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Excitation of low-frequency residual currents at combination frequencies of an ionising two-colour laser pulse

N.V. Vvedenskii, V.A. Kostin, I.D. Laryushin, A.A. Silaev

Abstract. **We have studied the processes of excitation of low-frequency residual currents in a plasma produced through ionisation of gases by two-colour laser pulses in laser-plasma schemes for THz generation. We have developed an analytical approach that allows one to find residual currents in the case when one of the components of a two-colour pulse is weak enough. The derived analytical expressions show that the effective generation of the residual current (and hence the effective THz generation) is possible if the ratio of the frequencies in the two-colour laser pulse is close to a rational fraction with a not very big odd sum of the numerator and denominator. The results of numerical calculations (including those based on the solution of the three-dimensional time-dependent Schrödinger equation) agree well with the analytical results.**

Keywords: *two-colour laser pulses, terahertz radiation, ionisation, combination frequencies, time-dependent Schrödinger equation.*

1. Introduction

Laser-plasma schemes based on ionisation of gases by two-colour laser pulses attract much attention due to the possibility of generating high-power and broadband terahertz (THz) radiation [1–14]. These schemes have been studied for pulses which contain a strong field at the fundamental frequency and an additional weaker field at the second harmonic frequency generated by a frequency doubling crystal [1–12]. The use of such laser pulses has made it possible to obtain THz fields with a spectrum covering tens of terahertz [2–4] and with electric strengths of up to a few MV cm^{-1} [5–7]. In addition, the possibilities of controlling the properties of generated THz radiation (polarisation, shape of the spectrum, energy) are demonstrated in manipulating the polarisation and phase of the ionising two-colour pulse [8], or in changing the gas pressure and focusing conditions [3, 9, 10], as well as in changing the type of the ionised gas and in the presence of pre-pulses ionising or orienting gas molecules [3, 11, 12]. Recently, another scheme has been experimentally and theoretically investigated in which a two-colour laser pulse contains, apart from a fundamental-frequency main field, an OPA-generated additional weak field with a frequency tunable near the half-harmonic frequency [13, 14].

The THz spectrum in laser-plasma schemes usually has a low-frequency core (with frequencies around 1 THz, which are much smaller than the inverse duration of the ionising pulse), where the main energy of the THz pulse is concentrated [6, 9, 13]. This low-frequency radiation is attributed to plasma currents in the long wakefield of the laser pulse, and the amplitude of the radiation is proportional to the residual current density (RCD) excited by the laser pulse behind the ionisation front [13, 15–17].

In this paper we study the processes of RCD excitation, when the ratio of the frequencies in a two-colour laser pulse is different from two, and the amplitude ratio of its components is arbitrary. For this purpose, the RCD is found analytically and numerically using semi-classical and quantum-mechanical approaches, and the dependences of the RCD on the laser pulse parameters are analysed in detail. Particular attention is paid to the dependences on the frequency ratio in a two-colour laser pulse. These dependences present a set of resonance-like peaks at frequency ratios that are close to rational fractions with an odd sum of the numerator and denominator.

2. Statement of the problem and derivation of analytical expressions for the residual current density

In this section, we describe the analytical model developed for finding the RCD and derive closed formulas for the dependences of the RCD on all the parameters of the laser pulse in the case when one of the components of the two-colour laser pulse (additional field) is sufficiently small in comparison with the other component (main field). The model is based on the semi-classical approach that uses the solution of the balance ionisation equation for the concentration N of free electrons in the plasma and the equation for the free-electron current density $\boldsymbol{j}$ [5, 13, 15–18]:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (N_{\mathrm{g}} - N)\,w(|\boldsymbol{E}|), \tag{1}$$

$$\frac{\mathrm{d}\boldsymbol{j}}{\mathrm{d}t} = \frac{e^2}{m} N\boldsymbol{E}. \tag{2}$$

