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METAMATERIALS

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Linear guided waves in a hyperbolic planar waveguide. Dispersion relations

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Abstract. **We have theoretically investigated waveguide modes propagating in a planar waveguide formed by a layer of an isotropic dielectric surrounded by hyperbolic media. The case, when the optical axis of hyperbolic media is perpendicular to the interface, is considered. Dispersion relations are derived for the cases of TE and TM waves. The differences in the characteristics of a hyperbolic and a conventional dielectric waveguide are found. In particular, it is shown that in hyperbolic waveguides for each TM mode there are two cut-off frequencies and the number of propagating modes is always limited.**

Keywords: *metamaterials, hyperbolic dispersion, waveguide modes, TE and TM waves.*

1. Introduction

Currently, metamaterials and their optical properties are in the focus of attention of many researchers. Metamaterials are artificially created media that consist of components of micrometer or nanometer (subwavelength) size and demonstrate as a rule an unusual, not found in nature, interaction with electromagnetic radiation. The most famous are negative index metamaterials [1–6], which in a certain frequency range exhibit simultaneously a negative permittivity and permeability [7, 8].

The authors of Refs [9–13] have shown that a negative refractive index and its related phenomena may occur in anisotropic media. It should be noted that uniaxial anisotropy, as a rule, is a typical property of metamaterials. Anisotropy of metamaterials can manifest itself in very unusual electrodynamic characteristics of these media. For example, they can demonstrate negative refraction in one direction and positive refraction in the orthogonal direction.

We assume that in an infinite uniaxial anisotropic medium, the coordinate axes X, Y and Z are chosen to coincide with the principal axes of the permittivity tensor in such a way that relations $\varepsilon_{xx} = \varepsilon_{\rm e}$ and $\varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{\rm o}$ are satisfied for principal permittivities. The dispersion relation for an extraordinary wave between the frequency and Cartesian components of the wave vector $\boldsymbol{k}$ has the form:

$$\frac{k_z^2 + k_y^2}{\varepsilon_{\rm e}(\omega)} + \frac{k_x^2}{\varepsilon_{\rm o}(\omega)} = \frac{\omega^2}{c^2}. \tag{1}$$

One can see from this relation that in the case when the value of $\varepsilon_{\rm e}$ or $\varepsilon_{\rm o}$ is negative, isofrequency contours defined by expression (1) are single-sheeted (at $\varepsilon_{\rm e} > 0$, $\varepsilon_{\rm o} < 0$) and two-sheeted (at $\varepsilon_{\rm e} < 0$, $\varepsilon_{\rm o} > 0$) hyperboloids [14–18]:

$$\frac{k_z^2 + k_y^2}{\varepsilon_{\rm e}(\omega)} - \frac{k_x^2}{|\varepsilon_{\rm o}(\omega)|} = \frac{\omega^2}{c^2}, \quad \frac{k_x^2}{\varepsilon_{\rm o}(\omega)} - \frac{k_z^2 + k_y^2}{|\varepsilon_{\rm e}(\omega)|} = \frac{\omega^2}{c^2}. \tag{2}$$

Accordingly, these anisotropic media are called hyperbolic.

Hyperbolic media are typically represented by structures composed of alternating planar layers of a conductor and a dielectric [14, 19, 20], or an array of conductive wires inside a dielectric [15, 21, 22].

Unlimited values of the wave vector are possible in (2). The result is a giant Purcell factor [23–26], which determines an increase in the spontaneous emission intensity, and the effect of superresolution [14, 27]. The attention of researchers is also drawn to optical phenomena at the interface of a conventional dielectric and a hyperbolic medium. Zapata-Rodriguez et al. [28] have considered surface waves. The Goos–Hänchen shift, significantly greater than an analogous shift for ordinary media, has been discussed by Jing Zhao et al. [29]. The authors of Refs [30, 31] have considered and studied a planar waveguide, whose core is a conventional isotropic dielectric, whereas a substrate and coating layer are hyperbolic media (Fig. 1). The properties of this waveguide as a guiding structure for surface plasmons have been examined.

