

# Interaction of elliptically polarised cross-degenerate cnoidal waves in an isotropic gyrotropic medium with spatial dispersion of cubic nonlinearity

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**Abstract.** Three unusual classes of particular analytical solutions to a system of four nonlinear equations are found for slowly varying complex amplitudes of circularly polarised components of the electric field. The system describes the self-action and interaction of two elliptically polarised plane waves collinearly propagating in an isotropic medium with second-order frequency dispersion and spatial dispersion of cubic nonlinearity. The solutions correspond to self-consistent combinations of two elliptically polarised cnoidal waves whose mutually orthogonal polarisation components vary in accordance with pairwise identical laws during propagation. At the same time, the amplitudes of the component with the same circular polarisation are proportional to two different elliptic Jacobi functions with the same periods.

**Keywords:** cubic nonlinearity, spatial and frequency dispersion, elliptically polarised cnoidal wave, cross degeneration.

## 1. Introduction

Investigation of peculiarities of multisoliton complexes formed due to the interaction of stable single soliton-like nonlinear waves, i.e., self-consistent solutions to nonlinear problems of different types (solitons, breathers and cnoidal waves [1–4]), has recently aroused considerable interest [2–6], which is associated with the possibility of using such complexes for fast transfer of optical information. Moreover, these complexes are of interest as electromagnetic field structures formed by self-organisation processes in nonlinear systems with many degrees of freedom. The propagation problems to be solved are always multiparametric and in most cases described by nonintegrable systems of nonlinear differential equations [7–10]. An example of such problems is the interaction of two mutually orthogonal circularly polarised components of an electromagnetic wave propagating through a nonlinear medium with frequency dispersion and spatial dispersion of cubic nonlinearity [11–19]. Peculiarities of their possible interaction have been previously analysed on the basis of the found exact particular [11–13] and approximate [14–19] solutions, obtained through various approaches (perturbation theory [14, 15], adiabatic approximation [16–19]).

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In this paper we present for the first time unusual particular analytical solutions to a system of four nonlinear equations for slowly varying complex amplitudes of circularly polarised components of the electric field, which describes the self-action and interaction of two elliptically polarised plane waves collinearly propagating in an isotropic medium with second-order frequency dispersion and spatial dispersion of cubic nonlinearity. These amazing solutions represent self-consistent combinations of two cnoidal waves for which time-dependent moduli of slowly varying amplitudes of circularly polarised components of two different elliptically polarised plane waves  $A_{\pm}$  and  $B_{\pm}$  change during the propagation in accordance with pairwise identical laws ( $A_{\pm}$  is proportional to  $|B_{\mp}|$ ) given by different elliptic Jacobi functions [20]. Solutions of this cross-degenerate type are not exotic, but form three classes of analytic solutions existing in the rather wide range of values of the material parameters of a nonlinear medium.

## 2. Initial system of equations and approximations used

The system of equations describing the evolution of slowly varying complex amplitudes  $A_{\pm}(z, \tau)$  and  $B_{\pm}(z, \tau)$  of circularly polarised orthogonal components (subscripts  $\pm$ ) of two elliptically polarised waves having the same carrier frequency  $\omega$  and propagating almost collinear to the  $z$  axis in an isotropic gyrotropic medium with spatial dispersion of cubic nonlinearity and second-order group velocity dispersion can be easily obtained as a generalisation (frequency dispersion in the case of propagation of the waves in the same direction is additionally taken into account) of a system of equations given in [21,22]:

$$\left(\frac{\partial}{\partial z} - i\frac{k_2}{2}\frac{\partial^2}{\partial \tau^2}\right)A_{\pm} = i\left[\pm\rho_0 - \left(\frac{\sigma_1}{2} \mp \rho_1\right)(|A_{\pm}|^2 + 2|B_{\pm}|^2) - \left(\frac{\sigma_1}{2} + \sigma_2\right)(|A_{\mp}|^2 + |B_{\mp}|^2)\right]A_{\pm} - i\left(\frac{\sigma_1}{2} + \sigma_2\right)A_{\mp}B_{\mp}^*B_{\pm}, \quad (1)$$

