

PACS numbers: 42.25.Bs
DOI: 10.1070/QE2015v045n11ABEH015800

Description of high-power laser radiation in the paraxial approximation

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Abstract. **We consider the feasibility of an adequate description of a laser pulse of arbitrary shape within the framework of the paraxial approximation. In this approximation, using a parabolic equation and an expansion in the small parameter, expressions are obtained for the field of a sufficiently intense laser radiation given in the form of axially symmetric Hermite–Gaussian beams of arbitrary mode and arbitrary polarisation. It is shown that in the case of sufficiently short pulses, corrections to the transverse components of the laser field are the first-order rather than the second-order quantities in the expansion in the small parameter. The peculiarities of the description of higher-mode Hermite–Gaussian beams are outlined.**

Keywords: *laser pulse, paraxial approximation, Hermite–Gaussian beams, higher modes.*

1. Introduction

An adequate representation of the vector field of laser radiation plays an important role in a variety of problems, in particular in studies on acceleration of charged particles. The most well-developed is the description of laser radiation in the form of Gaussian (or Hermite–Gaussian) beams propagating along the z axis in the quasi-optical paraxial approximation [1–7]. In this case, it is assumed that the beam radius a in the plane $z = 0$, corresponding to the centre of the waist of a Gaussian beam, markedly exceeds the wavelength λ_0. Then, the opening angle ϑ of the cone, in which the wave propagates along the z axis, is quite small in the far zone ($z \gg z_R$): $\vartheta \approx \lambda_0(\pi a)^{-1} = a/z_R \ll 1$. Here, $z_R = k_0 a^2/2$ is the Rayleigh length determining the diffraction spreading of a wave beam; $k_0 = 2\pi/\lambda_0 = \omega_0/c$ is the wave number; ω_0 is the carrier frequency of the wave; and c is the velocity of light. Thus, in the paraxial approximation, there exists a small parameter

$$\mu = 2/(k_0 a) = a/z_R \ll 1. \qquad (1)$$

Note that in the paraxial approximation, the transverse component of the wave vector is small compared to the length of this vector [6]: $(k_x^2 + k_y^2)/k_0^2 \ll 1$.

The presence of a small parameter allows the use of perturbation theory to solve the wave equation for the electromagnetic field vectors. In this case, the wave equation yields an approximate equation of parabolic type. In the zeroth approximation, the fields have only transverse components, and corrections to them normally occur in the second approximation. The longitudinal components of the field vectors in the direction of the wave propagation turn to be the first-order quantities.

Note that in a number of papers, which considered the motion of particles in the laser field, the longitudinal component of the field is neglected [8], which, naturally, leads to incorrect results.

Due to the creation of high-power lasers generating ultrashort pulses [9], there has appeared a problem of describing their radiation [10–12]. In particular, the authors of Refs [10, 11] considered tightly focused laser radiation with an intensity of 10^{22} W cm^{-2} or higher, for which the focal spot size may be less than the wavelength. In this case, parameter (1) is not small, so that the paraxial approximation becomes inapplicable. However, Quesnel and Mora [13] showed that the paraxial approximation can be used to describe femtosecond fundamental-mode pulses with an intensity of about 10^{18} W cm^{-2}.

In this paper, the field of high-power laser radiation is considered within the paraxial approximation. In contrast to [13], we discuss various cases of the relationship between the wavelength and the pulse length $c\Delta t$ (Δt is the pulse duration), and in addition, we consider Hermite–Gaussian beams of arbitrary mode and arbitrary polarisation. The field vectors are found through a direct solution to the parabolic equation. It is shown that in the case of sufficiently short pulses, corrections to the transverse components of the field vectors arise not in the second but in the first order in parameter (1). These corrections are calculated in the case of axially symmetric Hermite–Gaussian beams with an arbitrary pulse shape.

2. The field of laser radiation

We will describe laser radiation in a vacuum by using the field strengths $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$. The electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ satisfies the wave equation

$$\Delta \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \qquad (2a)$$

given that

$$\mathrm{div}\boldsymbol{E} = 0. \qquad (2b)$$

We assume that the radiation propagates in the direction of the z axis, and the electric field vector is dependent on the

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Received 9 March 2015; revision received 24 June 2015
Kvantovaya Elektronika **45** (11) 1063–1068 (2015)
Translated by I.A. Ulitkin

'fast' time $\tau = t - z/c$ and dimensionless parameter $\sigma = (t - z/c)/\Delta t$, which determines the pulsed nature of radiation: $\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}(\boldsymbol{r}, \tau, \sigma)$. We represent $\boldsymbol{E}(\boldsymbol{r}, \tau, \sigma)$ in the form of the Fourier expansion over the 'fast' time:

$$\boldsymbol{E}(\boldsymbol{r},\tau,\sigma) = (2\pi)^{-1}\int \mathrm{d}\omega \boldsymbol{E}_\omega(\boldsymbol{r},\sigma)\exp(-\mathrm{i}\omega\tau). \tag{3}$$