Here N_{g} is the initial concentration of neutral gas particles; $w(|\boldsymbol{E}(t)|)$ is the probability of ionisation of an atom per unit time in an electric field with an intensity $\boldsymbol{E}$; e and m are the

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electron charge and mass; and $\boldsymbol{E}(t)$ is the given dependence of the electric field of the laser pulse on time t. We consider linearly polarised two-colour laser pulses whose electric field is parameterised as follows:

$$\boldsymbol{E}(t) = [A_0(t)\cos(\omega_0 t) + A_1(t)\cos(\omega_1 t + \phi)]\boldsymbol{x}_0, \tag{3}$$

where $A_0(t)$ and $A_1(t)$ are the slowly varying envelopes of the main and additional fields, respectively; ϕ is the phase shift between the fields; $\boldsymbol{x}_0$ is the unit vector directed along the x axis; and ω_0 and ω_1 are the carrier frequencies of the fields. The exact solution of system (1), (2) has the form

$$N(t) = N_{\rm g}\left[1 - \exp\left(-\int_{-\infty}^{t} w(|\boldsymbol{E}(t')|)\,{\rm d}t'\right)\right], \tag{4}$$

$$\boldsymbol{j}(t) = \frac{e^2}{m}\int_{-\infty}^{t} N(t')\boldsymbol{E}(t')\,{\rm d}t'. \tag{5}$$

The RCD is determined by the free-electron current density at $t \to +\infty$, or, in other words, by the zero harmonic of the time derivative ${\rm d}\boldsymbol{j}/{\rm d}t$ of the current density:

$$\boldsymbol{j}_{\rm RCD} = \lim_{t\to\infty}\boldsymbol{j}(t) = \frac{e^2}{m}\int_{-\infty}^{+\infty} N(t')\boldsymbol{E}(t')\,{\rm d}t'. \tag{6}$$

The concentration of free electrons in the field of a linearly polarised laser pulse varies in a stepwise manner, and the moments of jumps in the concentration coincide with the maxima of the electric field magnitude. We consider laser pulses which produce sufficiently slow ionisation (over several periods of the main field), so that the value of each jump is small as compared with the total concentration of free electrons. In this case, in the right-hand side of equation (1), the concentration $N(t)$ can be replaced by its averaged (over the main field period) value $\bar{N}(t)$ [16]:

$$\frac{{\rm d}N}{{\rm d}t} = (N_{\rm g} - \bar{N})\,w(|\boldsymbol{E}(t)|), \tag{7}$$

where the averaged value itself satisfies the equation

$$\frac{{\rm d}\bar{N}}{{\rm d}t} = (N_{\rm g} - \bar{N})\,\bar{w}(|\boldsymbol{E}(t)|). \tag{8}$$

In the last equation, $\bar{w}(E)$ is the period-averaged ionisation probability. Under the tunnelling ionisation conditions, the ionisation probability is typically a strong function of the field, and so we can use the expression $\bar{w}(E) = [2/(\pi n_0)]^{1/2} w(E)$ for the averaged ionisation probability, where $n_0(E) = Ew'(E)/w(E) \gg 1$ characterises the steepness of the ionisation probability w as a function of the field E. The number $n_0(E)$ is equal to an exponent of the power function that best approximates the function $w(E)$ in the vicinity of a particular value of E. The square root of this number at a maximum field strength determines how many times the duration of the corresponding jump of the plasma concentration is less than the period of the main field.

As was done in [13, 15], for obtaining a closed-form formula for the RCD, we need to analyse the spectral composition of the electron concentration N. This analysis can be performed using the perturbation theory with respect to the small additional field. Taylor's formula for the ionisation probability as a function of the electric field allows one to write the relation

$$w(|\boldsymbol{E}(t)|) = w_0 + w_1 + \ldots, \tag{9}$$

where the expression for the contribution w_l to the ionisation probability in the lth order of the perturbation theory has the form

