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INTERACTION OF WAVES DOI: 10.1070/QEL16013

Spectral representation of adiabatic interaction of cnoidal waves in an isotropic gyrotropic nonlinear medium

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Abstract. **We analyse the evolution of the spectra of an approximate solution of the adiabatic interaction problem of two waves with orthogonal circular polarisations in an isotropic gyrotropic nonlinear medium with the second-order group velocity dispersion. It is shown that a gradual 'chaotisation' of the approximate solution for the amplitude of a rapidly changing wave is associated with its non-equidistant frequency spectrum.**

Keywords: *cubic nonlinearity, gyrotropy, cnoidal waves, adiabatic interaction, spectrum.*

1. Introduction

In our papers [1 – 5], the nonintegrable problem of interaction of a rapidly changing, plane, circularly polarised cnoidal wave ('information' signal) with a slowly varying orthogonally polarised 'control' signal in the form of a cnoidal wave [1, 4] or a soliton [2, 3, 5] (we considered an isotropic gyrotropic medium with Kerr nonlinearity and second-order group velocity dispersion) was solved analytically with the help of the adiabatic approximation [6 – 8], which is widely used both in the quantum and classical (semi-classical) description of nonlinear dynamics of various systems [9, 10]. The solution to this problem, found in the approximation of a small range of times $t_1 = t - z/u$ in the coordinate system running along the z axis with the group velocity u, gave reason to assert that the information signal is subjected to amplitude and frequency modulation by the control signal. Thus, in the case of the control signal in the form of a cnoidal wave, with increasing time we observed a distortion of the 'fast' component of the electric field during its propagation, manifested in the 'chaotisation' of the latter. For the control signal in the form of a soliton, the information signal was not chaotised due to the fact that the time of the effective interaction of orthogonal field components was limited by the soliton duration. Cascade processes ensured that on the cubic nonlinearity not only a nonlinear shift of the resonance frequency of the fast subsystem can occur, but also effective attenuation, making the solution spectrum continuous, can appear. This occurs, for example, with a nonlinear response of a fast electronic subsystem in the case of its strong coupling with a slow phonon subsystem in complex molecules [11 – 15]. To analyse the reasons of the above-mentioned 'chaotisation', in the present work we consider the interaction of two plane cnoidal waves ('fast' information and 'slow' control signals) with orthogonal circular polarisations in an isotropic gyrotropic nonlinear medium with the second-order group velocity dispersion. It is shown that a gradually chaotisation of the approximate solution is determined by the non-equidistant spectrum of the amplitude of a rapidly changing wave. To correctly calculate the spectrum of the approximate solution and to perform its subsequent analysis, the adiabatic approximation is generalised in this paper to the case of large t_1.

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2. Approximate solution for large times

The nonintegrable system of differential equations in partial derivatives for the slowly varying amplitudes $A_\pm(z,t_1)$ of circularly polarised components of the electric field $E_\pm(z,t) = A_\pm(z, t - z/u)\exp[\mathrm{i}(\omega t - kz)]$ of an elliptically polarised wave propagating along the z axis in a nonlinear isotropic gyrotropic medium with the second-order group velocity dispersion is taken from [1]:

$$\frac{\partial A_\pm}{\partial z} - \mathrm{i}\frac{k_2}{2}\frac{\partial^2 A_\pm}{\partial t_1^2} + \mathrm{i}\Big[\mp\rho_0 + \Big(\frac{\sigma_1}{2}\mp\rho_1\Big)|A_\pm|^2 + \Big(\frac{\sigma_1}{2}+\sigma_2\Big)|A_\mp|^2\Big]A_\pm = 0. \quad (1)$$

Here, $k_2 = \partial^2 k/\partial^2\omega \neq 0$, $\sigma_1 = 4\pi\omega^2\chi_{xyxy}/(kc^2)$ and $\sigma_2 = 2\pi\omega^2\chi_{xxyy} \times (kc^2)^{-1}$ are associated with the independent components of the tensor of the local cubic nonlinearity $\hat{\chi}^{(3)}(\omega; -\omega,\omega,\omega)$; and $\rho_{0,1} = 2\pi\omega^2\gamma_{0,1}/c^2$ are defined through the pseudoscalar constants $\gamma_{0,1}$ of linear and nonlinear gyration. Substituting $A_\pm(z,t_1) = r_\pm(t_1)\exp(\mathrm{i}\kappa_\pm z)$ into (1) and implementing a standard procedure of separation of variables, we obtain a system of ordinary differential equations [1]:

$$\frac{\mathrm{d}^2 r_\pm}{\mathrm{d}t_1^2} - \frac{2}{k_2}\Big[\Delta\kappa_\pm + \Big(\frac{\sigma_1}{2}\mp\rho_1\Big)r_\pm^2 + \Big(\frac{\sigma_1}{2}+\sigma_2\Big)r_\mp^2\Big]r_\pm = 0. \quad (2)$$