Introducing the dimensionless spatial coordinates $X, Y, Z = x/a$, y/a, z/z_R, in view of (1) we obtain an equation for the Fourier amplitudes of the strength $\boldsymbol{E}_\omega(\boldsymbol{r}, \sigma)$ in the form

$$\Delta_\perp \boldsymbol{E}_\omega + 4\mathrm{i}\frac{\omega}{\omega_0}\frac{\partial \boldsymbol{E}_\omega}{\partial Z} + \mu^2\frac{\partial^2 \boldsymbol{E}_\omega}{\partial Z^2} - \frac{2\lambda_0}{\pi c\Delta t}\frac{\partial^2 \boldsymbol{E}_\omega}{\partial\sigma\partial Z} = 0. \tag{4}$$

Here, $\Delta_\perp$ is the Laplace operator with respect to the transverse coordinates X, Y. One can see that the third term in equation (4) has second-order smallness and can be omitted in the zero-order approximation. The role of the last term in (4) essentially depends on the duration of the pulse under consideration. It is usually assumed [13] that the pulse length far exceeds the wavelength $\lambda_0/(c\Delta t) \ll 1$, or that the pulse duration is longer than the period of oscillations $\Delta t \gg T = 2\pi/\omega_0$. At the same time, the order of smallness is not specified. Consider in detail the possible relationship between the wavelength and the pulse length. If the last term in equation (4) is on the order of μ^2 or less, then this situation corresponds to sufficiently long pulses with a length of $c\Delta t \geqslant 2z_R$. For example, at a wavelength $\lambda_0 = 1$ μm and $a = 10$ μm the pulse duration is $\Delta t \geqslant 2$ ps. In this case, the last term in equation (4) does not affect the evolution of the Fourier amplitudes in the zeroth approximation. Then the vectors of the field can be represented as known expansions [3] in even powers of the parameter μ:

$$\boldsymbol{E}_\omega = \boldsymbol{E}^0_\omega + \mu^2\boldsymbol{E}^2_\omega + \ldots .$$

If the last term in equation (4) is small, on the order of μ, then in the zeroth approximation it can also be omitted. In this case, the pulses can be shorter: $c\Delta t \approx 2a$. For example, at $a = 10$ μm the duration of the pulse is $\Delta t \approx 60$ fs. Pulses of this duration are easily generated by modern high-power lasers. Then, the field vectors can be represented as a series of expansions:

$$\boldsymbol{E}_\omega = \boldsymbol{E}^0_\omega + \mu\boldsymbol{E}^1_\omega + \mu^2\boldsymbol{E}^2_\omega \ldots .$$

In the case of ultrashort pulses, when $c\Delta t \approx 2\lambda_0/\pi$, the last term in equation (4) is comparable to the main terms and the equation in the zeroth approximation becomes much more complicated. However, in this case, which is not considered here, the paraxial approximation is inapplicable.

Thus, as discussed in [13], the paraxial approximation can be quite adequate in describing a femtosecond laser.

In the zeroth approximation the laser field $\boldsymbol{E}^0_\omega$ can be treated in the form of transverse electromagnetic waves. Therefore, the vector $\boldsymbol{E}^0_\omega$ has only transverse components $\boldsymbol{E}^0_\omega \equiv \boldsymbol{E}^0_{\omega\perp}$, so that in the zeroth approximation the longitudinal component in the direction of the wave propagation is absent: $E^0_{\omega z} = 0$. It is assumed that equation (4) describes only transverse components of the vector $\boldsymbol{E}_\omega$, not only in the zeroth but also in all the subsequent approximations. This is due to the fact that the field components must satisfy equation (2b). This equation at given transverse components defined by equation (4) can be used to find, by successive approximations, the longitudinal component of the vector $\boldsymbol{E}_\omega(\boldsymbol{r}, \sigma)$.

3. Transverse component of the electric field

According to (4) the vector $\boldsymbol{E}^0_{\omega\perp}$ in the zeroth approximation is described by the parabolic equation

$$\Delta_\perp \boldsymbol{E}^0_{\omega\perp} + 4\mathrm{i}\frac{\omega}{\omega_0}\frac{\partial \boldsymbol{E}^0_{\omega\perp}}{\partial Z} = 0. \tag{5a}$$

Formally, this means neglecting the second derivative with respect to the longitudinal coordinate z.

Obviously, equation (5a) does not contain the parameter σ in the explicit form and therefore the vector $\boldsymbol{E}^0_{\omega\perp}(\boldsymbol{r},\sigma)$ can be represented as the product:

$$\boldsymbol{E}^0_{\omega\perp}(\boldsymbol{r},\sigma) = \boldsymbol{E}_\omega(\boldsymbol{r})f(\sigma), \tag{5b}$$

where the function $f(\sigma)$ determines the pulse shape. The function $f(\sigma)$ may be specified differently, but it should be rather smooth. For example, it is assumed in [13] that $f(\sigma) = \cos^2[\pi(t - z/c)/(2\Delta t)]$, in [14] $f(\sigma) = \exp\{-[(t - z/c - z_0/c)/\Delta t]^s\}$ and so on. Note that Bel'skii and Khapalyuk [15] considered the propagation of a two-dimensional pulse limited in space and time.