Apart from surface waves, guided (waveguide) waves, whose electromagnetic field is concentrated mainly in a dielectric layer and held there due to total internal reflection from

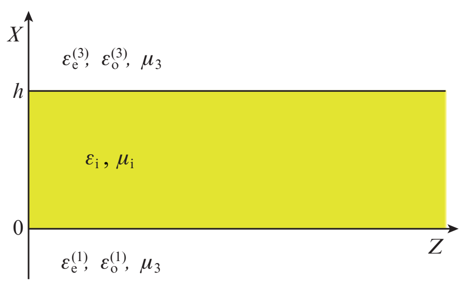

Figure 1. Scheme of a planar waveguide in question.

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the boundaries of the surrounding media (the substrate and the coating layer) can propagate in a waveguide. In this paper, we investigate theoretically the propagation of linear guided waves in a planar waveguide discussed earlier in [30]. The axis of anisotropy of the substrate and the coating layer is perpendicular to the surface between the media (along the X axis) (see Fig. 1). For this geometry, Maxwell's equations are separated into two uncoupled systems, which describe differently polarised waves, called TE and TM waves [32]. Analysis of the modes of TE- and TM-type waves has been performed independently. In each case, we have analytically obtained dispersion relations between the effective refractive index of the waveguide and the frequency of a guided wave. The case of a symmetric waveguide has been investigated in detail.

2. Field distributions for guided TE and TM waves

We consider a planar waveguide (Fig. 1), the core of which is an isotropic dielectric with permittivity $\varepsilon_{\rm i}$ and permeability $\mu_{\rm i}$. The thickness of the dielectric layer is equal to h. The substrate and the coating layer are hyperbolic media characterised by principal permittivities $\varepsilon_{\rm o}^{(1)}$, $\varepsilon_{\rm e}^{(1)}$ and $\varepsilon_{\rm o}^{(3)}$, $\varepsilon_{\rm e}^{(3)}$ and permeabilities μ_1 and μ_3. All the permeabilities are positive. The axis of anisotropy of hyperbolic media is perpendicular to the surfaces of the interface between the media, i.e. along the X axis (see Fig. 1). The axes Y and Z are parallel to the interface. The Z axis can be selected directed along the direction of the wave propagation. In this case, Maxwell's equations are invariant with respect to the shift of the system of coordinates along the Y axis. Thus, the electric and magnetic field strengths of a guided wave do not depend on the variable y; as a result, Maxwell's equations are decoupled into two independent systems of equations describing TE and TM waves [32].

The TE wave is specified by the components of the electric (E_y) and magnetic (H_x, H_z) field strengths. The strengths are assumed to be harmonic functions of time. The wave equation for the complex field amplitude $E = E_y(x,z,\omega)$ has the form

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial z^2} + k_0^2\varepsilon_{\rm o}(x)\mu(x)E = 0.$$

where $k_0 = \omega/c$, and ω is the radiation frequency. Permittivity and permeability are given by piecewise continuous functions (see Fig. 1):

$$\varepsilon_{\rm o}(x) = \begin{cases}\varepsilon_{\rm o}^{(1)} & x<0,\\ \varepsilon_{\rm i} & 0\leqslant x\leqslant h,\\ \varepsilon_{\rm o}^{(3)} & x>h,\end{cases}\qquad \varepsilon_{\rm e}(x) = \begin{cases}\varepsilon_{\rm e}^{(1)} & x<0,\\ \varepsilon_{\rm i} & 0\leqslant x\leqslant h,\\ \varepsilon_{\rm e}^{(3)} & x>h,\end{cases}$$

$$\mu(x) = \begin{cases}\mu_1 & x<0,\\ \mu_{\rm i} & 0\leqslant x\leqslant h,\\ \mu_3 & x>h.\end{cases}$$

Magnetic field components can be obtained from the relations

$$H_x = \frac{\rm i}{k_0\mu(x)}\frac{\partial E}{\partial z},\quad H_z = -\frac{\rm i}{k_0\mu(x)}\frac{\partial E}{\partial x}. \tag{3}$$

For the selected direction of the optical axis, the TE waves are ordinary, and therefore the case $\varepsilon_{\rm o} > 0$ is reduced to the well-known problem. However, in hyperbolic media the case $\varepsilon_{\rm o} < 0$ can be of interest.