$$w_l = \frac{\tilde{w}^{(l)}(E_0(t))\,E_1^l(t)}{l!}. \tag{10}$$

Here $E_0(t) = A_0(t)\cos(\omega_0 t)$ and $E_1(t) = A_1(t)\cos(\omega_1 t + \phi)$ are the projections of the main and additional fields on the x axis; $\tilde{w}^{(l)}(E) = {\rm sgn}^l(E) w^{(l)}(|E|)$; and $w^{(l)}(E)$ is the lth field derivative of the ionisation probability. Substituting (9) and (10) into equations (2) and (7), we obtain

$$N(t) = N_0 + N_1 + \ldots,\quad \boldsymbol{j}(t) = (j_0 + j_1 + \ldots)\boldsymbol{x}_0, \tag{11}$$

where the contributions N_l and j_l to the concentration of free electrons and current density in lth order satisfy the equations

$$N_l = \int_{-\infty}^{t}[N_{\rm g} - \bar{N}(t')]\,w_l(t')\,{\rm d}t', \tag{12}$$

$$\frac{{\rm d}j_l}{{\rm d}t} = \frac{e^2}{m}(N_l E_0 + N_{l-1}E_1), \tag{13}$$

where $N_{-1} \equiv 0$.

In long and not too intense laser pulses, when the ionisation continues during a significant number of the main field periods, each of the functions $\tilde{w}^{(l)}$ is a sequence of narrow peaks near the moments of the $E_0(t)$ extrema, i.e. near the moments of time $t_s = s\pi/\omega_0$, where s is an integer. In this case, the amplitudes of adjacent peaks do not differ greatly in absolute value. It allows us to represent $\tilde{w}^{(l)}(t)$ as a sum of harmonics of the frequency ω_0 with slowly varying envelopes $W_k^{(l)}(t)$:

$$\tilde{w}^{(l)}(E_0(t)) = \sum_{k=0}^{\infty} W_k^{(l)}(t)\cos(k\omega_0 t), \tag{14}$$

where

$$W_k^{(l)} = \frac{\omega_0}{\pi}\int_{t-\pi/\omega_0}^{t+\pi/\omega_0}\tilde{w}^{(l)}(E_0(t'))\cos(k\omega_0 t')\,{\rm d}t'. \tag{15}$$

Since the ionisation probability is a function of the absolute value of the field strength, at even l the amplitudes of all the peaks have the same sign (positive), and at odd l positive peaks alternate with the negative ones. Accordingly, in expansion (14), the terms with odd k at even l are small and can be neglected, and, vice versa, the terms with even numbers can be discarded at odd l. Integral (15) for the remaining terms can be calculated approximately using the Laplace method:

$$W_k^{(l)}(t) \approx \frac{\tilde{w}^{(l)}(A_0(t))}{\sqrt{2\pi n_l}}\exp\left(-\frac{k^2}{2n_l}\right) \tag{16}$$

for even $k + l$. Here $n_l = w^{(l+1)}(|A_0|)|A_0|/w^{(l)}(|A_0|)$ characterises the steepness of the lth field derivative of the ionisation probability. In calculating integral (15) by the Laplace method, we used the condition $n_l \gg 1$. For commonly encountered functions $w(E)$ at $n_l \gg 1$, it turns out that $n_l - n_{l+1} \approx 1$. For these functions strong inequalities $n_l \gg 1$ and $n_0 \gg l+1$ are equivalent. The latter inequality actually sets the upper limit on the order of the perturbation theory, where further calculations are valid.

