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Generalised model of a sheath of a plasma bubble excited by a short laser pulse or by a relativistic electron bunch in transversely inhomogeneous plasma

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Abstract. **An analytical model of a plasma bubble (a wake wave in the strongly nonlinear regime) in transversely inhomogeneous plasma is generalised to an arbitrary profile of an electron sheath at its boundary. Within the framework of this generalisation we have found the potential within the bubble and shown that its envelope is described by a second-order equation, similar to the equation of a less general theory. We have also determined the domain of parameters at which this equation is considerably simplified and no longer depends on the profile of the electron sheath.**

Keywords: *wakefield acceleration, strongly nonlinear regime.*

1. Introduction

Recently, much attention has been paid to plasma-based methods for electron acceleration [1, 2]. The essence of these methods is that a high-power laser pulse [3] or a bunch of charged particles [4] propagating through plasmaexcites a plasma wakefield with a large longitudinal electric field which can be used to accelerate charged particles. This makes it possible to attain an acceleration rate that is several orders of magnitude higher, as compared to conventional methods. One of the most promising ways of wakefield excitation is the strongly nonlinear regime in which a plasma structure is formed in the form of a bubble from which virtually all plasma electrons are expelled [5].

Of interest is also a theoretical study of the strongly nonlinear regime. For example, Lu et al. [6] managed to construct an analytical model that describes the envelope of a bubble for the case of homogeneous plasma. Later, Thomas et al. [7] showed that this theory can be generalised to the case of transversely inhomogeneous plasma with an axially symmetric density profile. The need for such a generalisation arose from the analysis of the numerical results demonstrating that the use of a plasma channel allows one to produce electron bunches with a small energy spread [8].

The disadvantage of existing models describing the shape of a plasma bubble is that they allow for various assumptions that do not follow from first principles. In particular, these models postulate a specific profile of the electron sheath at the boundary of the bubble. Rectangular [6, 7] and exponential [9] profiles have been previously used. In this paper, we get rid of this assumption and consider an arbitrary profile. Using it, we generalise the model of a bubble in transversely inhomogeneous plasma [7] and derive an equation for the envelope of the bubble, as well as show that, under certain conditions, the choice of the profile does not affect the shape of the bubble. We find a domain of parameters for which the equation for the envelope is represented in a simple form, and show that these domains are similar to the parameters of significantly different profiles of the electron sheath.

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2. General equations

Let a laser pulse or an electron bunch exciting a bubble propagate in plasma with a transversely inhomogeneous density profile $n(r)$ along the longitudinal axis z. For convenience, we will consider the dimensionless quantities, in which the time is normalised by ω_p^{-1}; the coordinates by $k_p^{-1} = c/\omega_p$; the momenta and energies by mc and mc^2, respectively; the charges by e; the concentrations by n_p; and the electric and magnetic fields by $mc\omega_p/e$. Here, $e > 0$ is the elementary charge; m is the electron mass; c is the speed of light in vacuum; n_p is the typical plasma concentration (for example, for the case of a plasma with a deep channel it can be the concentration outside the channel); and $\omega_p = (4\pi e^2 n_p/m)^{1/2}$ is the characteristic plasma frequency. The electromagnetic field of the wake is conveniently described by the vector potential $\boldsymbol{A}$ and the wake potential $\Psi = \phi - A_z$, where ϕ is the scalar potential of the electromagnetic field.

Because the plasma profile is cylindrically symmetric, we assume that the fields in the wake wave are also cylindrically symmetric, i.e. do not depend on the azimuthal angle φ. Also, we will use the quasi-static approximation, which assumes that the bubble propagates in plasma with a velocity close to the speed of light, while its structure virtually does not change over time. In this approximation, all the fields depend only on the combination of the time and longitudinal coordinate $\xi = t - z$. We also use the Lorentz gauge for the wake potential Ψ and the vector potential $\boldsymbol{A}$

$$\frac{1}{r}\frac{\partial}{\partial r}(rA_r) = -\frac{\partial \Psi}{\partial \xi}. \tag{1}$$

Under the Lorentz gauge, the equations for Ψ and A_z are written in the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = -J_z, \tag{2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) = J_z - \rho. \tag{3}$$

These potentials can be used to find the fields in the wake [6, 7, 9].