In the axially symmetric case with cylindrical coordinates $\rho \equiv r/a$ and Z, where r is the distance from the beam axis, equation (5a) takes the form:

$$\frac{\partial^2 \boldsymbol{E}_{\omega\perp}(\boldsymbol{r})}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial \boldsymbol{E}_{\omega\perp}(\boldsymbol{r})}{\partial\rho} + 4\mathrm{i}\frac{\omega}{\omega_0}\frac{\partial \boldsymbol{E}_{\omega\perp}(\boldsymbol{r})}{\partial Z} = 0. \tag{6}$$

Here, relation (5b) is taken into account. Equation (6) has a set of solutions. Usually, use is made of a self-similar solution of form [6]:

$$\boldsymbol{E}_{\omega\perp m}(r,z) = \frac{\boldsymbol{E}^0_{\omega m}(0)}{(1 + \mathrm{i}Z\omega_0/\omega)^{m+1}}g_m(\zeta). \tag{7}$$

Here, $\boldsymbol{E}^0_{\omega\perp}$ is the field amplitude in the focus on the beam axis; m is a positive integer, including zero; $\zeta = \rho^2/(1 + \mathrm{i}Z\omega_0/\omega)$; and $g_m(\zeta)$ is the desired function, which, according to (6) and (7), satisfies the ordinary differential equation

$$\zeta\frac{\mathrm{d}^2 g}{\mathrm{d}\zeta^2} + (1 + \zeta)\frac{\mathrm{d}g}{\mathrm{d}\zeta} + (m + 1)g = 0. \tag{8}$$

This equation has the solution [6]

$$g_m(\zeta) = \mathrm{e}^{-\zeta}L_m(\zeta), \tag{9}$$

where $L_m(\zeta)$ is the mth-order Laguerre polynomial. Polynomials $L_m(\zeta)$ form an orthonormal system of functions with weight $\exp(-\zeta)$ [16]. Thus, in order not to go beyond the parabolic approximation, the number m cannot be too large [6].

Thus, the solution of the parabolic equation (6) is represented in the form of the Hermite–Gaussian beam of the mth-order mode [6, 7]:

$$\boldsymbol{E}^0_{\omega\perp} \to \boldsymbol{E}_\omega(\boldsymbol{r})f(\sigma) = \frac{\boldsymbol{E}^0_{\omega m}(0)f(\sigma)}{(1 + \mathrm{i}Z\omega_0/\omega)^{m+1}}L_m(\zeta)\mathrm{e}^{-\zeta}. \tag{10}$$

Due to the linearity of equation (6), a superposition of modes (10) is its solution.

If the frequency spectrum of radiation, $\Delta\omega$, is quite narrow ($\Delta\omega/\omega_0 \ll 1$), the Fourier spectrum of the amplitude $\boldsymbol{E}^0_{\omega\perp}$ has a sharp peak in the vicinity of the carrier frequency ω_0. Then, the transverse components of the laser field (3) in the zeroth approximation can be written as

$$\boldsymbol{E}_{\perp m}(\boldsymbol{r},\tau,\sigma) \to \boldsymbol{E}^0_{\omega_0 \perp m}(r,\tau,\sigma)\exp(-\mathrm{i}\omega_0\tau).$$

We write the expression for the transverse components of the electric field in the equivalent form (assuming $\omega_0 = \omega$):

$$\boldsymbol{E}^0_{\perp m}(\boldsymbol{r},\tau,\sigma) = f(\sigma)\boldsymbol{E}_m(r,z)\exp(\mathrm{i}\theta), \tag{11a}$$

where the complex amplitude of the mth-order mode

$$\boldsymbol{E}_m(r,z) = \frac{\boldsymbol{E}^0_m(0)}{(1+\mathrm{i}Z)^{m+1}} L_m(\zeta)\mathrm{e}^{-\zeta}, \tag{11b}$$

and the phase

$$\theta = \omega(z/c - t), \quad \zeta = \rho^2/(1+\mathrm{i}Z). \tag{11c}$$

The Laguerre polynomial of order m, in general, has an expansion [16]:

$$L_m(\zeta) = \sum_{s=0}^{m} \frac{(-1)^s \Gamma^2(m+1)}{\Gamma^2(s+1)\Gamma(m-s+1)}\zeta^s,$$

where $\Gamma(m)$ is the gamma function. Because the argument $\zeta = \rho^2/[(1+Z^2)^{1/2}[\exp(-\mathrm{i}\chi)$, where $\chi = \arctan Z$, is complex, then each term of the Laguerre polynomial contains a phase factor. Therefore, formula (11a) can be rewritten in the form

$$\boldsymbol{E}(r,z,\tau,\sigma) = f(\sigma)\sum_{s=0}^{m}\boldsymbol{E}_{ms}(r,z)\exp(\mathrm{i}\psi_{ms}). \tag{12a}$$

