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Theory of repetitively pulsed operation of diode lasers subject to delayed feedback

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Abstract. **Repetitively pulsed operation of a diode laser with delayed feedback has been studied theoretically at varying feedback parameters and pump power levels. A new approach has been proposed that allows one to reduce the system of Lang–Kobayashi equations for a steady-state repetitively pulsed operation mode to a first-order nonlinear differential equation. We present partial solutions that allow the pulse shape to be predicted.**

Keywords: *diode laser subject to delayed feedback, repetitively pulsed operation, Lang–Kobayashi equations.*

1. Introduction

A diode laser subject to delayed feedback (DFB) has a variety of operation modes under continuous pumping, which is caused by the interference of the laser field reflected from the external mirror with the field circulating in the internal cavity. The simplified equations proposed by Lang and Kobayashi (LK) [1] for describing the dynamics of diode lasers with an external mirror are also applicable for describing the variety of dynamic operation modes [2]. Owing to this, they have been widely used in later studies concerned with the theory of diode lasers subject to DFB [3, 4]. Lasers with DFB are of interest because they are the simplest systems modelling the dynamics of two optically coupled identical lasers [5]. Note that, in all studies, laser operation modes are analysed using numerical methods because of the complexity of the equations involved.

Of particular practical interest is the study of steady-state repetitively pulsed operation modes. In terms of nonlinear dynamics, the development of these means the existence of stable limit cycles in the phase space of the system, which attract its integral curves for a wide class of initial conditions.

In this paper, we formulate a semi-analytical approach for finding strictly periodic steady-state field pulses in a diode laser subject to DFB. Comparison of solutions found in this approach with numerical calculation results obtained previously [6] demonstrates quantitative agreement between numerical and analytical solutions.

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2. Basic equations

The nonlinear dynamics of a semiconductor laser subject to DFB due to the reflection of some of the light from an external mirror (Fig. 1) can be described by the LK equations [1]

$$\frac{\partial E}{\partial t} = (1 - \mathrm{i}R)NE + \mathrm{i}M\mathrm{e}^{\mathrm{i}\kappa}E(t - \tau_\mathrm{d}),$$
$$T\frac{\partial N}{\partial t} = P - N - (1 + 2N)|E|^2. \qquad (1)$$

The former equation describes the dynamics of the slowly varying envelope of the field amplitude, $E(t)$, where R is the linewidth enhancement factor, and M and κ are the modulus and phase of the feedback (FB) coefficient, respectively. The time delay τ_d is equal to the external cavity round-trip time. Note that the existence of modes in the cavity formed by the front facet of the laser chip and the external mirror is left out of consideration in the LK model.

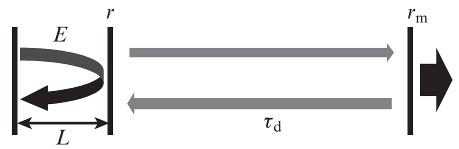

Figure 1. Schematic of a diode laser subject to DFB: L is the chip length; τ_d is the round-trip time in the feedback system; r and r_m are the amplitude reflection coefficients of the chip front facet and external mirror, respectively (the rear facet has high reflectivity).

The latter equation describes the dynamics of the inverse population $N(t)$, proportional to the pump power above threshold [7, 8] and normalised by the photon lifetime in the internal cavity: $\tau_\mathrm{ph} = (n_\mathrm{a}L/c)(\ln r^{-1} + \alpha_\mathrm{c}L)^{-1}$, where α_c is the distributed loss; L is the internal cavity length in the diode; r is the amplitude reflection coefficient of its facets; c is the speed of light in vacuum; and n_a is the effective refractive index of the active layer. For the gain coefficient in the active layer, we used a linear approximation near the lasing threshold, with a differential gain coefficient $g = \partial G/\partial N$. The threshold carrier density is determined by the optical losses in the bulk and on the facets of the chip: $N_\mathrm{th} = N_\mathrm{tr} + (gc\tau_\mathrm{ph}/n_\mathrm{a})^{-1}$. Here N_tr is the carrier density at which the active medium becomes transparent. In this approximation, the pump power P can be expressed through the carrier injection rate $j/(ed)$ (where e is the electron charge; d is the thickness of the active layer; and j is the current density) and the carrier lifetime τ_s in the absence of emission: $P = \frac{1}{2}(gc\tau_{ph}/n_a)[j\tau_s/(ed) - N_{th}]$. The dimensionless field amplitude is proportional to the physical amplitude $\mathcal{E}$: $E = (\frac{1}{2}gc\tau_s/n_a)^{1/2}\mathcal{E}$. Time t in (1) is normalised to the photon lifetime τ_{ph}. Note that a typical dimensionless inversion relaxation time, $T = \tau_s/\tau_{ph}$, is of the order of 1000. With the above notation, the field intensity can be expressed through the flux density of photons of energy $h\nu$: $2E^2/(g\tau_s)$.

