boldsymbol{\rho})$]. Using (3) along with expressions (10)–(12), by means of algebraic and statistical transformations we find

$$\begin{gathered} \Psi_\xi(\Psi_\xi(\Phi)) = \Psi_\xi(\Phi), \\ \Psi_{\mu\nu}(\Phi) = \Psi_\mu(\Psi_\nu(\Phi)) = \Psi_\nu(\Psi_\mu(\Phi)). \end{gathered} \tag{13}$$

The second relation in (13) shows that the optimal compensation algorithm III takes into account both algorithm I and algorithm II.

Consider the relationship between the algorithms at a correlation level. From general considerations, it is clear that the distribution $\Psi_\xi(\boldsymbol{\rho})$ and the error $\Delta_\xi(\boldsymbol{\rho})$ must be statistically independent:

$$\begin{gathered} \langle\overline{\Delta_\xi(\boldsymbol{\rho})\,\Psi_\xi(\boldsymbol{\rho})}\rangle = 0, \quad \langle\overline{\Delta_\xi(\boldsymbol{\rho})\,\Phi(\boldsymbol{\rho})}\rangle = \Delta_\xi^2, \\ \langle\overline{\Psi_\xi(\boldsymbol{\rho})\,\Phi(\boldsymbol{\rho})}\rangle = \sigma_\xi^2. \end{gathered} \tag{14}$$

Equations (14) are fulfilled by means of substituting $\Delta_\xi(\boldsymbol{\rho}) = \Phi(\boldsymbol{\rho}) - \Psi_\xi(\boldsymbol{\rho})$ into the expression for the correlation functions and subsequent averaging with regard to (10)–(12). In the same way can be found the relations coupling different algorithms:

$$\begin{gathered} \langle\overline{\Delta_\xi(\boldsymbol{\rho})\,\Psi_{\mu\nu}(\boldsymbol{\rho})}\rangle = 0, \\ \langle\overline{\Psi_\xi(\boldsymbol{\rho})\,\Psi_{\mu\nu}(\boldsymbol{\rho})}\rangle = \langle\overline{\Psi_\mu(\boldsymbol{\rho})\,\Psi_\nu(\boldsymbol{\rho})}\rangle = \sigma_{\mu\nu}^2. \end{gathered} \tag{15}$$

The analysis of control algorithms I, II and III given in Appendix 2 on the basis of relations (13)–(15) shows that, given (A2.5), the compensation error variance can be presented in the form

$$\Delta_{\mu\nu}^2 = \Delta_\mu^2 + \Delta_\nu^2 - \Delta_\mu \Delta_\nu K_{\mu\nu}, \tag{16}$$

where $K_{\mu\nu}$ is the aperture-averaged correlation coefficient of the errors $\Delta_\mu(\boldsymbol{\rho})$ and $\Delta_\nu(\boldsymbol{\rho})$. The value of $K_{\mu\nu}$ varies in the interval [0, 1]. The nonnegativity of the correlation coefficient is related to the fact that the algorithms assume that the statistical properties of the measured phase distribution and the WF sensor counts are known.

According to (A2.7), the compensation error lies within the following limits:

$$(\Delta_\mu^2 + \Delta_\nu^2)/2 \leqslant \Delta_{\mu\nu}^2 \leqslant \Delta_\mu^2 + \Delta_\nu^2,$$
$$\Delta_{\mu\nu} \approx 0.85(1 \pm 0.18)\sqrt{\Delta_\mu^2 + \Delta_\nu^2}. \tag{17}$$

The upper limit for $\Delta_{\mu\nu}$ is attained when errors I and II do not correlate with each other. The lower limit corresponds to the total correlation of errors I and II. From general considerations, it is clear that this takes place when $\Delta_\mu^2 \approx \Delta_\nu^2$.