Here, $\Delta\kappa_\pm = \kappa_\pm \mp \rho_0$; and $\kappa_\pm$ are the free parameters of the problem (separation constants). The value of the latter is limited only by the condition of applicability of the adiabatic approximation. By implementing it, we find an approximate solution $r_\pm(t_1)$ of system (2), which is valid not only for large but also for small t_1. In the latter case, it must coincide with those in [1]. In (2) we assume $r_+(t_1)$ to be slowly varying compared to the $r_-(t_1)$ function and will seek an approximate solution of the equation for r_- in the form of a Jacobi elliptic function [11]: $r_-(t_1) = C_-(t_1)\mathrm{dn}[\varphi_-(t_1), \mu_-]$. Here the amplitude $C_-(t_1)$ and the instantaneous frequency $\mathrm{d}\varphi_-(t_1)/\mathrm{d}t_1$ are slowly varying functions, and μ_- is a free parameter. Let us substitute $r_-(t_1)$ into the second equation of system (2), neglect the terms containing the derivatives of the slowly varying functions $C_-(t_1)$, $\mathrm{d}\varphi_-(t_1)/\mathrm{d}t_1$, and find $C_-(t_1)$ and $\varphi_-(t_1)$ in the form:

$$C_-(t_1) = C_0\sqrt{1 + \frac{\sigma_1 + 2\sigma_2}{2\Delta\kappa_-}r_+^2(t_1)}\,,$$

$$\varphi_-(t_1) = v_0\int_0^{t_1}\mathrm{d}t'\sqrt{1 + \frac{\sigma_1 + 2\sigma_2}{2\Delta\kappa_-}r_+^2(t')}\,. \tag{3}$$

Here

$$C_0 = 2\sqrt{\frac{\Delta\kappa_-}{(\mu_-^2 - 2)(\sigma_1 + 2\rho_1)}}\,; \quad v_0 = \sqrt{\frac{2\Delta\kappa_-}{k_2(2 - \mu_-^2)}}\,. \tag{4}$$

From (3) we see that at short times,

$$t_1 \ll \frac{\mathrm{d}\varphi_-(t_1)/\mathrm{d}t_1}{\mathrm{d}^2\varphi_-(t_1)/\mathrm{d}t_1^2},$$

the phase is

$$\varphi_-(t_1) \approx v_0 t_1\sqrt{1 + \frac{(\sigma_1 + 2\sigma_2)r_+^2(t_1)}{2\Delta\kappa_-}}\,,$$

which coincides with the results of [1–5]. The procedure for finding the slow component of the electric field does not differ from that used in [1–5]. As in [1], we obtain $r_+(t_1) = C_+ \times \mathrm{cn}(v_+ t_1, \mu_+)$. Here,

$$v_+^2 = \frac{2}{k_2(2\mu_+^2 - 1)}\left[\Delta\kappa_+ - \frac{2\mathrm{E}(\mu_-)(\sigma_1 + 2\sigma_2)\Delta\kappa_-}{\mathrm{K}(\mu_-)(2 - \mu_-^2)(\sigma_1 + 2\rho_1)}\right]; \tag{5}$$

$$C_+^2 = \frac{4\mu_+^2[2\mathrm{E}(\mu_-)(\sigma_1 + 2\sigma_2)\Delta\kappa_- - \mathrm{K}(\mu_-)(2 - \mu_-^2)(\sigma_1 + 2\rho_1)\Delta\kappa_+]}{(2\mu_+^2 - 1)[\mathrm{K}(\mu_-)(2 - \mu_-^2)(\sigma_1^2 - 4\rho_1^2) - 2\mathrm{E}(\mu_-)(\sigma_1 + 2\sigma_2)^2]}; \tag{6}$$