Here, we have introduced the amplitudes that determine the polarisation of the wave,

$$\boldsymbol{E}_{ms}(r,z) = \frac{\boldsymbol{E}^0_m(0)}{(1+Z^2)^{(m+1)/2}} \frac{(-1)^s\Gamma^2(m+1)}{\Gamma^2(s+1)\Gamma(m-s+1)} \times \frac{\rho^{2s}}{(1+Z^2)^{s/2}}\exp\left(-\frac{\rho^2}{1+Z^2}\right), \tag{12b}$$

and the phases

$$\psi_{ms} = \theta + \frac{\rho^2 Z}{1+Z^2} - (m+1+s)\chi. \tag{12c}$$

Note that the phase velocity of radiation in question exceeds the speed of light in vacuum.

In considering a particular mode, it is more convenient to use the description of the electric field of a Gaussian beam in the zeroth approximation in form (11a) rather than in (12a). Here are some examples. The field of the fundamental mode with an arbitrary polarisation is

$$\boldsymbol{E}^0_{\perp 0}(\rho,Z,\tau,\sigma) = \frac{f(\sigma)\boldsymbol{E}^0_0(0)}{2(1+Z^2)^{1/2}}\exp\left(-\frac{\rho^2}{1+Z^2}\right)\exp(\mathrm{i}\varphi_0). \tag{13a}$$

The field of the first mode is

$$\boldsymbol{E}^0_{\perp 1}(\rho,Z,\tau,\sigma) = \frac{f(\sigma)\boldsymbol{E}^0_1(0)}{2(1+Z^2)^{3/2}}\sqrt{(1-\rho^2)^2+Z^2} \times \exp\left(-\frac{\rho^2}{1+Z^2}\right)\exp[\mathrm{i}(\varphi_1+\xi_1)]. \tag{13b}$$

The field of the second mode is

$$\boldsymbol{E}^0_{\perp 2}(\rho,Z,\tau,\sigma) = \frac{f(\sigma)\boldsymbol{E}^0_2(0)}{2(1+Z^2)^{5/2}} \times \sqrt{[\rho^4 - 4\rho^2 + 2(1-Z^2)]^2 + 16Z^2(1-\rho^2)^2} \times \exp\left(-\frac{\rho^2}{1+Z^2}\right)\exp[\mathrm{i}(\varphi_2+\xi_2)]. \tag{13c}$$

In these formulas, $\varphi_m = \theta + Z\rho^2/(1+Z^2) - (2m+1)\chi$; $\xi_1 = \arctan[Z/(1-\rho^2)] + \pi\eta(1-\rho^2)$; $\xi_2 = \arctan 4Z(1-\rho^2)/[\rho^4 - 4\rho^2 + 2(1-Z^2)] + \pi\eta\{4Z(1-\rho^2)/[\rho^4 - 4\rho^2 + 2(1-Z^2)]\}$; and $\eta(x)$ is the Heaviside step function.

One can see from (13b) that at $Z = 0$ the electric field of the first mode disappears on a circle with a radius equal to the radius of a Gaussian beam.

In the case of sufficiently long pulses, a consistent solution of the parabolic equation (6) is given as an expansion in even powers of the parameter μ [3, 17]. In contrast to long pulses, the transverse components of the electric vector of sufficiently short and smooth pulses should have, according to equation (4), first-order approximation corrections. These corrections are found from the equation

$$\Delta_\perp \boldsymbol{E}^1_{\perp\omega} + 4\mathrm{i}\frac{\partial \boldsymbol{E}^1_{\perp\omega}}{\partial Z} - \frac{2\lambda_0}{\pi c\Delta t}\frac{\partial \ln f(\sigma)}{\partial\sigma}\frac{\partial \boldsymbol{E}^0_{\perp\omega}}{\partial Z} = 0. \tag{14}$$

If the field in the zeroth approximation, $\boldsymbol{E}^0_{\perp\omega}$, describes a beam of the mth mode by formula (11a), then using the recurrence relation for the Laguerre polynomials [16],

$$\zeta\frac{\partial L_m(\zeta)}{\partial\zeta} = (\zeta - m - 1)L_m(\zeta) + L_{m+1}(\zeta), \tag{15}$$

we obtain

$$\frac{\partial \boldsymbol{E}^0_{\perp m}}{\partial Z} = -\frac{\mathrm{i}\boldsymbol{E}^0_m(0)f(\sigma)}{(1+\mathrm{i}Z)^{m+2}}L_{m+1}(\zeta)\exp(-\zeta). \tag{16}$$