In the case of steady-state lasing, we have $E = E_{st}\exp(i\beta t)$, where β is the small difference between the laser frequency and the eigenfrequency of the cavity formed by the chip facets. The parameters β and E_{st} are given by $\beta = M\times\sqrt{1+R^2}\sin(\beta\tau_d - \kappa + \arctan(R^{-1}))$, $E_{st}^2 = (P - N_{st})/(1 + 2N_{st})$ and $N_{st} = M\sin(\kappa - \beta\tau_d)$.

For subsequent analysis, we represent the field in the form $E = E_0\exp[i\beta t + \psi(t)]$, where E_0 is the normalisation factor ($\overline{|E|^2} = E_0^2$) and the function ψ (complex-valued in the general case) determines the nonlinear dynamics of the field.

The delay term in (1) makes it impossible to use conventional approaches to solving ordinary differential equations. To simplify subsequent calculations, we introduce a function proportional to the delayed feedback signal:

$$\Lambda(t) = M\exp(i\kappa)\frac{E(t - t_d)}{E(t)} = M\exp[-i\chi + \psi(t-\tau_d) - \psi(t)], \quad (2)$$

where $\chi \equiv \beta\tau_d - \kappa$ is the phase shift in the FB loop in the case of steady-state lasing. For regular oscillations, the function $\Lambda(t)$ is also periodic and describes the DFB effect. The period-averaged population inversion and frequency detuning can then be expressed through the average real and imaginary parts of a new function:

$$\overline{N} = \mathrm{Im}\overline{\Lambda},\ \beta = \mathrm{Re}\overline{\Lambda} - R\,\mathrm{Im}\overline{\Lambda}. \quad (3)$$

Here $\overline{\Lambda}$ is the average of function (2) over one oscillation period. With the new notation, Eqns (1) for the functions ψ and $n = N - \overline{N}$ reduce to

$$\frac{\partial}{\partial t}\psi = (1 - iR)n + i[\Lambda(t) - \overline{\Lambda}],$$
$$\frac{\partial n}{\partial t} = -\frac{n}{T_1} - \frac{\omega_r^2}{2}[\exp(2\mathrm{Re}\psi) - 1]\left(1 + \frac{2n}{1+2\overline{N}}\right). \quad (4)$$

In the latter of Eqns (4), we introduce two parameters: $\omega_r^2 = 2(P - \overline{N})/T$ and $T_1 = T(1 + 2\overline{N})/(1 + 2P)$. The former parameter ($\omega_r$) is the relaxation oscillation frequency and the latter (T_1) is of the same order as the inversion relaxation time. Note that, in the case of diode lasers, the condition $\omega_r T \gg 1$ is usually satisfied. In our calculations, we took P = 0.8, T = 1000 and $M = 0.02$. The dimensionless frequency was $\omega_r = 0.04$. The average dimensionless inversion $\overline{N}$ is close to N_{st} (i.e. of the order of M), the inversion oscillation frequency is close to ω_r, and the amplitude $|n|$ is also of the order of ω_r. Therefore, $|n| \ll 1 + 2\overline{N}$. In subsequent analysis, we use the approximation $1 + 2n/(1 + 2\overline{N}) \approx 1$, whose validity was verified by numerical calculations for periodic conditions.

3. An approach to finding repetitive-pulse solutions

The interference of a delayed signal entering the active medium after being reflected from the external mirror with the field circulating within the chip gives rise to coupling between the fields emitted at different instants in time. The complete evolution of the field, described by the LK equations [1], can be found uniquely only if the field and population inversion are known over the external cavity round-trip time, which requires using a continuum data set. Nevertheless, among the possible solutions to the LK equations, there is a class of solutions asymptotically corresponding to a periodic attractor for a variety of initial conditions [6, 9]. Our purpose is to find a strictly periodic solution. Under such conditions, field and inversion dynamics can be described using a single periodic function, Φ. The existence of strictly periodic stable solutions at certain parameters of the system has been demonstrated by numerical calculations in a number of studies (see e.g. Refs [6, 10]).

There are no standard approaches to the general analysis of nonlinear differential equations containing delay terms. In the case of linear equations containing delay terms, the sought function is usually expanded in terms of exponential functions of time [11]. The exponents are then found by solving transcendental equations.

The requirement that the sought solution be strictly periodic makes the problem much simpler, which allows us to formulate a semi-analytical approach for analysing it. One significant simplification is related to the fact that, in the case of periodic conditions, the $\Lambda(t)$ term, corresponding to delayed feedback, is also periodic. Function (2), which appears in (4), can then be represented as a power series in the sought periodic solution. To find the coefficients in the expansion of the feedback term into a power series in the sought solution, we rely on the following additional considerations: 1. A small tangential perturbation of the solution, corresponding to a small displacement along the limit cycle, is obviously proportional to the first time derivative of the sought solution. 2. In the case of a strictly periodic solution, an arbitrary time origin can be chosen. Both considerations are valid provided that the repetitively pulsed operation mode is stable. Whether the sought solution is stable to small perturbations should be found out by numerical calculation and is not the subject of the analytical approach.