Expressions (17) factorise the compensation error dependence on the parameters μ, ν, which gives grounds for a separate study of the sensor and corrector in the AOS development. When matching the parameters μ and ν, it is advisable to proceed from the equality $\Delta_\mu^2 \approx \Delta_\nu^2$, which allows avoiding unnecessary complication of the system in the sketchy development of an AOS.

The expressions given above contain no explicit time dependence. To take it into account, for example, in the case of discrete time counts, it suffices to perform the following conversion in the formulas: to replace $\Phi(\boldsymbol{\rho})$, $\Psi(\boldsymbol{\rho})$, $\{M_m\}$ and $\{C_n\}$ by $\Phi(\boldsymbol{\rho}, t)$, $\Psi(\boldsymbol{\rho}, t)$, $\{M_m(t_m)\}$ and $\{C_n(t)\}$, respectively. Here t is the current time, and t_m is the time moment of registering the mth count. Herewith, the statistical averaging in all relevant expressions is performed with regard to the correlation in time, and all the above relations preserve their form.

3. Results of the computational experiment

In a computational experiment, a circular deformable mirror is considered as a WF corrector, and the sensor is based on a Hartmann sensor (HS). The deformable mirror's actuators are placed at the nodes of an equidistant grating, and the response functions are described by the Gaussian function: $r_n(\boldsymbol{\rho}) = \exp[-(\boldsymbol{\rho} - \boldsymbol{p}_n)^2/\omega^2]$, where $\boldsymbol{p}_n$ ($n = 1, 2, ..., \nu$) are the coordinates of nodes of an equidistant grating with the spacing p, and ω is the response function half-width. The grating nodes are located within the corrector's light aperture with a radius $R_{\rm L}$.

The HS possesses $\mu_{\rm g}$ sub-apertures. The HS counts are proportional to the WF local slopes measured at the nodes of a square grating with coordinates $\boldsymbol{q}_m = (q_{xm}, q_{ym})$, the points of counts are located within the light aperture. The grating axes $\boldsymbol{p}$ and $\boldsymbol{q}$ are co-directed; their centres of symmetry coincide with the light aperture's centre. The HS counts are $M_m = D_u\Phi(\boldsymbol{q}_m) + N_m$, where N_m is the noise component; D_u is the operator of differentiation with respect to the variable u, where $u = q_{xm}$ at $m = 1, 2, ..., \mu_{\rm g}$ and $u = q_{ym}$ at $m = \mu_{\rm g} + 1, \mu_{\rm g} + 2, ..., 2\mu_{\rm g}$; and the total number of counts is $\mu = 2\mu_{\rm g}$.

The correlation function of the random, normally distributed field $\varphi(\boldsymbol{\rho})$ with a zero average value is specified in the Gaussian form:

$$K_\varphi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \overline{\varphi(\boldsymbol{\rho}_1)\varphi(\boldsymbol{\rho}_2)} = \sigma^2 \exp[-(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2/\rho_{\rm c}^2],$$

where $\rho_{\rm c}$ is the correlation radius; σ^2 is the phase variance. Quantity (7) of the square of the aperture-averaged phase is

$$\sigma_\Phi^2 = \sigma^2\{1 - (2/a)\{1 - \exp(-a)[I_0(a) + I_1(a)]\}\}, \tag{18}$$

where $a = 2/c^2$; $c = \rho_{\rm c}/R_{\rm L}$ is the relative radius of correlation; and $I_0(a)$ and $I_1(a)$ are the modified [12] Bessel functions of imaginary argument. As a result, we have the approximate relation $\sigma_\Phi^2 \approx \sigma^2/(1 + c^2)$. This relation is obtained from (18) by matching the asymptotics at $c \gg 1$ and $c \ll 1$ and provides an underestimated value of σ_Φ^2 with a relative error up to 5%.