$$\left(\frac{\partial}{\partial z} - i\frac{k_2}{2}\frac{\partial^2}{\partial \tau^2}\right)B_{\pm} = i\left[\pm\rho_0 - \left(\frac{\sigma_1}{2} \mp \rho_1\right)(|B_{\pm}|^2 + 2|A_{\pm}|^2) - \left(\frac{\sigma_1}{2} + \sigma_2\right)(|B_{\mp}|^2 + |A_{\mp}|^2)\right]B_{\pm} - i\left(\frac{\sigma_1}{2} + \sigma_2\right)B_{\mp}A_{\mp}^*A_{\pm}. \quad (2)$$

The right-hand side of the system contains terms describing both self- and cross-modulation of all components, as well as their parametric four-wave mixing. Here,  $\tau$  is the time in the

moving coordinate system;  $k_2 = \partial^2 k / \partial \omega^2 = \text{const}$  characterises frequency dispersion; and  $k$  is the wave number. The values of the parameters  $\sigma_1 = 4\pi\omega^2 \chi_{xyxy}^{(3)} / (kc^2)$  and  $\sigma_2 = 2\pi\omega^2 \chi_{xyxy}^{(3)} / (kc^2)$  are given by two independent components of the tensor of local cubic nonlinearity  $\chi^{(3)}(\omega; -\omega, \omega, \omega)$ , and  $\rho_{0,1} = 2\pi\omega^2 \times \gamma_{0,1} / c^2$  are given by pseudoscalar constants  $\gamma_{0,1}$  of linear and nonlinear gyration. It is easy to see that systems (1) and (2) are completely symmetric: the equations for the ‘+’ and ‘-’ components of the field transform into each other during the simultaneous substitution  $\rho_{0,1} \leftrightarrow -\rho_{0,1}$  and  $+ \leftrightarrow -$  and the equations for the waves  $A_{\pm}$  and  $B_{\pm}$  transform into each other during the substitution  $A_{\pm} \leftrightarrow B_{\pm}$ .

We will seek a particular solution to system (1), (2) in the form:

$$A_{\pm}(z, \tau) = \rho_{a\pm}(\tau) \exp[i\varphi_{a\pm} + i(k_{\pm}^{(a)} \pm \rho_0)z], \quad (3)$$

$$B_{\pm}(z, \tau) = \rho_{b\pm}(\tau) \exp[i\varphi_{b\pm} + i(k_{\pm}^{(b)} \pm \rho_0)z]. \quad (4)$$

Here, the constants  $k_{\pm}^{(a,b)}$  and constant phases  $\varphi_{a\pm}$  and  $\varphi_{b\pm}$  providing the separation of the variables satisfy respectively the conditions  $k_{\pm}^{(a)} - k_{\mp}^{(a)} = k_{\pm}^{(b)} - k_{\mp}^{(b)}$ ,  $\varphi_{a+} - \varphi_{a-} - \varphi_{b+} + \varphi_{b-} = l\pi$  ( $l = 0, \pm 1, \pm 2, \dots$ ), arising due to the parametric interaction of circularly polarised components of the field in system (1), (2). Substituting (3), (4) into (1), (2), we obtain the system of second-order differential equations for  $\rho_{a\pm}(\tau)$  and  $\rho_{b\pm}(\tau)$ :

$$\begin{aligned} \ddot{\rho}_{a+}(\tau) - \frac{2}{k_2} \{ k_+^{(a)} \rho_{a+}(\tau) + \left( \frac{\sigma_1}{2} - \rho_1 \right) \rho_{a+}(\tau) \\ \times [\rho_{a+}^2(\tau) + 2\rho_{b+}^2(\tau)] + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \rho_{a+}(\tau) \\ \times [\rho_{a-}^2(\tau) + \rho_{b-}^2(\tau)] \pm \left( \frac{\sigma_1}{2} + \sigma_2 \right) \rho_{a-}(\tau) \rho_{b+}(\tau) \rho_{b-}(\tau) \} = 0, \end{aligned} \quad (5)$$