$\mathrm{E}(\mu_-)$ and $\mathrm{K}(\mu_-)$ are complete elliptic Legendre integrals of the second and first kind; and μ_+ is a free parameter. Substituting $r_+(t_1)$ into (3), we shall specify the expressions obtained for the modulation of the amplitude and instantaneous frequency,

$$\frac{C_-(t_1)}{C_0} = \frac{1}{v_0}\frac{\mathrm{d}\varphi_-}{\mathrm{d}t_1} = \sqrt{1 + m\mathrm{cn}^2(v_+ t_1,\mu_+)}\,, \tag{7}$$

where $m = (\sigma_1 + 2\sigma_2)C_+^2/(2\Delta\kappa_-)$, as well as the phases

$$\varphi_-(t_1) = v_0\int_0^{t_1}\mathrm{d}t'\sqrt{1 + m\mathrm{cn}^2(v_+ t',\mu_+)}\,. \tag{8}$$

Recall that the adiabatic approximation requires the fulfilment of the inequality $T_- \ll T_+$, superimposing constraints on the parameters of the problem. Here, $T_+ = 4\mathrm{K}(\mu_+)/v_+$ and $T_- = 2\mathrm{K}(\mu_-)/(\mathrm{d}\varphi_-/\mathrm{d}t_1)$ are the periods with which the components of the electric field change.

3. Spectrum of the approximate solution

To calculate the spectra of

$$S_\pm(\omega,\omega',z) = \int_{-\infty}^{\infty} r_\pm(z, t - z/u)\exp[\mathrm{i}(\omega - \omega')t - \mathrm{i}(k - \kappa_\pm)z]\mathrm{d}t \tag{9}$$

of the found approximate solution of $E_\pm(z,t)$, in the expression for $r_+(t_1)$ we present the Jacobi elliptic cosine as a trigonometric series [16–18]:

$$\mathrm{cn}(v_+ t_1,\mu_+) = \frac{\pi}{\mu_+\mathrm{K}(\mu_+)} \times\sum_{n=1}^{\infty}\cosh^{-1}\left[\frac{(2n-1)\pi\mathrm{K}([1-\mu_+^2]^{1/2})}{2\mathrm{K}(\mu_+)}\right]\cos\left[\frac{(2n-1)\pi v_+ t_1}{2\mathrm{K}(\mu_+)}\right]. \tag{10}$$

Substituting (10) into (9), we find that

$$S_+(\omega,\omega',z) = \frac{\pi^2 C_-}{\mu_+\mathrm{K}(\mu_+)}\exp\left[\mathrm{i}\left(\frac{\omega-\omega'}{u} - k + k_+\right)z\right] \times\sum_{n=1}^{\infty}\cosh^{-1}\left[\frac{(2n-1)\pi\mathrm{K}([1-\mu_+^2]^{1/2})}{2\mathrm{K}(\mu_+)}\right]$$
$$\times\delta\left[\omega - \omega' + \frac{(2n-1)\pi v_+}{2\mathrm{K}(\mu_+)}\right] + \delta\left[\omega - \omega' - \frac{(2n-1)\pi v_+}{2\mathrm{K}(\mu_+)}\right]. \tag{11}$$

Equation (11) shows that the spectrum $S_+(\omega,\omega',z)$ is equidistant and its components are arranged symmetrically with respect to the carrier frequency ω. The distance between adjacent lines is equal to $\pi v_+/\mathrm{K}(\mu_+)$. The modulus of the spectral amplitude of the harmonic decreases exponentially with distance from the centre frequency, and the phase shift of the harmonic depends on its numbers. The spatial period of spectrum (11), $\Delta z_+ = 4u\mathrm{K}(\mu_+)/v_+$, is determined by the requirement of multiplicity 2π of the phase shifts for all the harmonics of the spectrum.

To calculate the spectrum of $S_-(\omega,\omega',z)$ we represent the radicals from (7) and (8) in the form of a series in powers of $m \ll 1$. In order to simplify the form of the obtained formulas, we restrict our consideration below by a small modulation depth, assuming

$$\sqrt{1 + m\mathrm{cn}^2(v_+ t_1,\mu_+)} \approx \tfrac{1}{2}m\mathrm{cn}^2(v_+ t_1,\mu_+).$$

Substituting the elliptic Jacobi cosine in the form of (10) into the above relation, we obtain [determined in (7)] expressions for $C_-(t_1)$ and $\mathrm{d}\varphi_-/\mathrm{d}t_1$ in the form of trigonometric series. Now we can calculate the integral in (8) and write the phase of the information signal $\varphi_-(t_1)$ associated with the frequency v_+ in the form:

$$\varphi_-(t_1) = v_0\left[1 + \frac{m}{2\mu_+^2}\frac{\mathrm{E}(\mu_+)}{\mathrm{K}(\mu_+)} - \frac{m(1-\mu_+^2)}{2\mu_+^2}\right]t_1 + \frac{mv_0\mathrm{K}(\mu_+)}{2\pi v_+}\sum_{q=1}^{\infty}\frac{a_q}{q}\sin\left[\frac{q\pi v_+ t_1}{\mathrm{K}(\mu_+)}\right]. \tag{12}$$