3. Electron sheath model

According to equation (3), the source of the wake potential Ψ is the quantity $S(\xi, r) = J_z - \rho$. We consider the bubble model which assumes that within this bubble bounded by the curve $r_b(\xi)$, plasma electrons are completely absent, and so $J_z - \rho = -\rho_{ion}(r)$. Besides, there is a thin electron sheath at the bubble boundary. Within this model, the source $S(\xi, r)$ can be written as follows:

$$\begin{aligned} S &= s_{ion}(r), \quad r < r_b(\xi), \\ S &= s_0(\xi) g\left(\frac{r - r_b(\xi)}{\Delta}\right), \quad r > r_b(\xi), \end{aligned} \tag{4}$$

where $s_{ion}(r) = -\rho_{ion}(r)$; $r_b(\xi)$ is the bubble envelope; Δ is the characteristic thickness of the electron sheath at the bubble boundary; and $g(x)$ is an arbitrary function describing the shape of the boundary. For example, the analytic models from Refs [6, 7] make use of a rectangular profile $g(x) = \theta(1 - x)$, where $\theta(x)$ is the Heaviside function, while the model from [9] considers an exponential profile $g(x) = \exp(-x)$.

For further analysis it is convenient to introduce functions $M_n(x)$ associated with the electron sheath profile:

$$M_n(x) = \int_x^\infty y^n g(y)\,\mathrm{d}y. \tag{5}$$

We also assume that the function $g(x)$ decreases rapidly enough (i.e. all the integrals M_n converge) and is normalised: $M_0(0) = 1$.

The function $s_0(\xi)$ from model (4) can be found from the continuity equation in the same way as in [7]:

$$s_0(\xi) = \frac{-S_{ion}(r_b(\xi))}{r_b(\xi)\Delta[1 + \varepsilon(r_b(\xi))M_1(0)]}, \tag{6}$$

where $\varepsilon(r_b) = \Delta/r_b$, and the function S_{ion} is defined as follows:

$$S_{ion}(r) = \int_0^r s_{ion}(r')r'\,\mathrm{d}r'. \tag{7}$$

It should be noted that $S_{ion}(r) < 0$, because $s_{ion}(r) = -\rho_{ion} < 0$.

4. Solution for the potential

Within the framework of model (4) the solution to equation (3) can be written as $\Psi(\xi, r) = I(\xi, r) + \Psi_0(\xi)$, where

$$I(\xi, r) = \int_0^r \frac{\mathrm{d}r'}{r'}\int_0^{r'} S(\xi, r'')r''\,\mathrm{d}r''. \tag{8}$$

Outside the bubble, i.e., for $r > r_b$, the integral I can be represented as a sum of two integrals over the regions inside and outside of the bubble, respectively: $I = I_1 + I_2$, where

$$I_1(\xi) = \int_0^{r_b}\frac{S_{ion}(r')}{r'}\mathrm{d}r', \tag{9}$$

and $I_2(\xi, r)$ with (6) taken into account can be written as

$$I_2 = S_{ion}(r_b)\int_0^x \frac{\varepsilon\,\mathrm{d}y}{1 + \varepsilon y}\left[\frac{M_0(y) + \varepsilon M_1(y)}{1 + \varepsilon M_1(0)}\right], \tag{10}$$

where $x(\xi, r) = [r - r_b(\xi)]/\Delta$; and $\varepsilon = \varepsilon(r_b(\xi))$. Given that at $r \to \infty$ the potential Ψ must tend to zero, we can find $\Psi_0(\xi)$:

$$\Psi_0(\xi) = -\int_0^{r_b}\frac{S_{ion}(r')}{r'}\mathrm{d}r' - \frac{S_{ion}(r_b)\beta(r_b)}{2}, \tag{11}$$

where the function $\beta(r_b)$ is introduced as follows:

$$\beta(r_b) = 2\int_0^\infty \frac{\varepsilon\,\mathrm{d}y}{1 + \varepsilon y}\left[\frac{M_0(y) + \varepsilon M_1(y)}{1 + \varepsilon M_1(0)}\right]. \tag{12}$$

In the case of a rectangular profile this definition of β coincides with that from [7]. In [9], where the exponential profile $g(x) = \exp(-x)$ was considered, this notation is not used, but the calculation of the function $\beta(r_b)$ for this profile allows one to obtain the same expression for the potential, as in [9].