The former consideration allows us to derive an additional relationship by finding how the amplitude of a small tangential perturbation is related to the sought periodic solution. Placing the condition that the amplitude of the perturbation is proportional to the derivative of the sought solution, we find the desired relationship. The system of linear equations for arbitrary small perturbations is presented in the Appendix section. This is a system of three ordinary differential equations (ODEs) with coefficients that have a periodic time dependence through the sought function. The matrix of coupling coefficients, $\mathbf{A}$, has three eigenvalues: $\lambda_0 = 0$ and $\lambda_{1,2} = -\frac{1}{2}T_1 \pm i\omega_r\exp(\mathrm{Re}\psi)$ [note that $\exp(\mathrm{Re}\psi) = |E|/E_0$]. It follows from general considerations that a tangential perturbation corresponds to a combination of eigenvectors with an eigenvalue whose real part is zero. The combination can be found in explicit form provided that $\omega_r T_1 \gg 1$, which is usually fulfilled. The time variation of the tangential perturbation is only determined by $\exp(\mathrm{Re}\psi) = |E|/E_0$. Since a small perturbation is proportional to the derivative of the sought function Φ, it can be found to within a constant from $\mathrm{d}\Phi/\mathrm{d}t = |E|/E_0$. The function thus determined has the dimensions of time. For subsequent calculations, it is convenient to introduce the dimensionless function $\Theta = \omega\Phi$, where ω is a characteristic frequency to be determined, of the same order as the relaxation oscillation frequency ω_{r}:

$$\Theta(\tau) - \Theta(t) = \omega[\Phi(\tau) - \Phi(t)] = \omega\int_t^\tau \mathrm{d}t'(|E|/E_0). \quad (5)$$

The sought function characterises the rate of the time evolution of the system, and this rate is proportional to the field amplitude, rather than to its intensity, in contrast to the rate of variation in the population inversion on a laser transition [see (1)]. The period of the solution is determined by the increase in $\Theta = \omega\Phi$ as a function of the upper limit by 2π. Since the $|E|/E_0$ ratio is always positive, we have

$$\int_t^{t+2\pi/\Omega}\mathrm{d}t'(|E|/E_0) = 2\pi/\omega = \int_t^{t+2\pi/\Omega}\frac{\mathrm{d}\Phi}{\mathrm{d}t'}\mathrm{d}t',$$

where the pulse repetition rate is expressed through frequency as $2\pi/\Omega$. Here we take into account that, in the case of a strictly periodic function, an arbitrary time origin can be chosen.

The periodic function $\Lambda(t)$, related to the effect of DFB [see Eqn (2)], can be represented as a power series in $\exp(\mathrm{i}\Theta)$:

$$\Lambda(t) - \overline{\Lambda} = M\exp(\mathrm{i}\kappa)\frac{E(t-t_{\mathrm{d}})}{E(t)} - \overline{\Lambda} = \sum_{k=-\infty}^{\infty} h_k\exp(\mathrm{i}k\Theta), \quad (6)$$

where k is an integer. The constants h_k play the role of the DFB spectrum in this representation. Such a series is a natural generalisation of the Fourier series. The quantity $k\Theta$ will be referred to as the kth harmonic. The condition that small tangential perturbations are proportional to the derivatives of the corresponding nonlinear functions ($\delta\psi \propto \dot{\psi}$, $\delta\Phi \propto \dot{\Phi}$ and $\mathrm{Re}\psi = \ln\dot{\Phi}$) allows us to derive a key nonlinear equation for the function Φ (see Appendix):

$$\mathrm{d}\Phi/\mathrm{d}t = s_0 + 2\,\mathrm{Re}\sum_{k=1}^{\infty} s_k\exp(\mathrm{i}k\omega\Phi), \quad (7)$$

where the coefficients of the series can be expressed through the coefficients h_k in (6):

$$s_{k\neq 0} = \frac{h_k - h_{-k}^*}{2}\,\frac{\omega k}{\omega^2k^2 - \omega_{\mathrm{r}}^2} = s_{-k}^*. \quad (8)$$

It is worth pointing out that Eqn (7) can be interpreted as the first integral of the equation of motion for a classic nonlinear pendulum. To find s_0, we use the exact (by definition) relation $\overline{|E|^2} = E_0^2$ (here and in what follows, an overbar denotes averaging over the pulsation period), which allows an additional relation to be derived from Eqn (7):

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\Theta\Big[s_0 + \sum_{k\neq 0} s_k\exp(\mathrm{i}k\Theta)\Big]^{-1} = \frac{1}{\overline{\dot{\Phi}}} = s_0. \quad (9)$$

The parameter s_0 determines the deviation of the characteristic frequency ω from the pulse repetition rate: $\omega = \Omega s_0$.

The coefficients s_k of the series in (7) are determined in explicit form by (8) through the coefficients h_k in the expansion of the function $\Lambda(t) - \overline{\Lambda}$.