It is seen that differentiation of (11a) in Z increases the Laguerre polynomial by one order of magnitude. At the same time, function (16), as well as (11a), describes the Hermite–Gaussian beam and satisfies the parabolic equation (6). Given these properties of such beams, we obtain the general solution to equation (14):

$$\boldsymbol{E}^1_{\perp m}(r,z,t) = -\frac{\lambda_0}{2\pi c\Delta t}Z\frac{\boldsymbol{E}^0_m(0)f'(\sigma)}{(1+\mathrm{i}Z)^{m+2}} \times L_{m+1}(\zeta)\exp(-\zeta + \mathrm{i}\theta). \tag{17}$$

One can see from (17) that for any mode in the waist plane of the Hermite–Gaussian beam, the corrections to the transverse field components at $Z = 0$ are absent. We note also that the first-order corrections are entirely related to the pulsed nature of radiation. In the case of a Gaussian beam ($m = 0$), equation (17) yields an expression for the correction:

$$\boldsymbol{E}^1_{\perp 0}(r,z,t) = -\frac{\lambda_0}{4\pi c\Delta t} Z \frac{\boldsymbol{E}^0_0(0) f'(\sigma)}{(1+Z^2)^{3/2}} \sqrt{(1-\rho^2)^2 + Z^2} \times \exp\left(-\frac{\rho^2}{1+Z^2}\right) \exp(\mathrm{i}\psi_0), \quad (18a)$$

where the phase

$$\psi_0 = \theta - 3\chi + \frac{Z\rho^2}{1+Z^2} + \arctan\frac{Z}{1-\rho^2} + \pi\eta(1-\rho^2);$$

$$f'(\sigma) \equiv \frac{\partial f}{\partial \sigma}.$$

In the particular case of a smooth pulse given by $f(\sigma) = \cos^2[\pi(t - z/c)/(2\Delta t)]$, formula (18a) coincides with the results of [13], obtained for linearly polarised radiation.

The first-order correction for a beam with $m = 1$, is defined, according to (17), by the formula

$$\boldsymbol{E}^1_{\perp 1}(r,z,t) = -\frac{\lambda_0}{2\pi c\Delta t} Z \frac{\boldsymbol{E}^0_1(0) f'(\sigma)}{(1+Z^2)^{5/2}} \times \sqrt{[2(1-Z^2) + \rho^2(\rho^2-4)]^2 + 16Z^2(1-\rho^2)^2} \times \exp\left(-\frac{\rho^2}{1+Z^2}\right) \exp(\mathrm{i}\psi_1), \quad (18b)$$

where the phase

$$\psi_1 = \theta - 5\chi + \frac{Z\rho^2}{1+Z^2} + \arctan\frac{4Z(1-\rho^2)}{\rho^2(\rho^2-4) + 2(1-Z^2)} + \pi\eta\left[\frac{1-\rho^2}{\rho^2(\rho^2-4)+2(1-Z^2)}\right].$$

Comparing formulas (13) and (18a), one can see a common property: the field in the zeroth approximation for the first mode and the first-order correction to the fundamental-mode field exhibit the same spatial dependence; in addition, the same spatial dependence is demonstrated by the field of the second mode in the zeroth approximation and the first-order correction to the field of the first mode.

In the case of higher-order modes, the form of the first-order corrections (17) becomes more complex. Corrections of higher-order approximations μ^q ($q \geqslant 2$) to the transverse components of the electric field are determined in accordance with (4) by the equation

$$\Delta_\perp \boldsymbol{E}^q_{\omega m} + 4\mathrm{i}\frac{\partial \boldsymbol{E}^q_{\omega m}}{\partial Z} + \frac{\partial^2 \boldsymbol{E}^{q-2}_{\omega m}}{\partial Z^2} - \frac{2\lambda_0}{\pi c\Delta t}\frac{\partial^2 \boldsymbol{E}^{q-1}_{\omega m}}{\partial\sigma\partial Z} = 0. \quad (19)$$

It follows from this equation that the higher-order corrections can be calculated if the pulse function is differentiated the necessary number of times.

We find the second-order correction ($q = 2$) to the transverse components of the electric field strength of arbitrary mode. In view of formulas (10), (16a) and (17), from (19) we obtain the general expression:

$$\boldsymbol{E}^2_{\perp m}(r,z,t) = \frac{\boldsymbol{E}^0_m(0)}{4\mathrm{i}(1+\mathrm{i}Z)^{m+3}} \times \left\{ Z\left[f(\sigma) L_{m+2}(\zeta) - \frac{\lambda_0^2 f''(\sigma)}{(\pi c\Delta t)^2}(1+\mathrm{i}Z) L_{m+1}(\zeta)\right] - \frac{Z^2\lambda_0^2 f''(\sigma)}{2(\pi c\Delta t)^2} L_{m+2}(\zeta)\right\} \exp(-\zeta + \mathrm{i}\theta). \quad (20)$$