Thus, the expression for the potential inside the bubble can be written in the form

$$\Psi(\xi, r) = -\int_r^{r_b}\frac{S_{ion}(r')}{r'}\mathrm{d}r' - \frac{S_{ion}(r_b)\beta(r_b)}{2}. \tag{13}$$

Here, the dependence of the potential on the coordinate ξ is determined by the bubble envelope $r_b(\xi)$. The expression obtained for the potential Ψ inside the bubble almost completely coincides with the expression obtained in [7], differing only in the definition of the function $\beta(r_b)$, which is generally given by expression (12). Accordingly, the equation for the bubble envelope obtained in [7] with the use of this potential will also be valid. Thus, the bubble envelope $r_b(\xi)$ is described by the equation

$$A(r_b)r_b'' + B(r_b)(r_b')^2 + C(r_b) = \Lambda(\xi)/r_b, \tag{14}$$

where $\Lambda(\xi)$ is determined by the electron bunches within the bubble,

$$\Lambda(\xi) = -\int_0^{r_b(\xi)} J_z(\xi, r')r'\,\mathrm{d}r', \tag{15}$$

and coefficients depending on the functions $S_{ion}(r_b)$ and $\beta(r_b)$ generally have the same form as in [7], but for the function $\beta(r_b)$ given by relation (12). It should be noted that for simplicity, equation (14) takes into account the possibility of exciting a bubble only by an electron bunch. When the action of the laser field is taken into account, the coefficient C and the right-hand side of the equation contain additional terms that depend on the vector potential of the laser field, the form of which is found in [7].

Thus, we have shown that for the electron sheath profile of arbitrary shape we can find the second-order equation describing the bubble boundary $r_b(\xi)$.

5. Approximations in the equation for the envelope

In some cases, equation (14) can be greatly simplified. We consider a situation when the electron sheath at the bubble boundary is sufficiently thin compared to the size of the bubble, i.e. $\Delta \ll r_b$, which corresponds to $\varepsilon(r_b) = \Delta/r_b \ll 1$. In this case, the function $\beta(r_b)$ can be linearised in the small parameter $\varepsilon(r_b)$. From (12) it is easy to see that for $\varepsilon \ll 1$

$$\beta(r_b) \approx 2\varepsilon \int_0^\infty M_0(y)\,dy = 2\varepsilon M_1(0). \tag{16}$$

In particular, for a rectangular profile $g = \theta(1 - x)$ the linear expansion is written as $\beta \approx \varepsilon(r_b)$, and for an exponential profile $g = \exp(-x)$ – as $\beta \approx 2\varepsilon(r_b)$. Thus, $\beta(r_b) \to 0$ for $\varepsilon \to 0$ for any shape of the bubble boundary. In this limit two coefficients in equation (14) for the shape of the bubble can be written in a simpler form:

$$A(r_b) \approx 1 - \frac{S_{ion}(r_b)}{2},$$
$$B(r_b) \approx -\frac{s_{ion}(r_b) r_b}{2}. \tag{17}$$

The coefficient $C(r_b)$ in equation (14) has the form:

$$C(r_b) = -\frac{S_{ion}(r_b)}{2r_b}\left[1 + \left(1 - \frac{S_{ion}(r_b)\beta(r_b)}{2}\right)^{-2}\right]. \tag{18}$$

There are two limiting cases when the expression for this coefficient can also be simplified.

5.1. Approximation of an infinitely thin electron sheath

We assume that the value of Δ is so small that $|S_{ion}\beta/2| \ll 1$. With regard to expression (16), this condition can be rewritten as follows:

$$\Delta \ll \left|\frac{r_b}{S_{ion} r_b M_1(0)}\right|. \tag{19}$$

In this limiting case, $C(r_b) \approx - S_{ion}(r_b)/r_b$, and the equation for the envelope $r_b(\xi)$ will have the form

$$(S_{ion} - 2) r_b r_b'' + s_{ion} r_b^2 (r_b')^2 + 2S_{ion} = -2\Lambda(\xi). \tag{20}$$

Note that the coefficients of this equation do not depend on the function $g(x)$, i.e. on the electron sheath profile at the bubble boundary.

As an example we consider the domain of applicability of this approximation for the case of homogeneous plasma, i.e. $\rho_{ion}(r) = 1$. In this case, in accordance with definition (7), $S_{ion}(r_b) = - r_b^2/2$, so that condition (19) is written as $\Delta \ll 2 \times (r_b M_1(0))^{-1}$. Figures 1a and 1b show the domain of parameters in which this approximation is valid for the rectangular [$g = \theta(1 - x)$] and exponential [$g = \exp(-x)$] profiles of the electron sheath at the bubble boundary. It is seen that this approximation correctly describes the bubble only at very small values of Δ. In this case, the domains of parameters for the rectangular and exponential profiles visually do not differ if we use 2Δ instead of Δ as the second parameter for the exponential profile. This is explained by the fact that for the exponential profile the coefficient before the linear expansion