Because the waveguide is uniform along the Z axis, the solution of the wave equation can be found in the form $E(x,z) = \tilde{E}(x)\exp({\rm i}\beta z)$, where the parameter β is the propagation constant. The solution of the wave equation describing localised waves should be sought for by taking into account the boundary condition $\boldsymbol{E}\to 0$, $\boldsymbol{H}\to 0$ for $|x|\to\infty$. The solution procedure is known and described, for example, in [32]. The distribution of the electric field strength is given by the expressions:

$$E^{(1)} = A\exp(px + {\rm i}\beta z) + {\rm c.c.},\ \ x<0,$$

$$E^{(2)} = \frac{A}{2}[(1-{\rm i}\xi_p)\exp({\rm i}\kappa x + {\rm i}\beta z)$$

$$+ (1-{\rm i}\xi_p)\exp(-{\rm i}\kappa x + {\rm i}\beta z)] + {\rm c.c.},\ \ 0\leqslant x\leqslant h, \tag{4}$$

$$E^{(3)} = A\frac{1-{\rm i}\xi_p}{1+{\rm i}\xi_q}\exp[-q(x-h)]\exp({\rm i}\kappa h + {\rm i}\beta z) + {\rm c.c.},\quad x>h.$$

Here we used the parameters

$$p^2 = \beta^2 + k_0^2\mu_1|\varepsilon_{\rm o}^{(1)}|,\ \ q^2 = \beta^2 + k_0^2\mu_3|\varepsilon_{\rm o}^{(3)}|,\ \ \kappa^2 = k_0^2\mu_{\rm i}\varepsilon_{\rm i} - \beta^2.$$

With their help, we defined phase shifts ϕ_q and ϕ_p:

$$\xi_q = -\tan\frac{\phi_q}{2} = \frac{q\mu_{\rm i}}{\kappa\mu_3},\ \ \xi_p = -\tan\frac{\phi_p}{2} = \frac{p\mu_{\rm i}}{\kappa\mu_1}.$$

The normalised electric field amplitude A at $x = 0$ is arbitrary.

The TM wave is specified by the components of the magnetic (H_y) and electric (E_x, E_z) field strengths. The wave equation for the complex field amplitude $H = H_y(x,z,\omega)$ has the form

$$\frac{1}{\varepsilon_{\rm e}(x)}\frac{\partial^2 H}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{1}{\varepsilon_{\rm o}(x)}\frac{\partial H}{\partial x}\right) + k_0^2\mu(x)H = 0. \tag{5}$$

The electric field components can be obtained from the relations

$$E_x = -\frac{\rm i}{k_0\varepsilon_{\rm e}(x)}\frac{\partial H}{\partial z},\ \ E_z = \frac{\rm i}{k_0\varepsilon_{\rm o}(x)}\frac{\partial H}{\partial x}. \tag{6}$$

The principal permittivities and permeabilities are piecewise continuous functions, considered previously for TE waves.

Solving in a standard way the wave equation (5), we can see that in hyperbolic media, if $\varepsilon_{\rm e}^{(a)} < 0$, $\varepsilon_{\rm o}^{(a)} > 0$ (a = 1, 3), there are no solutions decreasing at infinity. Hence, there are no waves localised in the waveguide. If $\varepsilon_{\rm e}^{(a)} > 0$ and $\varepsilon_{\rm o}^{(a)} < 0$, when the conditions

$$k_0^2\mu_1\varepsilon_{\rm e}^{(1)} > \beta^2,\quad k_0^2\mu_3\varepsilon_{\rm e}^{(3)} > \beta^2 \tag{7}$$

are met, the wave equation allows for the solutions describing the waves confined by the waveguide. The magnetic field distribution is described by the functions:

$$H^{(1)} = A\exp(px + {\rm i}\beta z) + {\rm c.c.},\quad x<0,$$

$$H^{(2)} = \frac{A}{2}[(1+{\rm i}\xi_p)\exp({\rm i}\kappa x + {\rm i}\beta z) +$$

$$+ (1 - \mathrm{i}\xi_p)\exp(-\mathrm{i}\kappa x + \mathrm{i}\beta z)] + \text{c.c.}, \quad 0 \leqslant x \leqslant h, \qquad (8)$$

$$H^{(3)} = A\frac{1 + \mathrm{i}\xi_p}{1 - \mathrm{i}\xi_q}\exp[-q(x-h)]\exp(\mathrm{i}\kappa h + \mathrm{i}\beta z) + \text{c.c.}, \quad x > h.$$