Both series in (6) and (7) are expansions in terms of the functions $P_k = \exp[\mathrm{i}k\omega\Phi(t)]$. The system of these functions is complete as a power series, but they are nonorthogonal over the oscillation period $2\pi/\Omega$. In this case, the coefficients in the expansion of some function in terms of a nonorthogonal basis depend as well on the overlap integrals of the basis functions. As a consequence, to find the coefficients of series (6) and (7) it is necessary to know the entire matrix Π of the overlap integrals of the basis functions: $\Pi_{mk} = \langle P_m | P_k\rangle$ (here the angle brackets denote the average of the product of the basis functions P_m^* and P_k over the oscillation period):

$$\Pi_{mk} = f_{k-m} = (\Omega/2\pi)\int_{-\pi/\Omega}^{\pi/\Omega}\mathrm{d}t\exp[\mathrm{i}\omega(k-m)\Phi(t)] = f_{m-k}^*. \quad (10)$$

It is seen from (10) that the matrix elements $\Pi_{mk} = f_{k-m}$ depend on the index difference, i.e. this is a Toeplitz matrix and its inverse matrix can be found in explicit form [12]. The inverse matrix is also a Toeplitz matrix: $(\Pi^{-1})_{nm} = b_{n-m}$. The inverse matrix elements can be expressed through the coefficients s_k in the expansion of the function $\mathrm{d}\Phi/\mathrm{d}t$ as follows: multiplying both sides of (7) by the function P_{-n} and averaging over the oscillation period, we obtain the equation

$$(\omega/\Omega)\sum_{k=-\infty}^{\infty} s_k f_{k-n} = \delta_{0n}, \quad (11)$$

from which it follows that $b_{n-m} = (\omega/\Omega)s_{n-m}$ [δ_{mn} in (11) is the identity matrix].

According to definition (6), the numbers h_k are the coefficients in the expansion of the periodic function $\Lambda(t) - \overline{\Lambda}$ in terms of the basis chosen. Since the basis functions are nonorthogonal, the following system of equations should be solved [13]:

$$\langle P_m|(\Lambda(t) - \overline{\Lambda})\rangle = \sum_{k=-\infty}^{\infty} h_k\langle P_m|P_k\rangle = \sum_{k=-\infty}^{\infty}\Pi_{mk}h_k,$$

where the overlap integrals determine the matrix $\Pi_{mk} = \langle P_m|P_k\rangle$. The solution to this system for the coefficients h_n can be found using the inverse matrix Π^{-1}:

$$h_n = \sum_{m=-\infty}^{\infty}(\Pi^{-1})_{nm}\langle P_m|(\Lambda(t) - \overline{\Lambda})\rangle.$$

The sum over m in this solution can be reduced to a simple expression:

$$\sum_{m=-\infty}^{\infty}(\Pi^{-1})_{nm}\langle P_m| = \sum_{m=-\infty}^{\infty}(\omega/\Omega)s_{n-m}\exp(-\mathrm{i}m\omega\Phi) = \frac{\omega}{\Omega}\dot{\Phi}\exp(-\mathrm{i}n\omega\Phi).$$

As a result, the coefficients h_n are determined by the integral over the oscillation period $2\pi/\Omega$:

$$h_n = \Big\langle\frac{\omega}{\Omega}\dot{\Phi}P_n\Big|(\Lambda(t) - \overline{\Lambda})\Big\rangle = \frac{\omega}{2\pi}\int_{-\pi/\Omega}^{\pi/\Omega}\mathrm{d}\tau\,\dot{\Phi}\exp(-\mathrm{i}n\omega\Phi)(\Lambda(t) - \overline{\Lambda}). \quad (12)$$

The function $\dot{\Phi}$ in the integrand is, by definition, related to the field amplitude: $\mathrm{d}\Phi/\mathrm{d}t = |E|/E_0 = \exp(\mathrm{Re}\psi)$. According to

(2), the FB function is also related to $E(t)$ values at two different instants in time: $\Lambda(t) = M\exp(\mathrm{i}\kappa)E(t - t_\mathrm{d})/E(t)$. Thus, the functions Φ, ψ and Λ are related to each other. Using Eqn (A3) (see Appendix), the following expression for the function ψ can be derived:

$$\psi(t) = \int \mathrm{d}t\left\{\frac{\ddot{\Phi}}{\dot{\Phi}}(1 - \mathrm{i}R) + \mathrm{i}\left[\mathrm{Re}(\Lambda(t) - \overline{\Lambda}) - R\,\mathrm{Im}(\Lambda(t) - \overline{\Lambda})\right]\right\}. \quad (13)$$

Further transformations of Eqns (3) and (13) allow us to obtain the following integrofunctional equation:

$$\ln(\Lambda(\tau)) = \ln M + \mathrm{i}\kappa + (1 - \mathrm{i}R)\ln\frac{\dot{\Phi}(\tau - \tau_\mathrm{d})}{\dot{\Phi}(\tau)} + \mathrm{i}\int_{\tau}^{\tau - \tau_\mathrm{d}} \mathrm{d}t\{\mathrm{Re}\Lambda(t) - R\,\mathrm{Im}\Lambda(t)\}. \quad (14)$$