It can be seen that the second-order corrections, as well as the first-order ones, in the waist plane ($Z = 0$) of the beam of arbitrary mode are absent. In the case of a Gaussian beam, the second-order correction according to (20) is described by the expression

$$E^2_{\perp 0}(r,z,t) = \frac{E^0_0(0)}{4}\left\{\left[Z - \frac{Z^2\lambda_0^2 f''(\sigma)}{2(\pi c\Delta t)^2}\right] \times \frac{\sqrt{[\rho^2(\rho^2-4) + 2(1-Z^2)]^2 + 16Z^2(1-\rho^2)^2}}{(1+Z^2)^{5/2}} \exp[-\mathrm{i}(5\chi - \xi_2)] - \frac{Z\lambda_0^2 f''(\sigma)}{(\pi c\Delta t)^2}\frac{\sqrt{(1-\rho^2)^2+Z^2}}{(1+Z^2)^{3/2}}\exp[-\mathrm{i}(3\chi - \xi_1)]\right\} \times \exp\left(-\frac{\rho^2}{1+Z^2}\right)\exp\left[\mathrm{i}\left(\theta + \frac{Z\rho^2}{1+Z^2} - \frac{\pi}{2}\right)\right]. \quad (21)$$

This shows that the spatial dependence of the correction to the field of the fundamental mode in the second approximation is a combination of field dependences in the zeroth approximation for the first and second modes. The second-order corrections for the beams of the first and other modes according to (20) are more complicated.

Obtaining corrections of higher-order approximations ($q > 2$) from equation (19) is associated with very cumbersome calculations. However, we can see that corrections of any order vanish in the waist plane of the beam of any mode. Note that high-order corrections to the parameter μ for the fundamental mode of a Gaussian linearly polarised beam have been considered in [17]. However, Salamin [17] has not taken into account the pulsed nature of radiation but obtained only the corrections of even orders (μ^2, μ^4, μ^6,...). In addition, the expansions have been assumed valid at a sufficiently large value of the parameter, $\mu = 0.8$, which is not correct. Note also that the corrections calculated in [17] do not disappear in the focal plane, as is the case of the derived expression (21). Perhaps, this is due to the fact that the corrections obtained in [17] are some particular solution to the parabolic equation of form (19) (without the last term), whereas in deriving (20) and (21) we used the general properties of Hermite–Gaussian beams.

Note that in [17] the parabolic equation was considered for the potentials of a linearly polarised Gaussian beam rather than for the electric field vector.

4. The longitudinal electric field component of a Hermite–Gaussian beam

Consider equation (2b) by assuming that the longitudinal component of the electric field strength for any mode has the form

$$E_{zm}(X,Y,Z,\tau,\sigma,) = E_z(X,Y,Z,\sigma)\exp(\mathrm{i}\theta).$$

In this case, we obtain

$$E_z = \frac{\mathrm{i}}{ka}\left(\frac{\partial E_{mx}}{\partial X} + \frac{\partial E_{my}}{\partial Y} + \frac{a}{z_R}\frac{\partial E_z}{\partial Z}\right). \tag{22}$$

It follows that the longitudinal component $\boldsymbol{E}(\boldsymbol{r}, \tau, \sigma)$ is a quantity of the first-order smallness, which, using formulas (11), can be written as

$$E^1_{zm}(X,Y,Z,t) = \frac{2\mathrm{i}\,f(\sigma)}{ka(1+Z^2)^{(m+2)/2}}[XE^0_{xm}(0) + YE^0_{ym}(0)]$$

$$\times\frac{1}{m+1}\frac{\mathrm{d}L_{m+1}}{\mathrm{d}\zeta}\exp(-\zeta + \mathrm{i}\varphi_m), \tag{23a}$$

where the phase

$$\varphi_m \equiv \theta - (m+2)\chi. \tag{23b}$$

In deriving (23a) we used the formula [16]

$$\frac{\mathrm{d}L_m}{\mathrm{d}\zeta} - L_m = \frac{1}{m+1}\frac{\mathrm{d}L_{m+1}}{\mathrm{d}\zeta}.$$

Note that formula (23a) describes the longitudinal field in the first-order approximation for an arbitrary polarisation of radiation. This formula shows that the longitudinal electric field is absent on the axis of a Hermite–Gaussian beam of any mode. For the fundamental mode we obtain from (23a)

$$E^1_{z0}(X,Y,Z,\sigma,t) = -\frac{2\mathrm{i}\,f(\sigma)}{ka(1+Z^2)}[XE^0_{0x}(0) + YE^0_{0y}(0)]$$

$$\times\exp\left[-\frac{\rho^2}{1+Z^2} + \mathrm{i}\left(\varphi_0 + \frac{Z\rho^2}{1+Z^2}\right)\right]. \tag{24a}$$

For the first mode

$$E^1_{z1}(X,Y,Z,\sigma,t) = -\frac{2\mathrm{i}\,f(\sigma)\sqrt{(2-\rho^2)^2 + 4Z^2}}{ka(1+Z^2)^2}$$

$$\times[XE^0_{1x}(0) + YE^0_{1y}(0)]\exp\left[-\frac{\rho^2}{1+Z^2} + \mathrm{i}(\varphi_2 + \xi)\right]. \tag{24b}$$

Here,

$$\xi = \frac{Z\rho^2}{1+Z^2} + \arctan\frac{2Z}{2-\rho^2} + \pi\eta(2-\rho^2).$$

One can see from (24b) that in the plane of the beam waist the longitudinal electric field of the first mode vanishes on the circle of radius $\rho = \sqrt{2}$.