$$\begin{aligned} \ddot{\rho}_{a-}(\tau) - \frac{2}{k_2} \{ k_-^{(a)} \rho_{a-}(\tau) + \left( \frac{\sigma_1}{2} + \rho_1 \right) \rho_{a-}(\tau) \\ \times [\rho_{a-}^2(\tau) + 2\rho_{b-}^2(\tau)] + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \rho_{a-}(\tau) \\ \times [\rho_{a+}^2(\tau) + \rho_{b+}^2(\tau)] \pm \left( \frac{\sigma_1}{2} + \sigma_2 \right) \rho_{a+}(\tau) \rho_{b+}(\tau) \rho_{b-}(\tau) \} = 0, \end{aligned} \quad (6)$$

$$\begin{aligned} \ddot{\rho}_{b+}(\tau) - \frac{2}{k_2} \{ k_+^{(b)} \rho_{b+}(\tau) + \left( \frac{\sigma_1}{2} - \rho_1 \right) \rho_{b+}(\tau) \\ \times [2\rho_{a+}^2(\tau) + \rho_{b+}^2(\tau)] + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \rho_{b+}(\tau) \\ \times [\rho_{a-}^2(\tau) + \rho_{b-}^2(\tau)] \pm \left( \frac{\sigma_1}{2} + \sigma_2 \right) \rho_{a+}(\tau) \rho_{a-}(\tau) \rho_{b-}(\tau) \} = 0, \end{aligned} \quad (7)$$

$$\begin{aligned} \ddot{\rho}_{b-}(\tau) - \frac{2}{k_2} \{ k_-^{(b)} \rho_{b-}(\tau) + \left( \frac{\sigma_1}{2} + \rho_1 \right) \rho_{b-}(\tau) \\ \times [2\rho_{a-}^2(\tau) + \rho_{b-}^2(\tau)] + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \rho_{b-}(\tau) \\ \times [\rho_{a+}^2(\tau) + \rho_{b+}^2(\tau)] \pm \left( \frac{\sigma_1}{2} + \sigma_2 \right) \rho_{a+}(\tau) \rho_{a-}(\tau) \rho_{b+}(\tau) \} = 0. \end{aligned} \quad (8)$$

The subscripts ‘+’ and ‘-’ in this system correspond to even and odd  $l$ . We emphasise that system (5)–(8) is fully equivalent to (1), (2) to within the above simplifications.

### 3. Cross-degenerate solutions

Consider the case of cross degeneration, at which  $\rho_{a+}(\tau) = \alpha \rho_{b-}(\tau)$  and  $\rho_{b+}(\tau) = \beta \rho_{a-}(\tau)$ . Substituting these relations in (5)–(8) we can see that the constants  $\alpha, \beta$  and functions  $\rho_{a-}(\tau)$  and  $\rho_{b-}(\tau)$  in this case must satisfy the system of equations

$$\begin{aligned} \alpha \ddot{\rho}_{b-}(\tau) - \frac{2}{k_2} \{ k_+^{(a)} \alpha + \left( \frac{\sigma_1}{2} - \rho_1 \right) \\ \times \alpha [\alpha^2 \rho_{b-}^2(\tau) + 2\beta^2 \rho_{a-}^2(\tau)] + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \\ \times \alpha [\rho_{a-}^2(\tau) + \rho_{b-}^2(\tau)] \pm \left( \frac{\sigma_1}{2} + \sigma_2 \right) \beta \rho_{a-}^2(\tau) \} \rho_{b-}(\tau) = 0, \end{aligned} \quad (9)$$

$$\begin{aligned} \ddot{\rho}_{a-}(\tau) - \frac{2}{k_2} \{ k_-^{(a)} + \left( \frac{\sigma_1}{2} + \rho_1 \right) \\ \times [\rho_{a-}^2(\tau) + 2\rho_{b-}^2(\tau)] + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \\ \times [\alpha^2 \rho_{b-}^2(\tau) + \beta^2 \rho_{a-}^2(\tau)] \pm \left( \frac{\sigma_1}{2} + \sigma_2 \right) \alpha \beta \rho_{b-}^2(\tau) \} \rho_{a-}(\tau) = 0, \end{aligned} \quad (10)$$