In (8) we define the parameters as

$$p^2 = k_0^2\mu_1|\varepsilon_{\mathrm{o}}^{(1)}| - \frac{|\varepsilon_{\mathrm{o}}^{(1)}|}{\varepsilon_{\mathrm{e}}^{(1)}}\beta^2, \quad q^2 = k_0^2\mu_3|\varepsilon_{\mathrm{o}}^{(3)}| - \frac{|\varepsilon_{\mathrm{o}}^{(3)}|}{\varepsilon_{\mathrm{e}}^{(3)}}\beta^2,$$

$$\kappa^2 = k_0^2\mu_{\mathrm{i}}\varepsilon_{\mathrm{i}} - \beta^2,$$

and use the expression for the phase shifts ϕ_q and ϕ_p.

3. Dispersion relations

Given that the electric and magnetic fields disappear at $|x| \to \infty$, solutions to Maxwell's equations describe waves confined by the waveguide. We must distinguish two cases: a coupled pair of surface waves and waveguide modes. In a linear waveguide, the amplitude of surface waves is maximal at the interface between a dielectric layer and surrounding media. In the case of a dielectric–metal–dielectric or metal–dielectric–metal waveguide, the coupled surface wave is said to be a plasmon polariton wave. A guided wave in the form of a coupled pair of surface waves at the interface between a dielectric and a hyperbolic material was considered in [30, 31]. It can be obtained from the above expressions, if we replace κ^2 by $\kappa^2 = \beta^2 - k_0^2\mu_{\mathrm{i}}\varepsilon_{\mathrm{i}}$. A planar waveguide, apart from surface waves, can confine a set of waves, called guided modes [32].

Localised waves are characterised by a relationship between the propagation constant β and the frequency ω, which is called the dispersion relation. Dispersion relations for the waveguide considered here follow from the requirement of continuity of the tangential components of the electric and magnetic fields at interfaces between media. It is convenient to obtain dispersion relations separately for TE and TM waves.

3.1. The case of a TE wave

The distribution of magnetic fields in a waveguide can be derived from the expressions found for the electric fields (4) using equations (3). The condition of continuity of tangential components of the vectors of the electric and magnetic fields leads to the relationship:

$$\exp(2\mathrm{i}\kappa h)\left(\frac{1 - \mathrm{i}\xi_q}{1 + \mathrm{i}\xi_q}\right)\left(\frac{1 - \mathrm{i}\xi_p}{1 + \mathrm{i}\xi_p}\right) = 1.$$

Using the expression for the phase shifts, we can write the dispersion relation in a form having a clear physical meaning:

$$2kh + \phi_p + \phi_q = 2\pi m, \quad m = 0, 1, 2, \ldots \qquad (9)$$

If we introduce the effective refractive index n_{eff}, following the formula $\beta = k_0 n_{\mathrm{eff}}$, then (9) can be rewritten as

$$hk_0\sqrt{n_{\mathrm{i}}^2 - n_{\mathrm{eff}}^2} = \arctan\left(\frac{\mu_{\mathrm{i}}}{\mu_1}\sqrt{\frac{n_1^2 + n_{\mathrm{eff}}^2}{n_{\mathrm{i}}^2 - n_{\mathrm{eff}}^2}}\right)$$

$$+ \arctan\left(\frac{\mu_{\mathrm{i}}}{\mu_3}\sqrt{\frac{n_3^2 + n_{\mathrm{eff}}^2}{n_{\mathrm{i}}^2 - n_{\mathrm{eff}}^2}}\right) + \pi m.$$

Here, n_1, n_{i} and n_3 are the refractive indexes of media that form the waveguide ($n_1^2 = \mu_1|\varepsilon_{\mathrm{o}}^{(1)}|$, $n_3^2 = \mu_3|\varepsilon_{\mathrm{o}}^{(3)}|$, $n_{\mathrm{i}}^2 = \mu_{\mathrm{i}}\varepsilon_{\mathrm{i}}$).