The expansion coefficients a_q can be determined using formula (10). Substituting (12) into the expansion of the elliptic function $\mathrm{dn}[\varphi_-(t_1),\mu_-]$ in series [16–18],

$$\mathrm{dn}[\varphi_-(t_1),\mu_-] = \frac{\pi}{2\mathrm{K}(\mu_-)}$$

$$\times\left\{1 + 2\sum_{p=1}^{\infty}\cosh^{-1}\left[\frac{p\pi\mathrm{K}([1-\mu_-^2]^{1/2})}{\mathrm{K}(\mu_-)}\right]\cos\left[\frac{p\pi}{\mathrm{K}(\mu_-)}\varphi_-(t_1)\right]\right\}, \quad (13)$$

and using formulas

$$\cos\alpha = [\exp(\mathrm{i}\alpha) + \exp(-\mathrm{i}\alpha)]/2,$$

$$\exp(\pm\mathrm{i}\alpha\sin\beta) = \sum_{r=-\infty}^{\infty}(\pm\mathrm{i})^r J_r(\alpha)\exp[\mathrm{i}r(\beta - \pi/2)],$$

where $J_r(\alpha)$ are the Bessel functions, and using the expression derived for $C_-(t_1)$, we can obtain the spectrum of $S_-(\omega,\omega',z)$ in the form of infinite sums of δ-functions:

$$S_-(\omega,\omega',z) = \exp\{\mathrm{i}[(\omega-\omega')/u - k + k_-]z\}$$

$$\times\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}R_{q,r,p}\delta[\omega - \omega' + (r+1)q\pi\nu_1 + p\pi\nu_2]. \quad (14)$$

Here, $R_{q,r,p}$ are the spectral amplitudes, the explicit form of which, due to the awkwardness, is not given; $\nu_1 = \nu_+/\mathrm{K}(\mu_+)$; and $\nu_2 = [\nu_0/\mathrm{K}(\mu_-)]\{1 + (m/2\mu_\mp^2)[\mathrm{E}(\mu_+)/\mathrm{K}(\mu_+) - 1 + \mu_\mp^2]\}]$. Formula (14) suggests that the resulting spectrum is discrete, but, in the general case, non-equidistant. Its splitting is due to two independent factors, i.e., the control signal (lower harmonic frequency ν_1) and information signal (lower harmonic frequency ν_2 renormalised by frequency modulation). All the spectral components in (14) are in phase only in the plane $z = 0$. As the distance from it increases, distortions are accumulated in the phases, and the approximate solution becomes aperiodic and resembles chaos. An exception is the case of multiple frequencies: $l\nu_1 = \nu_2$, where l is an integer that is greater than unity due to the adiabatic approximation. In this special case, with increasing running coordinate z the phases shift of all spectral harmonics in (14) can be a multiple of 2π. This will occur for the first time at a distance $\Delta z_- = 2u \times \mathrm{K}(\mu_+)/\nu_+$. On the period $\Delta z = \Delta z_+ = 2\Delta z_-$ the information and control signals will be phase-locked. With this choice of $\nu_{1,2}$ the emerging distortions are first accumulated and then completely compensated for, and the information signal is periodic.

We emphasise that in a similar manner we can generalise other approximate solutions of system (1), written in the form of numerous combinations of elliptic functions, to the case of large interaction times. For example, a combination considered in [4] with the solutions resulting from the permutation of the Jacobi elliptic functions in the 'fast' and 'slow' components, as well as solutions in the form of identical, but nonuniformly time-scaled, elliptic functions. For all of these solutions we can construct spectra and draw similar conclusions.

It is interesting to compare the found spectra of approximate solutions (11) and (14) of system (1) with those of its exact particular solutions $\tilde{A}_\pm(z,t_1) = \tilde{r}_\pm(t_1)\exp(\mathrm{i}\tilde{\kappa}_\pm z)$ [17], corresponding to the type of separation of variables used in this paper and expressed through the same elliptic functions: $\tilde{r}_+(t_1) = r_{0+}\mathrm{cn}(\nu t_1,\mu)$ and $\tilde{r}_-(t_1) = r_{0-}\mathrm{dn}(\nu t_1,\mu)$, where $r_{0\pm}$ are the constant amplitudes, and $\tilde{\kappa}_\pm$ are the constants of separation of variables for partial solutions [17]. Their temporal spectra are equidistant, and they can be obtained from (11) and (14) at $\mu_\pm = \mu$, $\nu_+ = \nu_0 = \nu$, $m = 0$ [in this case, in (14) the sums over q and r disappear). Distortions in them are also absent.