Neglecting the pulsed nature of radiation, the next correction to the longitudinal electric field arises in the third-order approximation with respect to the parameter μ [17]. In the case of a quite smooth and short pulse, the next correction term is a quantity of second-order smallness. It can be found from equation (22) using formula (17):

$$E^2_{zm}(X,Y,Z,t) = -\frac{\mathrm{i}\lambda_0 Zf'(\sigma)}{ka\pi c\Delta t(m+2)(1+Z^2)^{(m+3)/2}}$$

$$\times[XE^0_{mx}(0) + YE^0_{my}(0)]\frac{\mathrm{d}L_{m+2}}{\mathrm{d}\zeta}\exp(-\zeta + \mathrm{i}\varphi_{m+1}). \tag{25}$$

It can be seen that the longitudinal electric field in the second-order approximation is absent not only on the beam axis, but also in the entire focal plane.

5. The magnetic field of a Hermite–Gaussian beam

The components of the induction vector of the magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$ can be found using Maxwell's equation: $\mathrm{rot}\,\boldsymbol{E} = -1/c(\partial\boldsymbol{B}/\partial t)$. Representing the field in the form of the Fourier expansion (3), we obtain

$$\mathrm{i}\omega\boldsymbol{B}_\omega - \frac{1}{\Delta t}\frac{\partial\boldsymbol{B}_\omega}{\partial\sigma} = c\,\mathrm{rot}\,\boldsymbol{E}_\omega - \frac{1}{\Delta t}\frac{\partial[\boldsymbol{e}_1\boldsymbol{E}_\omega]}{\partial\sigma} + \mathrm{i}ck[\boldsymbol{e}_1\boldsymbol{E}_\omega]. \tag{26}$$

Here, $\boldsymbol{e}_1$ is the unit vector in the direction of the wave propagation, i.e., the z axis. It follows from equation (26) that in the zeroth approximation, the transverse components of the laser field of the mth mode are related by the expressions that are characteristic of the electromagnetic wave in a vacuum:

$$B^0_{xm}(r,z,\sigma,t) = -E^0_{ym}(r,z,\sigma,t),$$
$$B^0_{ym}(r,z,\sigma,t) = E^0_{xm}(r,z,\sigma,t). \tag{27}$$

The same relationships are preserved in the first-order approximation. In the second-order approximation we obtain

$$B^2_{xm}(r,z,\sigma,t) = -E^2_{ym}(r,z,\sigma,t) + \frac{1}{\mathrm{i}ka}\left(\frac{\partial E^1_{zm}}{\partial Y} - \frac{a}{z_R}\frac{\partial E^0_{ym}}{\partial Z}\right),$$
$$B^2_{ym}(r,z,\sigma,t) = E^2_{xm}(r,z,\sigma,t) - \frac{1}{\mathrm{i}ka}\left(\frac{\partial E^1_{zm}}{\partial X} - \frac{a}{z_R}\frac{\partial E^0_{xm}}{\partial Z}\right). \tag{28}$$

Here, the components of the electric field $\boldsymbol{E}^2_{\perp m}(r,z,\sigma,t)$ are found from the general formula (20). Using formulas (16) and (23a), we obtain the general expressions for the transverse components of the magnetic field in the second-order approximation:

$$B^2_{xm}(r,z,\sigma,t) = -E^2_{ym}(r,z,\sigma,t) + \frac{2f(\sigma)}{(ka)^2(m+1)(1+\mathrm{i}Z)^{m+2}}$$

$$\times\left\{E^0_{ym}(0)\frac{\mathrm{d}L_{m+1}}{\mathrm{d}\zeta} + \frac{2Y}{(m+2)(1+\mathrm{i}Z)}[XE^0_{xm}(0) + YE^0_{ym}(0)]\right.$$

$$\left.\times\frac{\mathrm{d}^2L_{m+2}}{\mathrm{d}\zeta^2}\right\}\exp(-\zeta + \mathrm{i}\theta) + \frac{f(\sigma)E^0_{ym}(0)L_{m+1}(\zeta)}{kz_R(1+\mathrm{i}Z)^{m+2}}\exp(-\zeta + \mathrm{i}\theta),$$

$$B^2_{ym}(r,z,\sigma,t) = E^2_{xm}(r,z,\sigma,t) - \frac{2f(\sigma)}{(ka)^2(m+1)(1+\mathrm{i}Z)^{m+2}}$$

$$\times\left\{E^0_{xm}(0)\frac{\mathrm{d}L_{m+1}}{\mathrm{d}\zeta} + \frac{2X}{(m+2)(1+\mathrm{i}Z)}[XE^0_{xm}(0) + YE^0_{ym}(0)]\right.$$

$$\left.\times\frac{\mathrm{d}^2L_{m+2}}{\mathrm{d}\zeta^2}\right\}\exp(-\zeta + \mathrm{i}\theta) + \frac{f(\sigma)E^0_{xm}(0)L_{m+1}(\zeta)}{kz_R(1+\mathrm{i}Z)^{m+2}}\exp(-\zeta + \mathrm{i}\theta).$$