This equation contains the phase of the FB coefficient, κ, indicating that it has a significant effect on the shape of periodic field oscillations. Jointly solving Eqns (7) and (14) allows us to determine the pulse shape at particular physical parameters of the laser. The functions $\dot{\Phi}(\omega,\boldsymbol{h};\tau)$ and $\Lambda(\omega,\boldsymbol{h};\tau)$ can be calculated explicitly through the characteristic frequency ω and the coefficients h_k ($\boldsymbol{h}$ is the vector made up of these coefficients). In numerically solving Eqn (14), substitution of the functions $\dot{\Phi}(\omega,\boldsymbol{h};\tau)$ and $\Lambda(\omega,\boldsymbol{h};\tau)$ gives a system of transcendental algebraic equations in unknown ω and $\boldsymbol{h}$ on a discrete time mesh with a preset step $\tau_i - \tau_{i-1}$ over one pulse period. The time delay and FB coefficient are physical parameters. The described formulation of the problem offers the possibility of examining multistability conditions, whereas in the approach based on direct integration of the LK equations the problem of finding such conditions remains open.

4. Solutions in the form of a single pulse per period

As pointed out earlier [2], a key parameter determining stability loss in a steady state is the combination of parameters $f = M\tau_\mathrm{d}\sqrt{1 + R^2}$, where f is referred to as the effective feedback strength. Numerical calculations demonstrate that instability arises when f is of the order of unity. Note that the feedback phase and pump power also influence the stability of steady-state operation.

It is of interest to analyse the simplest case, where the right-hand side of Eqn (7) comprises only two terms: the constant s_0 and the fundamental ($k = 1$) harmonic. Field dynamics are then characterised by three constants: ω, s_0 and E_0. To relate these constants to physical parameters (pump power, phase and modulus of the FB coefficient and signal delay time), it is necessary to solve the integral equation (14). This was beyond the purpose of our work at this stage. Instead, we compared numerical and analytical solutions, adjusting constants in the latter by fitting to the numerical solution. Equation (7) then has the form

$$\dot{\Phi} - S\cos(\omega\Phi) = s_0. \quad (15)$$

The function $\cos(\omega\Phi)$ can be found in explicit form:

$$\cos(\omega\Phi) = \frac{\cos(\Omega t) - S/s_0}{1 - (S/s_0)\cos(\Omega t)}. \quad (16)$$

The pulse repetition rate can also be represented explicitly: $\Omega = \omega\sqrt{s_0^2 - S^2}$. On the other hand, as shown above we have $\Omega = \omega/s_0$. Therefore, s_0 and S/s_0 are related by

$$s_0^{-4} = 1 - (S/s_0)^2.$$

As seen from (16), it is s_0 that determines the pulse shape.

It is easy to show that a change in the sign of the coefficient S corresponds to a displacement of the curve by half-period. The modulus of the field amplitude [function $f_4(t)$ in Table 1] is

$$|E(t)| = E_0\dot{\Phi} = \frac{E_0}{\sqrt[4]{1 - (S/s_0)^2}} \times\left\{1 + (S/s_0)\frac{\cos[\sqrt[4]{1 - (S/s_0)^2}\,\omega t] - S/s_0}{1 - (S/s_0)\cos[\sqrt[4]{1 - (S/s_0)^2}\,\omega t]}\right\}. \quad (17)$$

The field maximum to minimum ratio is $(1 + S/s_0)/(1 - S/s_0)$. A change in FB phase leads to slight changes in the characteristic frequency $\omega = (1.11{-}1.23)\omega_\mathrm{r}$ and characteristic amplitude $E_0 = (0.96{-}0.99)\sqrt{P}$. The two parameters $\omega \approx \omega_\mathrm{r} = \sqrt{2(P - \overline{N})/T}$ and $E_0 = \sqrt{(P - \overline{N})/(1 + 2\overline{N})}$ are determined primarily by the normalised pump power relative to the threshold P. They are weak, implicit functions of parameter S/s_0. At the same time, S/s_0 strongly influences the oscillation frequency (through the factor $s_0^{-1} = \sqrt[4]{1 - (S/s_0)^2}$) and pulse amplitude. For $|S/s_0| \to 1$, both the oscillation amplitude and period increase. Physically, this means that, for oscillations to have a high peak, it is necessary to accumulate more pump energy, which will lead to an increase in oscillation period at a given normalised pump power relative to the threshold.

To find out whether the partial solutions found here are applicable for describing steady-state oscillations in the LK model, we numerically integrated the LK equations for parameters of the system corresponding to a transition through a bifurcation point with hard oscillation excitation [14].