It is assumed that the sensor noises are not correlated with the slopes $D_u\Phi(\boldsymbol{q}_m)$, the errors for the counts with different numbers and also the errors along the axes x and y are statistically independent; $\overline{N_m} = 0$, $\overline{N_m^2} = \gamma(\overline{{\rm grad}^2\varphi(\boldsymbol{\rho})})/2$, where $\overline{{\rm grad}^2\varphi(\boldsymbol{\rho})} = 4\sigma^2/\rho_{\rm c}^2$ is the variance of the local slopes of the normal to the phase surface, and γ is the relative fraction of noise.

The system parameters in the calculations are varied within the following limits: $0.1 \leqslant c \leqslant 1.0$, $1 \leqslant \mu \leqslant 170$, $1 \leqslant \nu \leqslant 137$ and $0 \leqslant \gamma \leqslant 1.0$. Typical results of the calculations based on relation (7) with regard to (10)–(12) and accepted constraints on the system parameters are shown in Figs 1–3.

Figure 1 presents the dependence of Δ_ν/σ [curve (*1*), algorithm II] on the number ν of actuators and the dependences of Δ_μ/σ [curves (*2–6*), algorithm I] on the number of the counts μ for different relative fractions γ of noise. As expected, if the phase gradient is used as the counts, the approximation error Δ_μ/σ increases with increasing measurement error [curves (*3–6*)]. With increasing number μ of the counts, the magnitude of Δ_μ/σ begins to be determined by the noise component and approaches $\sigma_{\rm ns} \approx \sqrt{2\pi\gamma}/\sqrt{\mu c^2}$. This estimate has been obtained by considering the fact that the phase error for the mth count is proportional to the product $N_m(R_{\rm L}\sqrt{\pi}/\sqrt{2\mu})$. Let us denote the sensor error with regard to noises by $\Delta_\mu(\sigma_{\rm ns})$. Assuming the matrices $\langle RR\rangle$ and $\overline{MM}$

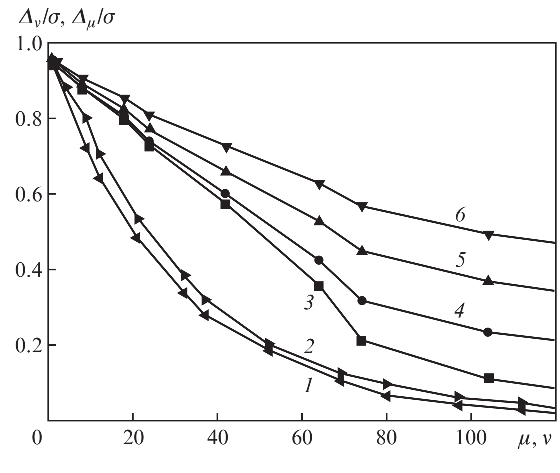

Figure 1. Dependences of the errors Δ_ν/σ on the number ν of the corrector actuators (*1*) and Δ_μ/σ on the number μ of the WF sensor counts (*2–6*) obtained in the phase measurements (*1*, *2*) and WF slopes for γ = (*3*) 0, (*4*) 0.05, (*5*) 0.2 and (*6*) 0.5. The correlation radius $c = 0.3$, the response function half-width $\omega = 2p$.

diagonal, given relations (6)–(8) and (10), it is easy to find the estimate for the WF sensor error:

$$\Delta_\mu^2(\sigma_{ns}) \approx \sigma_\Phi^2/(1 + 1/\sigma_{ns}^2) + \Delta_\mu^2(0)/(1 + \sigma_{ns}^2), \qquad (19)$$

where $\Delta_\mu^2(0)$ is the error in the absence of noises.

Figure 2 shows the dependence of $\chi_{\mu\nu} = \Delta_{\mu\nu}/(0.85\times\sqrt{\Delta_\mu^2 + \Delta_\nu^2})$ on μ for several values of γ. Given relation (17), this ratio should lie within the interval 0.82–1.18, which is fully consistent with the data of Fig. 2.