$$\begin{aligned} \beta \ddot{\rho}_{a-}(\tau) - \frac{2}{k_2} \{ k_+^{(b)} \beta + \left( \frac{\sigma_1}{2} - \rho_1 \right) \\ \times \beta [2\alpha^2 \rho_{b-}^2(\tau) + \beta^2 \rho_{a-}^2(\tau)] + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \\ \times \beta [\rho_{a-}^2(\tau) + \rho_{b-}^2(\tau)] \pm \left( \frac{\sigma_1}{2} + \sigma_2 \right) \alpha \rho_{b-}^2(\tau) \} \rho_{a-}(\tau) = 0, \end{aligned} \quad (11)$$

$$\begin{aligned} \ddot{\rho}_{b-}(\tau) - \frac{2}{k_2} \{ k_-^{(b)} + \left( \frac{\sigma_1}{2} + \rho_1 \right) \\ \times [2\rho_{a-}^2(\tau) + \rho_{b-}^2(\tau)] + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \\ \times [\alpha^2 \rho_{b-}^2(\tau) + \beta^2 \rho_{a-}^2(\tau)] \pm \left( \frac{\sigma_1}{2} + \sigma_2 \right) \alpha \beta \rho_{a-}^2(\tau) \} \rho_{b-}(\tau) = 0. \end{aligned} \quad (12)$$

We will seek its solution in the form:

$$\begin{aligned} \rho_{a+}(\tau) = \alpha \text{Gcn}(\gamma\tau, \mu), \quad \rho_{a-}(\tau) = F \text{dn}(\gamma\tau, \mu), \\ \rho_{b+}(\tau) = \beta F \text{dn}(\gamma\tau, \mu), \quad \rho_{b-}(\tau) = \text{Gcn}(\gamma\tau, \mu), \end{aligned} \quad (13)$$

where  $\gamma$  is the scale factor, and  $\mu$  is the modulus of the elliptic Jacobi functions [20]. Substituting (13) into equations (9)–(12) transforms the latter into polynomial functions with respect to one of the Jacobi functions [in our case  $\text{sn}(\gamma\tau, \mu)$ ]. By equating to zero the coefficients of these polynomials together with the condition  $k_{\pm}^{(a)} - k_{\mp}^{(a)} = k_{\pm}^{(b)} - k_{\mp}^{(b)}$ , we form a system of nine algebraic equations for  $\gamma, \mu, F, \alpha, \beta, G, k_{\pm}^{(a)}$  and  $k_{\pm}^{(b)}$ . Its nontrivial solutions define the relations limiting the allowable values of the introduced constants at which cross-degenerate solutions (13) really exist. Using five equations of this system, we can conveniently express  $k_{\pm}^{(a)}, k_{\pm}^{(b)}$  and  $\gamma$  through  $\mu, F, \alpha, \beta, G$  and material parameters of the medium:

$$\begin{aligned} k_+^{(a)} = -\frac{1}{2\mu^2} \left\{ \left( \frac{\sigma_1}{2} - \rho_1 \right) [2\beta^2 F^2 \mu^2 + \alpha^2 G^2 (2\mu^2 - 1)] \right. \\ \left. + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \left[ \left( 1 \pm \frac{\beta}{\alpha} \right) F^2 \mu^2 + G^2 (2\mu^2 - 1) \right] \right\}, \end{aligned} \quad (14)$$

$$k_-^{(a)} = \frac{1}{2} \left\{ \left( \frac{\sigma_1}{2} + \rho_1 \right) [2F^2(\mu^2 - 1) - 3G^2] + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \right. \\ \left. \times \left[ \beta^2 \left( 1 \pm \frac{\alpha}{\beta} \right) F^2 \mu^2 - 2\beta^2 F^2 - \alpha^2 \left( 1 \pm 2\frac{\beta}{\alpha} \right) G^2 \right] \right\}, \quad (15)$$