The solid lines in Fig. 2 represent three types of numerical solutions to the system of equations (1) for steady-state periodic radiation pulses, including the case where the FB phase is $\kappa = 180°$. At this FB phase, the oscillations acquire the nature of periodic solitons [15]. In our calculations, we used the following parameters of the laser system: $P = 0.8$, $M =$

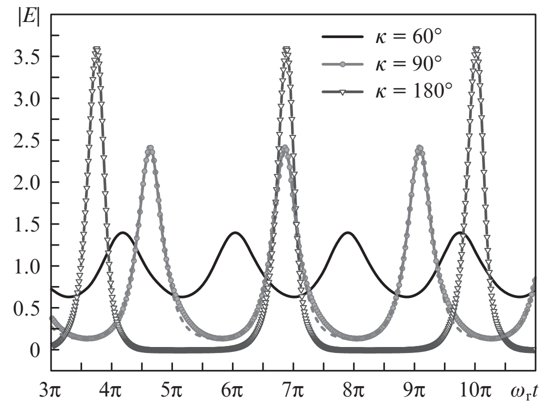

Figure 2. Steady-state wave field oscillations. The solid lines represent numerical calculation results for an effective FB strength $f = 0.63$ at phases $\kappa = 60°$, $90°$ and $180°$. The dashed line shows the analytical solution at $\kappa = 90°$.

0.02, $T = 1000$, $\tau_d = 10$ and $R = 3$. The effective FB strength parameter was $f = 0.63$. Generally, the phase κ is determined by the position of the external mirror and the output frequency. In our calculations, we took $\kappa = 60°$, 90° and 180°. At the parameters chosen, near $\kappa = 60°$ there is bifurcation with hard excitation of a repetitively pulsed operation mode. The oscillation shape at this κ differs markedly from a harmonic one, and the oscillation amplitude is not small. As κ increases to 90° and above, the pulse becomes sharper. The dashed line shows the analytical solution to Eqn (15) at $\kappa = 90°$. As seen in Fig. 2, the numerical and analytical solutions differ only slightly.

Since the relation between the parameters E_0, ω and s_0 of the analytical solution and the physical parameters of the laser cannot be represented in explicit form, the former parameters were evaluated by fitting the analytical solution to the numerical solution found by directly integrating the LK equations. For each combination of physical parameters, there is a corresponding combination of constants in Eqn (15). These constants can be found when in numerical calculation a repetitively pulsed operation mode sets in. For a steady state, the parameters E_0, ω and S/s_0 and the time origin are determined by nonlinear fitting. The procedure converges when the error (rms deviation) in calculation results over the period of function (17) reaches a minimum. The residual is the minimum mean square deviation $\chi^2_{\rm red}$ obtained by the fitting.

The fitting results are presented in Table 1 [formula (17) corresponds to the function $f_4(t)$]. We considered three FB phases ($\kappa = 60°$, 90° and 180°): the oscillation amplitude was found to rapidly increase with κ. Nevertheless, the pulse shape is similar to that described by function (17) in all instances. The goodness of fit can be quantified by the mean square deviation of the fitting function from the exact value: $\chi^2_{\rm red} = (n-p)^{-1}\times\sum_1^n[y_i - f(x_i)]^2$. Because there is a limited number of data points, n, in the data set, we use the test referred to as 'reduced chi squared' [16]. In this test, division by n is replaced by the reduced divisor $n - p$, where p is the number of fitting parameters ($p = 4$ in our case). The parameter $\chi^2_{\rm red}$ is presented in the third column of Table 1. The integrated relative error can be written as $\sqrt{\chi^2_{\rm red}/\overline{y^2}}$, where $\overline{y^2} = n^{-1}\sum_1^n y_i^2$. At $\kappa = 90°$, the relative error does not exceed 6% ($p = 4$). It is seen that there is very good fit quality.

One more analytical solution can be found when the terms of the series in (7) form an infinite power series in $\cos^k(\omega\Phi)$, with coefficients decreasing as z^k, where z is a constant less than unity. Remarkably, in this case the solution retains its structure:

$$f_5(t) = E_0 s_0\left[1 + (S/s_0)\frac{\cos(\Omega t) + V}{1 + V\cos(\Omega t)}\right]. \qquad (18)$$

In contrast to (17), here the rational expression contains an additional parameter, V, instead of S/s_0.

The coefficients of the analytical solution represented as $f_5(t)$ are intricate functions of the physical parameters of the system, such as the FB phase, but they can be extracted directly from the numerical solution if (18) does correspond to the solution. These coefficients at different κ values and other parameters maintained constant are listed in Table 1. The use of the function $f_5(t)$ allows one to more accurately determine the pulse shape as a function of time, which confirms that the analytical solution converges to the numerical one as the number of higher harmonics increases.