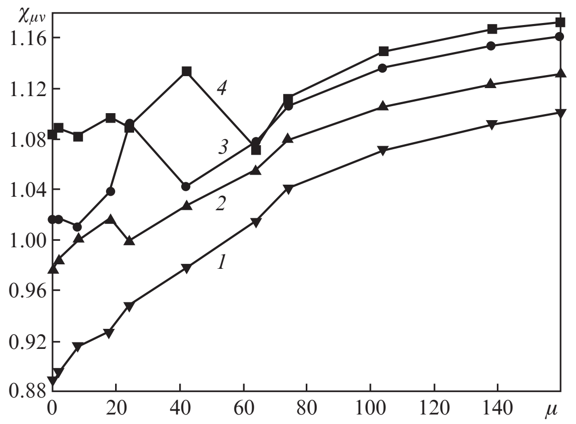

Figure 2. Dependences of the value $\chi_{\mu\nu}$ on the number μ of the counts at the actuators' number $\nu = 37$, the correlation radius $c = 0.3$, the response function half-width $\omega = 0.7p$ and γ = (*1*) 0, (*2*) 0.05, (*3*) 0.2 and (*4*) 0.5.

Based on the results of a computational experiment aimed at evaluation of the quantity $K_{\mu\nu}$ as applied to the AOS with a HS, the following approximate relation has been obtained: $K_{\mu\nu} \approx (\Delta_\mu \Delta_\nu/\sigma_\Phi^2)^{4/5}$. Herewith, the maximum error in the calculation according to formula (16) does not exceed 10%.

Figure 3 shows the dependence of $\mu(\nu)/\nu$ obtained by matching the values of μ and ν using the condition $\Delta_\mu \approx \Delta_\nu$. This dependence is nonmonotonic. As follows from the analysis of Fig. 3, the required number of HS counts turns out 1.5–5.5 and more times higher than the number of actuators ν. If the number of counts is predetermined, the corrector quality requirements are reduced with increasing measurement error and, consequently, the required number of actuators decreases in comparison with the number of counts.

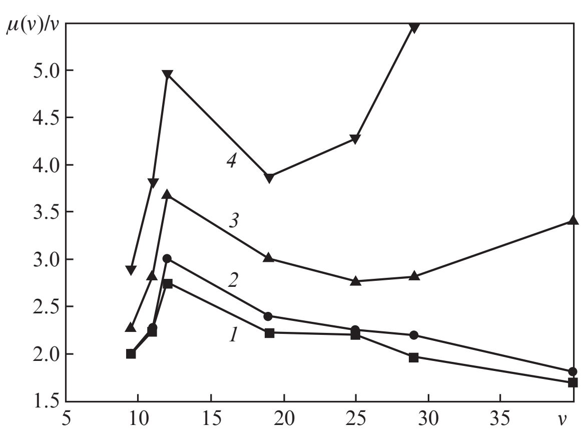

Figure 3. Dependence of the relative number of counts $\mu(\nu)/\nu$ on the number ν of actuators at the correlation radius $c = 0.7$, the response function half-width $\omega = 0.7p$ and γ = (*1*) 0, (*2*) 0.05, (*3*) 0.2 and (*4*) 0.5.

If the phase values $M_m = \Phi(\boldsymbol{q}_m)$ are used as counts [curves (*1*) and (*2*) in Fig. 1], the required number of actuators, provided that $\Delta_\mu \approx \Delta_\nu$ (neglecting the noise error of the WF sensor) is approximately equal to the number of counts. Such a difference from the case of the measurements employing a HS is explained as follows. Suppose we know the exact values of the phase $\Phi(\boldsymbol{q}_m)$ and the derivative $D_u\Phi(\boldsymbol{q}_m)$ at some point of the aperture of the beam under correction. In a vicinity of that point the values of the derivative may deviate significantly from the value $D_u\Phi(\boldsymbol{q}_m)$. This difference is the greater, the greater is the distance between the points of counts. Accordingly, the phase value at any neighbouring point, reconstructed by means of $D_u\Phi(\boldsymbol{q}_m)$, may differ significantly from the exact phase value. This error decreases with increasing number of counts $D_u\Phi(\boldsymbol{q}_m)$ and is absent if the phase values $\Phi(\boldsymbol{q}_m)$ are used as the counts. In the latter case the correlation coefficient $K_{\mu\nu} \approx (\Delta_\mu \Delta_\nu/\sigma_\Phi^2)^{2/5}$ in formula (16) is on average greater than in the case of measurements by means of the HS.