$$k_+^{(b)} = \frac{1}{2} \left\{ \left( \frac{\sigma_1}{2} - \rho_1 \right) [2\beta^2 F^2(\mu^2 - 1) - 3\alpha^2 G^2] + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \right. \\ \left. \times \left[ \left( 1 \pm \frac{\beta}{\alpha} \right) F^2 \mu^2 - 2F^2 - \left( 1 \pm 2\frac{\alpha}{\beta} \right) G^2 \right] \right\}, \quad (16)$$

$$k_-^{(b)} = -\frac{1}{2\mu^2} \left\{ \left( \frac{\sigma_1}{2} + \rho_1 \right) [2F^2 \mu^2 + G^2(2\mu^2 - 1)] \right. \\ \left. + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \left[ \beta^2 \left( 1 \pm \frac{\alpha}{\beta} \right) F^2 \mu^2 + \alpha^2 G^2(2\mu^2 - 1) \right] \right\}, \quad (17)$$

$$\gamma^2 = -\frac{1}{\mu^2 k_2} \left\{ \left( \frac{\sigma_1}{2} + \rho_1 \right) (2F^2 \mu^2 + G^2) \right. \\ \left. + \left( \frac{\sigma_1}{2} + \sigma_2 \right) [(\beta^2 \pm \alpha\beta) F^2 \mu^2 + \alpha^2 G^2] \right\}. \quad (18)$$

The remaining four algebraic equations of the system,

$$\left[ \left( \frac{\sigma_1}{2} + \rho_1 \right) \pm \left( \frac{\sigma_1}{2} + \sigma_2 \right) \alpha\beta \right] (F^2 \mu^2 - G^2) = 0, \quad (19)$$

$$\left[ \left( \frac{\sigma_1}{2} - \rho_1 \right) \alpha\beta \pm \left( \frac{\sigma_1}{2} + \sigma_2 \right) \right] (\beta^2 F^2 \mu^2 - \alpha^2 G^2) = 0, \quad (20)$$

$$\left( \frac{\sigma_1}{2} - \rho_1 \right) (2\beta^2 F^2 \mu^2 + \alpha^2 G^2) - \left( \frac{\sigma_1}{2} + \rho_1 \right) (2F^2 \mu^2 + G^2) \\ + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \left[ \left( 1 \pm \frac{\beta}{\alpha} \mp \alpha\beta - \beta^2 \right) F^2 \mu^2 + (1 - \alpha^2) G^2 \right] = 0, \quad (21)$$

$$\left( \frac{\sigma_1}{2} - \rho_1 \right) [2\beta^2 F^2 \mu^4 - \alpha^2 G^2(\mu^2 + 1)] + \left( \frac{\sigma_1}{2} + \rho_1 \right) \\ \times [2F^2 \mu^4 - G^2(\mu^2 + 1)] + \left( \frac{\sigma_1}{2} + \sigma_2 \right) \left[ \left( 1 \pm \frac{\beta}{\alpha} \right) F^2 \mu^2(\mu^2 + 1) \right. \\ \left. + \beta^2 \left( 1 \pm \frac{\alpha}{\beta} \right) F^2 \mu^2(\mu^2 + 1) - 2(1 + \beta^2) F^2 \mu^2 - \alpha^2 \left( 1 \pm 2\frac{\beta}{\alpha} \right) G^2 \mu^2 \right. \\ \left. - \left( 1 \pm 2\frac{\alpha}{\beta} \right) G^2 \mu^2 + (1 + \alpha^2) G^2(2\mu^2 - 1) \right] = 0, \quad (22)$$

relate the quantities  $\mu$ ,  $F$ ,  $\alpha$ ,  $\beta$  and  $G$  and should have non-trivial solutions. It turns possible under certain restrictions and relations between the constants defining problem (1), (2). It is convenient to start with equations (19), (20) having the simplest form, and then to substitute the quantities found during their solution into (21), (22) in order to check them for compatibility and to determine additional constraints on the values of the problem parameters. Finally, one must make sure that the value of  $\gamma^2$  found in (15) is positive, i.e., cross-degenerate solutions (13) do exist. Note that the values  $\mu = 1$  and  $\mu = 0$  are of no interest, because at  $\mu \rightarrow 1$  solutions (13) become completely degenerate bright solitons, and at  $\mu \rightarrow 0$  one of the waves does not change over time [20].