4. Conclusions

We have analysed the evolution of the spectra (obtained in [1]) for the approximate solution to the problem of adiabatic interaction of two plane cnoidal waves (information and control signals) with orthogonal circular polarisations in an isotropic gyrotropic nonlinear medium with the second-order group velocity dispersion. We have found a discrete equidistant spectrum of the control signal. It is shown that in the general case, the spectrum of the information signal is discrete and non-equidistant. It contains an infinite number of spectral lines – combinations of frequency harmonics of amplitudes of two interacting cnoidal waves. It is this form of the spectrum that determines aperiodicity and gradual 'chaotisation' of the approximate analytical solution for the information signal. If the frequency harmonics of the amplitudes of two interacting cnoidal waves are multiples, then on the first half of the spatial period of the approximate solution the resulting distortions are accumulated, and on the second – are fully compensated for. The polarisation state of the resultant electric field consisting of fast information and slow control signals in this case will also vary periodically.

The generalisation of the algorithm for constructing the solutions to nonintegrable systems of nonlinear Schrödinger equations of type (1) in the adiabatic approximation [1] for larger interaction times allows us to extend the findings of spectral analysis to other solutions of this system.

References

1. Makarov V.A. et al. *Laser Phys.*, **24**, 085405 (2014).
2. Makarov V.A. et al. *Laser Phys. Lett.*, **11**, 115402 (2014).
3. Makarov V.A. et al. *Opt. Express*, **22**, 26607 (2014).
4. Makarov V.A., Petnikova V.M., et al. *Kvantovaya Elektron.*, **45**, 35 (2015) [*Quantum Electron.*, **45**, 35 (2015)].
5. Makarov V.A., Petnikova V.M., Rudenko K.V., Shuvalov V.V. *Phys. Wave Phenomena*, **23**, 96 (2015).
6. Davydov A.S. *Quantum Mechanics* (Oxford: Pergamon Press, 1965; Moscow: Nauka, 1973).
7. Messiah A. *Quantum Mechanics* (Amsterdam: North-Holland Publishing Co., 1965; Moscow: Nauka, 1979) Vol. II.
8. Griffiths D.J. *Introduction to Quantum Mechanics* (Prentice Hall Inc., 1995).
9. Berry M.V. *Quantum, Classical and Semiclassical Adiabaticity*. In: *Theoretical and Applied Mechanics*. Ed. by F.I. Niordson and N. Olhoff (North-Holland: Elsevier Sci. Publ., 1985) pp 83–96.
10. Trubetskov D.I., Rozhnev A.G. *Lineinye kolebaniya i volny* (Linear Oscillations and Waves) (Moscow: Fizmatlit, 2001).
11. Kovarskii V.A., Perel'man N.F., Averbukh I.Sh. *Mnogokvantovye protsessy* (Multiquantum processes) (Moscow: Energoatomisdat, 1985).
12. Gribov L.A., Baranov V.I., Zelentsov D.Yu. *Elektronno-kolebatel'nye spektry mnogoatomnykh molekul. Teoriya i metody rasscheta* (Electron-Vibrational Spectra of Polyatomic Molecules. Theory and Calculation Methods) (Moscow: Nauka, 1997).
13. Burenin A.V. *Usp. Fiz. Nauk*, **180**, 745 (2010).
14. Grishanin B.A., Petnikova V.M., Shuvalov V.V. *J. Appl. Spectr.*, **47**, 1309 (1987).
15. Grishanin B.A., Petnikova V.M., Shuvalov V.V. *Vestnik Mosk. Univ., Ser. Fiz., Astronom.*, **29** (4), 58 (1988).
16. Gradshteyn I.S., Ryzhik I.M. *Tables of Integrals, Series and Products* (San Diego, CA: Academic Press, 2000; Moscow: Nauka, 1971).
17. Korn G., Korn T. *Mathematical Handbook for Scientists and Engineers: Definitions, Theorems, and Formulas for Reference and Review* (New York: Dover Publications, Inc., 2000; Moscow: Nauka, 1974).
18. Abramowitz M., Stegun I. (Eds) *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* (Washington: Nat. Bur. Standards, 1964; Moscow: Nauka, 1979).