The dispersion relation shows that the effective refractive index is limited by the condition $0 \leqslant n_{\mathrm{eff}}^2 < n_{\mathrm{i}}^2$. In the case of conventional dielectric media surrounding the waveguide core, a similar restriction has the form $\max(n_1^2, n_3^2) \leqslant n_{\mathrm{eff}}^2 < n_{\mathrm{i}}^2$. The difference between these inequalities is due to the fact that when the waveguide is surrounded by hyperbolic materials, TE waves do not propagate in the surrounding hyperbolic media: total internal reflection occurs at any angle of incidence, as if the core of the waveguide were a dielectric core surrounded by a metal.

Further analysis of the dispersion relation will be made for the case of a symmetric waveguide when $n_1^2 = n_3^2$. This dispersion relation takes the form

$$k_0 h\sqrt{n_{\mathrm{i}}^2 - n_{\mathrm{eff}}^2} = 2\arctan\left(\frac{\mu_{\mathrm{i}}}{\mu_1}\sqrt{\frac{n_1^2 + n_{\mathrm{eff}}^2}{n_{\mathrm{i}}^2 - n_{\mathrm{eff}}^2}}\right) + \pi m. \qquad (10)$$

We rewrite relation (10) in a normalised form. We introduce the parameter b as follows: $n_1^2 + n_{\mathrm{eff}}^2 = b\Delta$, where $\Delta = n_1^2 + n_{\mathrm{i}}^2$. Then, the normalised waveguide thickness is $V = k_0 h\sqrt{n_{\mathrm{i}}^2 + n_1^2}$, and relation (10) transforms to the form

$$V\sqrt{1 - b} = 2\arctan\left(\frac{\mu_{\mathrm{i}}}{\mu_1}\sqrt{\frac{b}{1 - b}}\right) + \pi m. \qquad (11)$$

This relationship specifies the dispersion dependence $b(V, m)$ and its form coincides with a similar expression for uniaxial anisotropic dielectrics with positive components of the permittivity tensor. However, the normalised effective refractive index b lies in the interval $[b_0, 1)$, where $b_0 = n_1^2/(n_1^2 + n_{\mathrm{i}}^2)$, whereas in a standard situation b lies in the interval $[0, 1)$. The dependences $b(V, m)$ at $\mu_{\mathrm{i}}/\mu_1 = 1.2$ and $b_0 = 0.2$ are shown in Fig. 2. The fact that unlike a conventional waveguide, $b_0 > 0$, means that the cut-off frequency $V_{\mathrm{c}0}$ for the TE_0 mode is nonzero. By setting $b_0 = 0$ in (11), we obtain

$$V_{\mathrm{c}0} = 2\sqrt{1 + \frac{n_1^2}{n_{\mathrm{i}}^2}}\arctan\frac{\mu_{\mathrm{i}} n_1}{\mu_1 n_{\mathrm{i}}}. \qquad (12)$$

In a conventional dielectric waveguide $V_{\mathrm{c}0} = 0$.

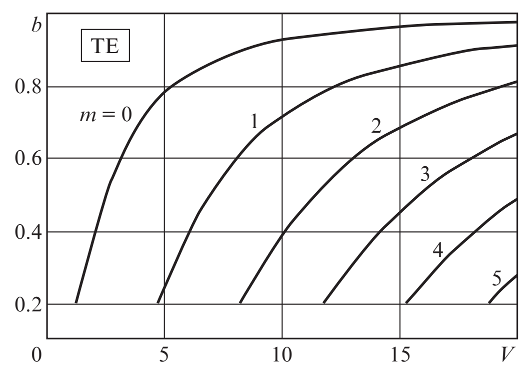

Figure 2. Dispersion curves for a TE wave in a planar waveguide with a hyperbolic medium ($\mu_{\mathrm{i}}/\mu_1 = 1.2$, $b_0 = 0.2$, $m = 0$–5).