Varying the FB phase in the range $60° < \kappa < 220°$ leads to drastic changes in the pulse amplitude and repetition rate, which can be well described analytically throughout the κ range corresponding to repetitively pulsed operation. Figure 3 shows an enlarged portion of Fig. 2, which demonstrates the pulse shape at $\kappa = 90°$. It is well seen that the curves obtained numerically and analytically agree well. The slight discrepancies are caused by symmetry breaking in the case of the calculated curve because of the finite inversion relaxation time. The analytical pulse shape (Fig. 2, dashed line) is symmetric with respect to the maximum and minimum of the function. The phase $\kappa = 90°$ corresponds to oscillations with a large amplitude and a markedly sharper pulse shape. Note that the standard small perturbation theory is only applicable near an Andronov–Hopf bifurcation for small-amplitude harmonic oscillations. Our approach allows one to analyse as well processes with pulse sharpening and a large oscillation amplitude.

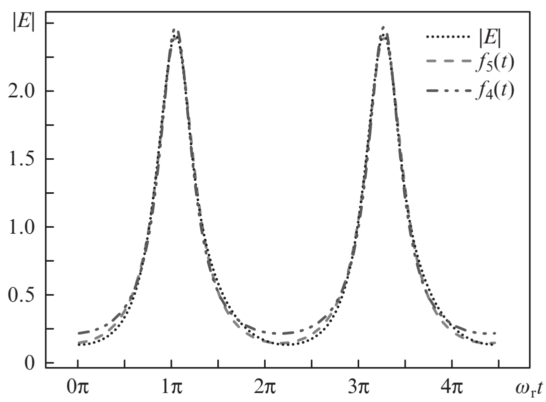

Figure 3. Theoretical (f_5, f_4) and calculated (dotted line) pulse shapes at $\kappa = 90°$ [f_5 is function (18) for five-parameter fitting; the fitting function f_4 is given by (17)].

Table 1. Coefficients of the analytical formulas (17) and (18) extracted from calculation results at $\kappa = 60°$, 90° and 180° (oscillation frequency $\omega_r = 0.04$, pump power $P = 0.8$).

κ	$\sqrt{\chi^2_{\rm red}/\overline{y^2}}$	$\chi^2_{\rm red}$	$A = s_0E_0/\sqrt{P}$	$-S/s_0$	V	Ω/ω_r
		$f_4(t)$; $V = -S/s_0$, $p = 4$				
60°	1.1%	5.6 E−5	1	0.375		1.075
90°	6%	1.85 E−3	1.33	0.84		0.90
180°	12.8%	8.14 E−3	1.91	0.963		0.64
		$f_5(t)$; $V \neq -S/s_0$, $p = 5$				
60°	1.0%	5.2 E−5	1	0.376	0.356	1.075
90°	3.4%	5.7 E−4	1.26	0.887	0.80	0.90
180°	8.6%	3.7 E−3	1.84	1	0.958	0.65

5. Conclusions

Based on the standard Lang–Kobayashi model for a semiconductor laser subject to DFB, we have proposed a method for a theoretical analysis of steady-state repetitively pulsed operation of such a laser. The dynamics of the laser subject to DFB is characterised by a great diversity of operation modes, among which modes with regular periodic pulsations are of practical interest. The proposed approach allows the description of periodic operation of a laser subject to DFB to be reduced to analysis of a first-order nonlinear differential equation containing an infinite power series of the expansion in terms of the sought periodic functions. We have found an analytical solution for the case where the infinite series can be reduced to the sum of the first terms of the expansion. For comparison with numerical results, we have chosen parameters of the laser near a bifurcation point differing from Andronov–Hopf bifurcation in that hard excitation of anharmonic oscillations occurs at this point. Moving farther away from the bifurcation point leads to an even sharper pulse shape. Despite the pulse narrowing, the analytical solution agrees well with numerical calculation results. The analytical solution provides a relation between the characteristics of the pulse: the pulsation period and amplitude in a wide range of parameters.

Thus, we have developed analytical theory for a laser subject to delayed feedback and found convenient expressions describing periodic laser operation modes and predicting the pulse shape. The proposed approach is applicable for examining steady-state repetitively pulsed laser operation modes corresponding to motion along limit cycles in phase space.

Appendix. Deduction of formula (7)

The system of linear equations for perturbations can be found by substituting $\psi \to \psi + \delta\psi$ and $n \to n + \delta n$ (where $\delta\psi$ and δn are small perturbations) in (4). In the case of periodic oscillations, small perturbations along a limit cycle correspond to further motion along the limit cycle. The coefficients in the system of linear differential equations for perturbations are purely periodic functions, including the DFB term. The system of equations for small perturbations can be represented in matrix form:

$$\frac{d\boldsymbol{v}}{dt} = \mathbf{A}\boldsymbol{v} + \delta\boldsymbol{b}, \text{ where } \boldsymbol{v} \equiv \begin{pmatrix} \delta\psi(\tau) \\ \delta\psi^*(\tau) \\ \delta n(\tau) \end{pmatrix}. \quad \text{(A1)}$$

The matrix $\mathbf{A}$ and vector $\boldsymbol{b}$ can be written as

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 - iR \\ 0 & 0 & 1 + iR \\ -(\omega_r^2/2)e^{2\mathrm{Re}\psi} & -(\omega_r^2/2)e^{2\mathrm{Re}\psi} & -1/T \end{pmatrix},$$

$$\boldsymbol{b} = \begin{pmatrix} i(\Lambda(t) - \overline{\Lambda}) \\ -i(\Lambda(t) - \overline{\Lambda})^* \\ 0 \end{pmatrix}. \quad \text{(A2)}$$

The components of the vector $\boldsymbol{b}$ can be expressed through the function $\Lambda(t)$ (6). The variation in $\delta\Lambda(t)$, written as the Fourier expansion (6) in terms of harmonics, is proportional to the perturbation of the phase $\delta\Phi$.