This implies that even for the fundamental mode of a Gaussian beam with an arbitrary polarisation the magnetic field in the second-order approximation is described by very complex formulas. They are simplified in the case of a linear polarisation.

The longitudinal component of the magnetic field, as well as of the electric field, occurs only in the first-order approximation:

$$B^1_{zm} = \frac{1}{\mathrm{i}ka}\left(\frac{\partial E^0_{ym}}{\partial X} - \frac{\partial E^0_{xm}}{\partial Y}\right) = \frac{2f(\sigma)[XE^0_{ym}(0) - YE^0_{xm}(0)]}{\mathrm{i}ka(m+1)(1+\mathrm{i}Z)^{m+2}}$$

$$\times \frac{\mathrm{d}L_{m+1}}{\mathrm{d}\zeta}\exp(-\zeta + \mathrm{i}\theta). \quad (29a)$$

It is easy to see that formula (29a) is similar to formula (23a) for the longitudinal electrical field. As in the case of the electric field, the longitudinal component of the magnetic field is absent on the axis of a Hermite–Gaussian beam of any mode. For the fundamental mode, equation (29a) yields

$$B^1_{z0} = \frac{2\mathrm{i}f(\sigma)[XE^0_{ym}(0) - YE^0_{xm}(0)]}{ka(1+Z^2)}$$

$$\times \exp\left[-\frac{\rho^2}{1+Z^2} + \mathrm{i}\left(\varphi_0 + \frac{Z\rho^2}{1+Z^2}\right)\right], \quad (29b)$$

where φ_0 is found from formula (23b).

The longitudinal component of the magnetic field of the first mode is

$$B^1_{z1} = \frac{2\mathrm{i}f(\sigma)[XE^0_{ym}(0) - YE^0_{xm}(0)]}{ka(1+Z^2)^2}\sqrt{(2-\rho^2)^2 + 4Z^2}$$

$$\times \exp\left[-\frac{\rho^2}{1+Z^2} + \mathrm{i}(\varphi_2 + \xi)\right], \quad (29c)$$

and in the second-order approximation it is calculated similarly to (25).

6. Conclusions

We have considered the field of high-power laser radiation in the paraxial approximation by expanding the wave equation in parameter (1). The pulsed nature of radiation has been taken into account by using the parameter σ as an additional argument of the field. In this case, we have not assumed beforehand the existence of the pulse function, by which the field vectors are multiplied. This function of sufficiently general form arises naturally in considering the sequential relationships between the pulse length and wavelength of radiation in a parabolic equation. It has been shown in accordance with [13] that the paraxial approximation is applicable to describe femtosecond laser pulses; however, in contrast to the longer pulses, the transverse field components have first-order corrections rather than the second-order ones in parameter (1). In the axially symmetric case, we have obtained general expressions for the laser field in the zeroth, first- and second-order approximations of expansions in the parameter μ in the form of Hermite–Gaussian beams of arbitrary mode and arbitrary polarisation. Specific formulas for the first and the fundamental modes have been presented.

Typically, laser radiation is considered as a Gaussian beam. However, higher modes change the spatial structure of pulsed radiation, which, of course, affects its propagation behaviour. It is shown that for the higher-order corrections to be calculated, it is required to differentiate the pulse function by the necessary number of times. We emphasise that the expressions obtained are limited by the applicability of the paraxial approximation for the case of short pulses. The spectral width of the pulse, small compared with the carrier frequency ($\Delta\omega/\omega_0 \ll 1$), is related to the duration of a transform-limited pulse by the general expression: $\Delta\omega\Delta t \approx 2\pi$.

Note that the found second-order corrections to the transverse components of the electric field strength of the fundamental mode do not coincide with the corrections presented in [17]. The main difference lies in the fact that corrections [17] do not vanish in the focal plane of radiation, i.e. according to [17], the radiation power in the focal plane is determined not by the field of the zeroth approximation, but the field with a predetermined accuracy. As possible reasons for the discrepancies we indicate that the corrections obtained in [17] are one of the particular solutions of a parabolic equation in the second-order approximation.

The resulting expressions for the field vectors can be used in the analysis of the ponderomotive acceleration of relativistic electrons and acceleration in the regime of cyclotron autoresonance [5].

Acknowledgements. The authors thank the reviewers for their valuable comments.

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