4. Conclusions

We have further developed a statistical model of a phase-conjugating AOS. In the framework of an integrated mathematical approach, three algorithms are considered, each of them intended for minimisation of one of the following indicators: the sensor error I (in the case of an ideal corrector), the corrector error II (in the case of ideal measurements) and the compensation error III (with regard to discreteness of measurements, sensor noises and incompleteness of the system of response functions of the corrector actuators). We have investigated the functional and statistical relationships between these algorithms and found an approximating relation that enables calculation, with the accuracy of no worse than 10%, of error III of the WF compensation as a function of errors I and II of the WF sensor and corrector. The relation derived allows factorisation of the compensation error dependence on the number μ of the sensor counts and the number ν of the corrector actuators, and makes it possible to study the WF sensor and corrector independently of each other in the AOS development. When matching the parameters μ and ν, it is advisable to proceed from equality of the errors of the sensor and corrector. Herewith, in the case of using a Hartmann sensor as a sensor, the number of actuators may be ~1.5 and more times smaller than the number of counts. When the noise component in the measurement error rises, this difference increases; if the error variance amounts to 10% of the variance of the slope of the WF under correction, at $\mu >$ 30–50 the value of ν turns out 3–5 times smaller.

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Appendix 1. Algorithms minimising the errors of the WF sensor and corrector as well as the compensation error

Problem I. The sensor error, i.e. the WF approximation error determined in accordance with the sensor counts in the case of an ideal corrector, depends on the values $\{M_m\}$, their functional relationship with $\Phi(\boldsymbol{\rho})$, and the measurement noises. The reconstructed phase distribution is represented as the sum

$$\Psi(\boldsymbol{\rho}) = \sum_{m=1}^{\mu} M_m Q_m(\boldsymbol{\rho}), \tag{A1.1}$$

where $Q_m(\boldsymbol{\rho})$ is the deterministic, spatially distributed weight coefficient with which the mth count of the WF sensor is taken into account in the phase distribution. When minimising error (2), variational calculus is used and the form of the functions $\{Q_m(\boldsymbol{\rho})\}$ constituting the column $Q(\boldsymbol{\rho}) = \|Q_m(\boldsymbol{\rho})\|$ is optimised. Given the results of Refs [1, 5, 8], we have

$$Q(\boldsymbol{\rho}) = \overline{MM}^{-1}\overline{M\Phi(\boldsymbol{\rho})}, \quad \overline{M\Phi(\boldsymbol{\rho})} = \|\overline{M_m\Phi(\boldsymbol{\rho})}\|. \tag{A1.2}$$

After substituting (A1.2) into (A1.1), we obtain the reconstruction algorithm description:

$$\Psi(\boldsymbol{\rho}) = M\,\overline{MM}^{-1}\overline{M\Phi(\boldsymbol{\rho})}. \tag{A1.3}$$

From comparison of (A1.3) and (3), it is clear that $H = \overline{MM}^{-1}$, and the functions $\{\overline{M_m\Phi(\boldsymbol{\rho})}\}$ are similar to the response functions $\{R_m(\boldsymbol{\rho})\}$ and can be conditionally considered as optimal response functions of the WF corrector. Thus, in problem I

$$R(\boldsymbol{\rho}) = \|\overline{M_m\Phi(\boldsymbol{\rho})}\|, \ M = \|M_m\|, \ H = \overline{MM}^{-1}. \tag{A1.4}$$