This procedure has allowed us to find three families of solutions. The first (for free parameters  $\sigma_{1,2}$ ,  $\mu$  and  $F$ ) has the form

$$G^2 = F^2 \mu^2, \quad \gamma^2 = \frac{4}{k_2} \frac{\sigma_1^2 + \sigma_1 \sigma_2 - 2\sigma_2^2}{\sigma_1 - \sigma_2 - 3\rho_1} F^2, \quad (23) \\ \alpha = (-1)^{l+1} \beta, \quad \alpha^2 = \beta^2 = \frac{\sigma_1^2 - \sigma_2 + 3\rho_1}{\sigma_1 - \sigma_2 - 3\rho_1}$$

and exists at

$$\rho_1^2 = \frac{2\sigma_1^2 + \sigma_1 \sigma_2 - 3\sigma_2^2}{3}, \quad \beta^2 \neq (-1)^l \frac{\sigma_1 + 2\sigma_2}{\sigma_1 - 2\rho_1}, \quad (24) \\ \beta^2 \neq (-1)^l \frac{\sigma_1 + 2\sigma_2}{\sigma_1 + 2\rho_1}.$$

In (23), (24) and below all the derived expressions are not equal to zero and do not tend to infinity. The relationships that ensure these trivial restrictions on the parameters of the medium are not additionally written out. The second family of solutions (for free parameters  $\sigma_{1,2}$ ,  $\rho_1 \neq 0$ ,  $\mu$  and  $F$ ) is given by

$$G^2 = F^2 \mu^2, \quad \gamma^2 = -\frac{4}{k_2} \frac{\sigma_2^2 + \sigma_1 \sigma_2 + \rho_1^2}{\sigma_1 + 2\sigma_2} F^2, \quad (25)$$

and the coefficients  $\alpha$  and  $\beta$  are the solutions to the system of equations

$$\alpha^2 + \beta^2 = -2 \frac{\sigma_1 + 2\rho_1}{\sigma_1 + 2\sigma_2}, \quad (-1)^{l+1} \alpha\beta = \frac{\sigma_1 + 2\sigma_2}{\sigma_1 - 2\rho_1}. \quad (26)$$

The third family of solutions is determined by the relations

$$\alpha^2 G^2 = \beta^2 F^2 \mu^2, \quad \gamma^2 = \frac{2}{k_2} \frac{\sigma_1 + 2\rho_1}{\alpha^2} \\ \times \frac{\sigma_2^2 + \sigma_1 \sigma_2 + \rho_1^2}{\left( \frac{3}{2}\sigma_1 + 2\sigma_2 - \rho_1 \right) (\sigma_2 - \rho_1) - \left( \frac{1}{2}\sigma_1 - 2\sigma_2 - 3\rho_1 \right) (\sigma_1 + \sigma_2 + \rho_1)} F^2, \quad (27)$$

and the coefficients  $\alpha$  and  $\beta$  are the solutions to the system of equations

$$\alpha^2 + \beta^2 \\ = \frac{\sigma_1^2 - 4\rho_1^2}{\left( \frac{3}{2}\sigma_1 + 2\sigma_2 - \rho_1 \right) (\sigma_2 - \rho_1) - \left( \frac{1}{2}\sigma_1 - 2\sigma_2 - 3\rho_1 \right) (\sigma_1 + \sigma_2 + \rho_1)}, \quad (28) \\ (-1)^{l+1} \alpha\beta = \frac{\sigma_1 + 2\rho_1}{\sigma_1 + 2\sigma_2}. \quad (29)$$

The parameters  $\sigma_{1,2}$ ,  $\rho_1$ ,  $\mu$  and  $F$  are thus free ones.