3.2. The case of a TM wave

Using expression (8) and relations (6) we find the electric field strengths. Then, the condition of continuity of tangential components of the vectors of the electric and magnetic fields allows us to obtain the dispersion relation for TM waves:

$$\exp(2i\kappa h)\left(\frac{1+i\xi_q}{1-i\xi_q}\right)\left(\frac{1+i\xi_p}{1-i\xi p}\right)=1.$$

Using the phase shifts ϕ_q and ϕ_p, the dispersion relation can be written in the form

$$2\kappa h+\phi_p+\phi_q=2\pi m,\quad m=0,1,2,\ldots \tag{13}$$

If we return to the original variables, expression (13) will take the form

$$h\sqrt{k_0^2(\mu_i\varepsilon_i-n_{\rm eff}^2)}+\arctan\sqrt{\frac{\varepsilon_i^2}{|\varepsilon_o^{(3)}|\,\varepsilon_e^{(3)}}\frac{\varepsilon_e^{(3)}-n_{\rm eff}^2}{\varepsilon_i-n_{\rm eff}^2}}$$
$$+\arctan\sqrt{\frac{\varepsilon_i^2}{|\varepsilon_o^{(1)}|\,\varepsilon_e^{(1)}}\frac{\varepsilon_e^{(1)}-n_{\rm eff}^2}{\varepsilon_i-n_{\rm eff}^2}}=\pi m.$$

Restricting our consideration to the case of a symmetric waveguide ($n_1^2=n_3^2$), the dispersion relation can be rewritten as

$$hk_0\sqrt{n_i^2-n_{\rm eff}^2}=-2\arctan\sqrt{\frac{\varepsilon_i^2}{|\varepsilon_o^{(1)}|\,\varepsilon_e^{(1)}}\frac{n_e^2-n_{\rm eff}^2}{n_i^2-n_{\rm eff}^2}}+\pi m, \tag{14}$$

where $n_i^2=\mu_i\varepsilon_i$ for an isotropic dielectric and $n_e^2=\mu_1\varepsilon_e$ for an extraordinary wave in a hyperbolic medium.

Equation (14) implies that the effective index must meet the conditions

$$0\leqslant n_{\rm eff}^2<n_i^2,\quad 0\leqslant n_{\rm eff}^2<n_e^2.$$

In the case of conventional dielectrics, this condition is different: $n\leqslant n_{\rm eff}<n_i$, where n is the refractive index of a substrate (or a coating layer).

The transition to normalised variables in (14) is performed as follows: $n_e^2-n_{\rm eff}^2=b\Delta>0$, where b is the normalised effective index of the waveguide, $\Delta=n_i^2-n_e^2$ and the normalised thickness of the waveguide is defined as $V=k_0h\sqrt{n_i^2-n_e^2}$.

As a result of this substitution of the variables we obtain

$$V\sqrt{1+b}=-2\arctan\sqrt{\frac{\varepsilon_i^2}{|\varepsilon_o^{(1)}|\,\varepsilon_e^{(1)}}\frac{b}{1+b}}+\pi m. \tag{15}$$

In this case, the parameter b belongs to the interval $[0,b_0]$, where $b_0=b(n_{\rm eff}=0)=n_e^2/(n_i^2-n_e^2)$.

The dispersion curves corresponding to equation (15) are presented in Fig. 3. Here, $\varepsilon_i^2/(|\varepsilon_o^{(1)}|\,\varepsilon_e^{(1)})=1.2$, $b_0=2$. For comparison, Fig. 4 shows the dispersion curves for a waveguide consisting of dielectrics with positive values of permittivity and permeability, satisfying the equation

$$V\sqrt{1-b}=2\arctan\sqrt{u\frac{b}{1-b}}+\pi m,\quad m=0,1,2,\ldots,$$

where for the anisotropic environment of the waveguide core we use the parameter $u=\varepsilon_i^2/(\varepsilon_o\varepsilon_e)$ (for the curves in the figure $u=1.2$).

Figures 3 and 4 show that for a TM wave of a hyperbolic waveguide the number of modes of a guided wave is always finite: with increasing thickness of the waveguide layer (or the radiation frequency) some modes disappear from the waveguide, while others appear. In such a waveguide there does not exist a zero mode ($m=0$). In waveguides made of conventional dielectrics (Fig. 4) the number of modes increases with increasing thickness h, and none of the waveguide modes disappears. Thus, for each TM mode of a hyperbolic waveguide there are two cut-off frequencies: the frequency at which the mode appears in the waveguide, i.e., at $b(V_{cm}^{(2)})=b_0$, and the frequency at which the mode disappears from the waveguide, i.e., at $b(V_{cm}^{(1)})=0$. For modes of a conventional dielectric waveguide, the second cut-off frequency $V_{cm}^{(2)}$ does not exist.