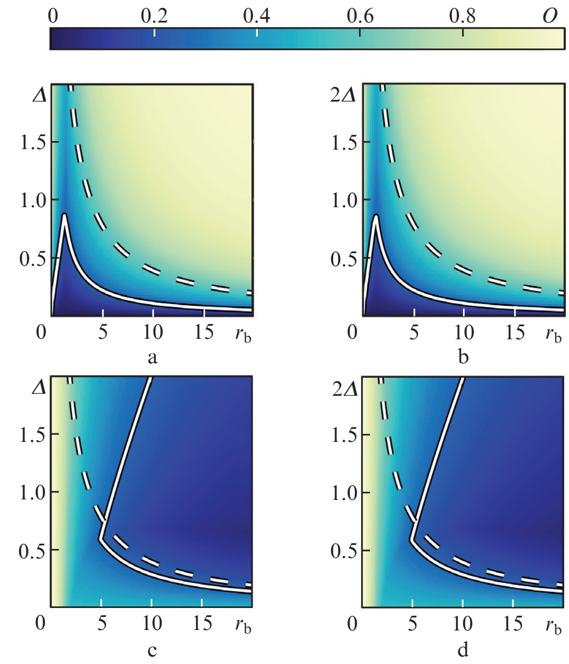

Figure 1. Maximum relative error O in calculation of the coefficients A, B and C in the approximation of an infinitely thin electron sheath (a, b) and in the relativistic approximation (c, d) as compared to the exact numerical solution in the space of parameters r_b and Δ for rectangular (a, c) and exponential (b, d) electron sheath profiles in the case of homogeneous plasma. Solid curves correspond to the level of 0.25, the dashed curves correspond to $\Delta = 4/r_b$ (a, c) and $2\Delta = 4/r_b$b (b, d).

of the function $\beta(r_b)$ is two times greater than that for the rectangular profile.

5.2. Relativistic approximation

Consider now the case that is opposite to the previous one, i.e.

$$\Delta \gg \left|\frac{r_b}{S_{ion}(r_b) M_1(0)}\right|. \tag{21}$$

Physically, this condition corresponds to the fact that the electrons at the bubble boundary are relativistic, and therefore this approach is called relativistic. This condition can be fulfilled simultaneously with the $\Delta \ll r_b$ if the plasma bubble is large enough. In this case, $C(r_b) \approx - S_{ion}(r_b)/(2r_b)$, and in the coefficient $A(r_b)$ we must also neglect unity, so that the equation for $r_b(\xi)$ is written as follows:

$$S_{ion} r_b r_b'' + s_{ion} r_b^2 (r_b')^2 + S_{ion} = -2\Lambda(\xi). \tag{22}$$

In this approximation, the coefficients of the equation do not depend on the profile of the electron sheath at the bubble boundary; therefore, the equation is identical to that obtained in [7]. This approximation often yields the results that correspond to the results of PIC simulations [6, 7].

As an example we also consider the domain of the parameters (where this approximation is valid) in the case of homogeneous plasma for the same two (rectangular and exponential) profiles of the electron sheath on the bubble boundary. These domains are shown in Figs 1c and 1d. As in the previous case, they are virtually indistinguishable when Δ and 2Δ are used as the second parameter for the exponential and rectangular profiles, respectively. At the same time it is clear that for the approximation to be fulfilled, inequality (21) can be replaced by a less stringent one, $\Delta \geqslant 2r_b/M_1(0)$. This approach is also invalid at small transverse sizes r_b. This means that within its framework, it is impossible to describe the process of bubble excitation because originally $r_b = 0$. It should be noted that for any given value of Δ we can find a sufficiently large bubble of size r_b, at which this approximation will be fulfilled, which explains why this approach is often in quite good agreement with the results of PIC simulations.

6. Conclusions

Using the already established theory of a bubble in transversely inhomogeneous plasma, we have considered a generalised model of the electron sheath at the bubble boundary in which the sheath has not only rectangular or exponential profiles, as in previous works, but also can be described by an arbitrary function.

Within the framework of this model, we have found the potential within the plasma bubble and have shown that it has the same form as in the particular case of the theory for the rectangular profile; however, the form of the function $\beta(r_b)$ depends on the shape of the electron sheath. This allows one to use a second-order differential equation to describe the bubble envelope in the general theory as well.

We have also shown that the equation for the bubble in the general case has two approximations within the framework of which it does not depend on the thickness and profile of the electron sheath at the bubble boundary. The domains of applicability of the approximations are substantially similar for different profiles (with a corresponding re-determination of thickness Δ). This leads to the conclusion that the selectable profile shape has little effect on the shape of the bubble envelope and justifies the description of the bubble with the help of a simple model with a rectangular profile of the electron sheath at the boundary.

However, the profile of the electron sheath has a significant effect on the electromagnetic field outside the plasma bubble, the knowledge of which is necessary for the analysis of injection processes and self-injection of electrons. To find this field for an arbitrary profile of the electron sheath is the subject of future work.

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