Problem II. In calculating the WF corrector error, it is assumed that the WF measurements are exact, the phase distribution error being caused by the finiteness of the number ν of the control channels. The reconstructed phase distribution is given by the sum:

$$\Psi(\boldsymbol{\rho}) = \sum_{n=1}^{\nu} C_n R_n(\boldsymbol{\rho}).$$

The coefficients $\{C_n\}$ are determined from the condition of $\langle\Delta^2(\boldsymbol{\rho})\rangle$ minimality [1, 4]:

$$C = \langle\Phi R\rangle\langle RR\rangle^{-1}, \ \langle\Phi R\rangle = \|\langle\Phi(\boldsymbol{\rho})R_n(\boldsymbol{\rho})\rangle\|. \tag{A1.5}$$

From comparison of (A1.5) and (3) it follows that $H = \langle RR\rangle^{-1}$, and the row matrix $\langle\Phi R\rangle$ represents the analogue of a row of the counts M.

As a result, in problem II

$$R(\boldsymbol{\rho}) = \|R_n(\boldsymbol{\rho})\|, \ M = \|\langle\Phi(\boldsymbol{\rho})R_n(\boldsymbol{\rho})\rangle\|,$$
$$H = \langle RR\rangle^{-1}. \tag{A1.6}$$

In this case, $\overline{MM} = \|\langle\langle K_\Phi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)R_n(\boldsymbol{\rho}_1)R_m(\boldsymbol{\rho}_2)\rangle\rangle\|$, where K_Φ is the correlation function of $\Phi(\boldsymbol{\rho})$.

Problem III. When analysing the WF compensation algorithm, the phase distribution $\Psi(\boldsymbol{\rho})$ formed by AOS is specified as a linear combination of the known response functions $\{R_n(\boldsymbol{\rho})\}$ of the corrector actuators, while the control actions are defined as a linear combination of the sensor counts $\{M_m\}$ [1, 2, 8]:

$$\Psi(\boldsymbol{\rho}) = \sum_{n=1}^{\nu} C_n R_n(\boldsymbol{\rho}), \ C_n = \sum_{m=1}^{\mu} M_m H_{mn}.$$

The values $\{H_{mn}\}$ are the sought-for ones and constitute the matrix $H = \overline{MM}^{-1}\langle\overline{M\Phi}R\rangle\langle RR\rangle^{-1}$ in the optimisation according to the criterion of minimum of indicator (2). Here, $\overline{MM} = \|\overline{M_n M_m}\|$; $\langle\overline{M\Phi}R\rangle = \|\langle\overline{M_m\Phi(\boldsymbol{\rho})}R_n(\boldsymbol{\rho})\rangle\|$ is the matrix of overlapping of the response functions $R_n(\boldsymbol{\rho})$ with the functions $\overline{M_m\Phi(\boldsymbol{\rho})}$ of joint correlation of the measured phase and the sensor counts; and $\langle RR\rangle = \|R_n(\boldsymbol{\rho})R_m(\boldsymbol{\rho})\|$. Thus, in relations (3)–(6) in the conditions of problem III

$$R(\boldsymbol{\rho}) = \|R_n(\boldsymbol{\rho})\|, \ M = \|M_m\|,$$
$$H = \overline{MM}^{-1}\langle\overline{M\Phi}R\rangle\langle RR\rangle^{-1}. \tag{A1.7}$$

Appendix 2. Compensation error estimation

Let us obtain and examine three statistical relationships.