Verification has shown that the cross-degenerate solutions cannot be constructed both on the basis of two other pairs of different elliptic Jacobi functions  $[\text{dn}(\gamma\tau, \mu)$  and  $\text{sn}(\gamma\tau, \mu)$ , and  $\text{cn}(\gamma\tau, \mu)$  and  $\text{sn}(\gamma\tau, \mu)$ ], which was possible for one pair of waves [11, 13], and on the basis of fundamental solutions of the second-order Lamé equation [6, 20]. Therefore, the above solutions (13) are unique and form three different classes of cross-degenerate solutions for all admissible values of material parameters  $\sigma_{1,2}$  and  $\rho_1$ .

We note here that although at first glance the above analysis applies only to situations in which  $\mu$  is real and changes in the interval  $[0, 1]$ , at  $\mu > 1$  and imaginary values of  $\mu$  the elliptic Jacobi functions transform into different combinations of the same elliptic functions  $\text{sn}(\gamma\tau', \mu')$ ,  $\text{cn}(\gamma\tau', \mu')$  and  $\text{dn}(\gamma\tau', \mu')$

with renormalised values of the argument  $\gamma\tau'$  and modulus  $\mu'$ , which is in the interval  $[0, 1]$  (see Tables 8.151 and 8.152 in [20]). Therefore, taking into account the possibility of such a renormalisation, the above expressions and conclusions cover those situations when  $\mu > 1$  or is imaginary.

We emphasise that under conditions (25)–(29) periodic cross-degenerate solutions (13) defined on a set of points  $\alpha^2 \neq \beta^2$  cease to exist at points  $\alpha = \pm\beta$ , since  $\gamma^2$  vanishes. Therefore, all the points at which  $\alpha^2 = \beta^2$  are singular for these solutions. The fact that the first of the derived periodic cross-degenerate solutions that exists at  $\gamma^2 \neq 0$  is not an asymptotic for the other two solutions is quite unexpected. In the known-to-us cases of existence of several branches of solutions the latter are usually ‘sewed’, i.e., under certain conditions they asymptotically transform into each other.

#### 4. Conclusions

We have found three unusual classes of particular analytical solutions to a system of four nonlinear equations for slowly varying complex amplitudes of circularly polarised components of the electric field. The system describes the self-action and interaction of two plane elliptically polarised waves during their collinear propagation in an isotropic medium with second-order frequency dispersion and spatial dispersion of cubic nonlinearity. The solutions represent self-consistent combinations of the components of two elliptically polarised plane cnoidal waves whose two pairs of mutually orthogonal circularly polarised components vary in accordance with pairwise identical laws during propagation. At the same time, the amplitudes of the component with the same circular polarisation are proportional to two different elliptic Jacobi functions with the same periods of change in time:  $\text{dn}(\gamma\tau, \mu)$  and  $\text{cn}(\gamma\tau, \mu)$ . It is found that these solutions form three different classes, because they exist in some region of variation of material parameters and do not transform into each other even asymptotically.

The resulting cross-degenerate solutions with separable variables  $\tau$  and  $z$ , whose phases vary linearly with respect to  $z$  and do not depend on  $\tau$ , are constructed on the basis of the eigenfunctions of the first-order Lamé equation [6, 20]. Therefore, the character of the evolution of elliptical polarisation of each of the two consistently propagating and interacting cnoidal waves is given by formulas that are similar to those expressions that describe the evolution of the polarisation ellipse of a cnoidal wave and are given earlier in [11]. For the solution of equation (23) in the case of  $\alpha = \mp\beta$ , the polarisation states of the two waves determined by the normalised Stokes vector [11, 12] differ only in the sign of their  $z$ th components, while other components of this vector are equal to each other (see the character of the evolution of the polarisation state of a single elliptically polarised cnoidal wave on the Poincaré sphere shown in Fig. 2 [12]). For cross-degenerate solutions (24)–(29) if  $\alpha \neq \mp\beta$ , the differences between the polarisation states of two propagating waves are more significant (all the components of their Stokes vectors are different).

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