Given that

$$\tilde{w}^{(l)}(A_0) = w(|A_0|)A_0^{-l}\prod_{i=0}^{l-1} n_i = \left(\frac{\pi n_0}{2}\right)^{1/2}\bar{w}(|A_0|)A_0^{-l}\prod_{i=0}^{l-1} n_i, \quad (17)$$

we obtain

$$\tilde{w}^{(l)}(E_0(t)) \approx \sqrt{\frac{n_0}{n_l}}\frac{\prod_{i=0}^{l-1} n_i}{2A_0^l(t)}\bar{w}(|A_0(t)|) \times \sum_{k=0}^{\infty}\exp\left[-\frac{(2k+1)^2}{2n_l}\right]\cos[(2k+1)\omega_0 t] \quad (18)$$

for odd l and

$$\tilde{w}^{(l)}(E_0(t)) \approx \sqrt{\frac{n_0}{n_l}}\frac{\prod_{i=0}^{l-1} n_i}{2A_0^l(t)}\bar{w}(|A_0(t)|) \times \sum_{k=0}^{\infty}\exp\left(-\frac{2k^2}{n_l}\right)\cos(2k\omega_0 t) \quad (19)$$

for even l. The contributions w_l are proportional to the product of $\tilde{w}^{(l)}(E_0)$ and $E_1^l = A_1^l\cos^l(\omega_1 t + \phi)$ and represent a sum of harmonics at certain combination frequencies of a two-colour pulse. In accordance with (12) and (13), the contributions j_l to the current density are also a sum of harmonics at some combination frequencies $\alpha\omega_0 + \beta\omega_1$ of the field of a two-colour pulse (others than those in the case of w_l), where α and β are even numbers, and $\alpha + \beta$ is an odd number. The contribution to the RCD can be only made by very low combination frequencies $\Delta\omega = b\omega_1 - a\omega_0$ that are small compared with the inverse time scales of the slowly varying envelopes of the corresponding harmonics, where a and b are the natural numbers of different parity. In order to find these low-frequency components of the current density, according to (13), one should find the harmonics of the electron concentration at frequencies $\omega_0 \pm \Delta\omega$ and $\omega_1 \pm \Delta\omega$ that are close to the frequencies of the ionising pulse.

Substituting (18) and (19) into (10) and (12) and analysing the resulting expressions, we find that the lowest order of the perturbation theory where there is a low-frequency component of the current density derivative at the frequency $\Delta\omega$ is the order $l = b$. This low-frequency current density in the bth order is determined by the correction N_b containing the harmonics $N_b^{[(1\mp a)\omega_0 \pm b\omega_1]}$ at frequencies $\omega_0 \pm \Delta\omega = (1\mp a)\omega_0 \pm b\omega_1$, and by the correction N_{b-1} containing the harmonic $N_{b-1}^{[a\omega_0-(b-1)\omega_1]}$ at the frequency $\omega_1 - \Delta\omega = a\omega_0 - (b-1)\omega_1$:

$$N_b^{[(1\mp a)\omega_0 \pm b\omega_1]} \approx \frac{2K_b}{\omega_0 \pm \Delta\omega}\exp\left[-\frac{(a\mp 1)^2}{2n_b}\right] \times\cos[(\omega_0 \pm \Delta\omega)t \pm b\phi], \quad (20)$$

$$N_{b-1}^{[a\omega_0-(b-1)\omega_1]} \approx \frac{2K_{b-1}}{\omega_1 - \Delta\omega}\exp\left(-\frac{a^2}{2n_{b-1}}\right) \times\cos[(\omega_1 - \Delta\omega)t - (b-1)\phi], \quad (21)$$

where

$$K_l = \sqrt{\frac{n_0}{n_l}}\left(\prod_{i=0}^{l-1} n_i\right)\frac{(N_g - \bar{N})\bar{w}(|A_0(t)|)}{2l!}\left[\frac{A_1(t)}{2A_0(t)}\right]^l. \quad (22)$$

In deriving these expressions, in integral (12) we took the slow factor $N_g - \bar{N}$ outside the sign of integration. Substituting expressions (20) and (21) into equation (13) with $l = b$, we find the equation for the current density harmonic at the frequency $\Delta\omega = b\omega_1 - a\omega_0$:

$$\frac{\mathrm{d}j_b^{(b\omega_1 - a\omega_0)}}{\mathrm{d}t} = \frac{e^2K_bA_0(t)}{m}\left\{\frac{\exp[-(a-1)^2/(2n_b)]}{\omega_0 + \Delta\omega} - \frac{\exp[-(a+1)^2/(2n_b)]}{\omega_0 - \Delta\omega} - \sqrt{\frac{n_b}{n_{b-1}}}\frac{2b^2}{n_b}\frac{\exp[-a^2/(2n_{b-1})]}{\omega_1 - \Delta\omega}\right\} \times\sin(\Delta\omega t + b\phi). \quad (23)$$