1. Using expression (1), (13) and linearity of the operators, we can formulate the following chain of equalities:

$$\Delta_{\mu\nu}(\Phi) = \Phi - \Psi_\nu(\Psi_\mu(\Phi)) = \Phi - [\Psi_\mu(\Phi) - \Delta_\nu(\Psi_\mu(\Phi))]$$
$$= \Delta_\mu(\Phi) + \Delta_\nu(\Phi - \Delta_\mu(\Phi)) = \Delta_\mu(\Phi) + \Delta_\nu(\Phi) - \Delta_\nu(\Delta_\mu(\Phi)),$$

or

$$\Delta_\nu(\Delta_\mu(\Phi)) = \Delta_\mu(\Phi) + \Delta_\nu(\Phi) - \Delta_{\mu\nu}(\Phi).$$

After squaring and averaging the last relation with regard to (14) and (15), we have

$$\langle\overline{\Delta_\nu^2(\Delta_\mu(\boldsymbol{\rho}))}\rangle = \Delta_{\mu\nu}^2 - \Delta_\nu^2 - \Delta_\mu^2 + 2\langle\overline{\Delta_\nu(\boldsymbol{\rho})\Delta_\mu(\boldsymbol{\rho})}\rangle \geqslant 0. \tag{A2.1}$$

2. The value $\sigma_{\mu\nu}^2 = \langle\overline{[\Phi(\boldsymbol{\rho}) - \Delta_\mu(\boldsymbol{\rho})][\Phi(\boldsymbol{\rho}) - \Delta_\nu(\boldsymbol{\rho})]}\rangle$. Transforming the product by means of formulas (14) and (15), we obtain

$$\Delta_{\mu\nu}^2 = \Delta_\mu^2 + \Delta_\nu^2 - \langle\overline{\Delta_\nu(\boldsymbol{\rho})\Delta_\mu(\boldsymbol{\rho})}\rangle \geqslant 0. \tag{A2.2}$$

3. Calculating the value $\langle\overline{[\Delta_\mu(\boldsymbol{\rho}) - \Delta_\nu(\boldsymbol{\rho})]^2}\rangle$. Expanding the square and performing averaging, we find

$$\Delta_\mu^2 + \Delta_\nu^2 - 2\langle\overline{\Delta_\nu(\boldsymbol{\rho})\Delta_\mu(\boldsymbol{\rho})}\rangle \geqslant 0. \tag{A2.3}$$

Considering relations (A2.1)–(A2.3) as a system of equations and inequalities, by means of algebraic transformations we obtain

$$\langle\overline{\Delta_\nu(\boldsymbol{\rho})\Delta_\mu(\boldsymbol{\rho})}\rangle = \langle\overline{\Delta_\nu^2(\Delta_\mu(\boldsymbol{\rho}))}\rangle \geqslant 0, \tag{A2.4}$$

$$\Delta_{\mu\nu}^2 = \Delta_\mu^2 + \Delta_\nu^2 - \langle\overline{\Delta_\nu(\boldsymbol{\rho})\Delta_\mu(\boldsymbol{\rho})}\rangle \geqslant 0, \tag{A2.5}$$

$$0 \leqslant \langle\overline{\Delta_\nu(\boldsymbol{\rho})\Delta_\mu(\boldsymbol{\rho})}\rangle \leqslant (\Delta_\mu^2 + \Delta_\nu^2)/2. \tag{A2.6}$$

Let us estimate the limits within which the value $\Delta_{\mu\nu}^2$ may vary. Substituting (A2.6) into (A2.5), we find

$$(\Delta_\mu^2 + \Delta_\nu^2)/2 \leqslant \Delta_{\mu\nu}^2 \leqslant \Delta_\mu^2 + \Delta_\nu^2,$$
$$\Delta_{\mu\nu} \approx 0.85(1 \pm 0.18)\sqrt{\Delta_\mu^2 + \Delta_\nu^2}. \tag{A2.7}$$

The lower boundary in (A2.7) is attained when special requirements to AOS parameters are fulfilled, for example, $\Phi_\nu(\Phi) = \Phi_\mu(\Phi)$ or $\Delta_\mu^2 \approx \Delta_\nu^2$.

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