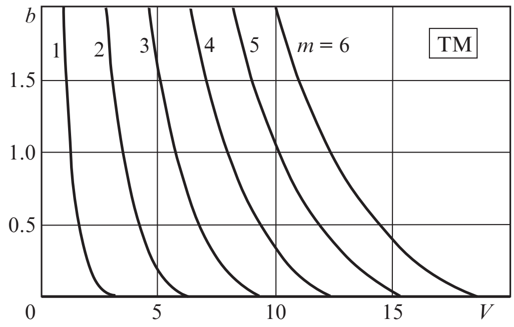

Figure 3. Dispersion curves for a TM wave in a planar waveguide with a hyperbolic medium [$\varepsilon_i^2/(|\varepsilon_o^{(1)}|\,\varepsilon_e^{(1)})=1.2$, $b_0=2$, $m=1-6$].

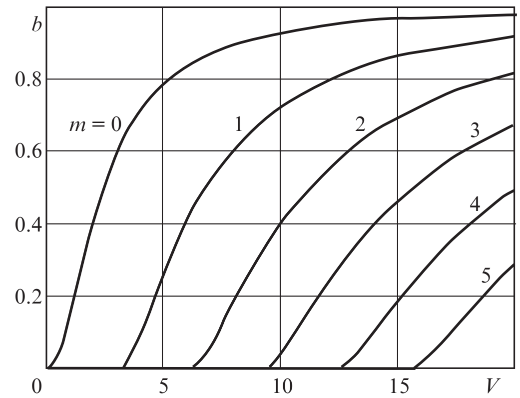

Figure 4. Dispersion curves for a TM (TE) wave in a conventional planar dielectric waveguide [$\varepsilon_i^2/(\varepsilon_o\varepsilon_e)=1.2$, $m=1-5$].

4. Conclusions

We have considered a planar waveguide in the form of an isotropic dielectric surrounded by a hyperbolic medium, the anisotropy axis of which is perpendicular to the interface between the media. We have shown that for a hyperbolic medium with $\varepsilon_o<0$, $\varepsilon_e>0$ the propagation of guided modes is possible. In the geometry in question, the TE wave is ordinary and the TM wave is extraordinary. At $\varepsilon_o>0$, $\varepsilon_e<0$, in the case of a TM wave, radiation is emitted from the waveguide, and in the case of a TE wave the situation does not differ from that in a conventional dielectric waveguide. For this reason, attention has been paid only to the case of a hyperbolic medium with $\varepsilon_o<0$, $\varepsilon_e>0$.

The effective refractive index of a TE wave in the waveguide layer satisfies $0\leqslant n_{\rm eff}<n_i$. In the case of a TM wave the effective index varies in the interval $0\leqslant n_{\rm eff}<n_e$ (it is assumed that $n_e<n_i$). Thus, in both cases, the effective index may be zero, which corresponds to small values of the projection of

the Poynting vector on the z direction or the group velocity of waves in hyperbolic waveguides of this type. In the case of a standard (elliptical) anisotropic dielectric waveguide, the effective index lies in the interval $n_e \leqslant n_{eff} < n_i$ for a TM wave and in the interval $n_o \leqslant n_{eff} < n_i$ for a TE wave.

We have found the dispersion relations for TE and TM waves and plotted the corresponding dispersion curves. It is shown that for a TM wave the number of guided modes is always finite. For each mode, there are two cut-off frequencies: one corresponds to the appearance of a mode in the waveguide, and the other – to its disappearance. For the first several modes there are intervals of the waveguide layer thicknesses at which only this mode propagates. Such phenomena do not occur in conventional dielectric waveguides, in which the number of modes increases continuously with increasing thickness of the waveguide layer or the emission frequency.

With regard to the application of the results obtained, we note the following. The authors of Refs [30, 31] have studied experimentally a plasmon waveguide (a dielectric surrounded by hyperbolic media) in the range of wavelengths from 800 to 1500 nm. In the visible range, transparent metamaterials are still absent. However, this does not mean that we will not have them in the future. Compensation for losses or creation of metamaterials without metal inclusions is two obvious ways to produce transparent metamaterials in the visible range. Nonetheless, the results presented are valid in the frequency range, in which losses are negligible and it is acceptable use of macroscopic electrodynamics.

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