The matrix $\mathbf{A}$ has three eigenvectors, $\boldsymbol{v}_0$, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$,

$$\boldsymbol{v}_0 = \begin{pmatrix} i \\ -i \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_{1,2} = \begin{pmatrix} 1 - iR \\ 1 + iR \\ \lambda_{1,2} \end{pmatrix},$$

with eigenvalues $\lambda_0 = 0$ and $\lambda_{1,2} = -\frac{1}{2}T_1 \pm i\omega_r \exp(\mathrm{Re}\psi)$. The roots $\lambda_{1,2}$ correspond to underdamped oscillations. Steady-state oscillations are determined as a partial solution to (A1) by varying the constants of the solution for a uniform system with the matrix $\mathbf{A}$:

$$\boldsymbol{v}(\tau) = \int^{\tau} dt\{K(\tau, t)\delta\boldsymbol{b}(t)\}.$$

Here $K(\tau, t) = U(\tau)U^{-1}(t)$ is the Cauchy matrix, where the matrix U is the resolvent of $\mathbf{A}$:

$$U(t) = E + \int_{t_0}^{t}\mathbf{A}(t)dt + \int_{t_0}^{t}\mathbf{A}(t)dt\int_{t_0}^{t}\mathbf{A}(t)dt + \ldots .$$

The convolution in t is performed in the ε vicinity of point τ with vector $\delta\boldsymbol{b}$. As a result of the damping of relaxation oscillations, the oscillating solutions are time-independent at the lower limit, corresponding to an instant in the distant past. In effect, the dynamics of steady-state oscillations are independent of the prior history. All is determined by a small vicinity of point τ, where the vector $\delta\boldsymbol{b}$ can be expanded in terms of the eigenvectors of the matrix $\mathbf{A}$ and the action of the kernel in the $t \to \tau$ limit satisfies the rule

$$K(\tau, t)\boldsymbol{v}_i = \exp\left(\int_t^{\tau}\lambda_i dt\right)\boldsymbol{v}_i.$$

A combination of the vectors $\boldsymbol{v}_0$ and $\boldsymbol{v}_1 + \boldsymbol{v}_2$ for $\omega_r T_1 \gg 1$ makes it possible to exclude one equation [for the $\delta n(\tau)$ inversion] [10], and oscillating functions for the $\delta\psi(\tau)$ and $\delta\psi^*(\tau)$ variations are the solution to the equations

$$\begin{pmatrix} \delta\psi(\tau) \\ \delta\psi^*(\tau) \end{pmatrix} = \int_t^{\tau} dt\delta\Phi$$

$$\times\left[q_+(t)\begin{pmatrix} 1 - iR \\ 1 + iR \end{pmatrix}\cos(\omega_r\Phi(\tau) - \omega_r\Phi(t)) + q_-(t)\begin{pmatrix} +i \\ -i \end{pmatrix}\right], \quad \text{(A3)}$$

$$q_+(t) = -\frac{d(\Lambda - \Lambda^*)/2i}{d\Phi}, \quad q_-(t) = Rq_+(t) + \frac{d(\Lambda + \Lambda^*)/2}{d\Phi}.$$

Summing the equations for $\delta\psi(\tau)$ and $\delta\psi^*(\tau)$ (A3) leads to an equation for the function $\delta\mathrm{Re}\psi(\tau)$. Using the relation $\delta\mathrm{Re}\psi = \delta\ln\dot{\Phi}$, the left-hand side of the equation can be reduced to the function $\dot{\Phi}$. The tangential perturbations are proportional to the derivatives of the functions: $\delta\psi \propto \dot{\psi}$ and $\delta\Phi \propto \dot{\Phi}$. The $\delta\Phi \to d\Phi$ substitution in the integral in (A3) allows the integral to be calculated analytically. After rather simple transformations, Eqn (A3) for linear perturbations converts into a nonlinear differential equation for the function Φ itself:

$$\frac{\ddot{\Phi}}{\dot{\Phi}} + \sum_k \frac{(k\omega)^2}{(k\omega)^2 - \omega_r^2}\frac{h_k - h_{-k}^*}{2i}\exp(ik\omega\Phi) = 0. \quad \text{(A4)}$$

Integrating (A4), we obtain (7).

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