If $a + b \ll n_0$, then $n_0 \approx n_1 \approx \ldots \approx n_b$, and the exponents in the previous expression can be approximated by their Taylor expansion around the value of $-a^2/(2n_0)$. In this case, $K_b \approx (N_g - \bar{N})[n_0A_1/(2A_0)]^b\bar{w}(|A_0|)/(2b!)$, and the expression for the harmonics of the current density is greatly simplified:

$$\frac{\mathrm{d}j_b^{(b\omega_1 - a\omega_0)}}{\mathrm{d}t} \approx g\frac{e^2N_gA_0(t)}{m\omega_0}\left(1 - \frac{\bar{N}}{N_g}\right) \times\bar{w}(|A_0(t)|)\left[\frac{n_0A_1(t)}{2A_0(t)}\right]^b\sin(\Delta\omega t + b\phi), \quad (24)$$

where

$$g = \frac{2}{b!}\exp\left(-\frac{a^2}{2n_0}\right)\left(\frac{a^2 - b^2}{n_0 a} - \frac{\Delta\omega}{\omega_0}\right). \quad (25)$$

Having integrated expression (24) over time with the use of the Laplace method, for the contribution to the RCD at a particular combination frequency $\Delta\omega = b\omega_1 - a\omega_0$ we obtain the expression

$$\boldsymbol{j}_{\mathrm{RCD}}^{(a,b)} \approx g\,j_{\mathrm{osc}}\sigma\exp\left(-\frac{\Delta\omega^2\tau_{\mathrm{i}}^2}{2}\right)\left(\frac{n_0F_1}{2F_0}\right)^b\sin(b\phi - \Delta\omega t_0)\boldsymbol{x}_0. \quad (26)$$

Here $j_{\mathrm{osc}} = e^2N_gF_0/(m\omega_0)$ is the amplitude of the oscillatory current induced by the main field in the plasma with concentration N_g; $F_{0,1} = A_{0,1}(t_0)$; t_0 is the position of the inflection point of the function $\bar{N}(t)$, i.e., the moment of time at which the averaged electron concentration grows most rapidly ('time moment of ionisation'); $\tau_{\mathrm{i}} = [-(\mathrm{d}\bar{N}/\mathrm{d}t)/(\mathrm{d}^3\bar{N}/\mathrm{d}t^3)]^{1/2}|_{t=t_0} \approx \tau/n_0^{1/2}$ is the characteristic ionisation time; $\tau = [-A_0/(\mathrm{d}^2A_0/\mathrm{d}t^2)]^{1/2}|_{t=0}$ is the duration of the ionising pulse [we assume here for convenience that the maximum of the envelope $A_0(t)$ occurs at $t = 0$]; and $\sigma = N(+\infty)/N_g = \bar{N}(+\infty)/N_g \approx 1 - \exp[-(2\pi)^{1/2}\times\bar{w}(|F_0|)\tau_{\mathrm{i}}]$ is the final (maximum) degree of ionisation.

3. Discussion of the analytical results and their comparison with the results of semi-classical calculations

Equation (26) describes the contribution to the RCD at specific values of a and b. The general expression for the RCD is obtained by summing over all admissible $a + b$:

$$\boldsymbol{j}_{\rm RCD} = \sum_{a,b} \boldsymbol{j}_{\rm RCD}^{(a,b)} \text{ (with odd } a + b). \quad (27)$$

In this sum the terms decrease rapidly with the growth of both a and b, and the summation can be done only over the terms with $a + b \leqslant n_0$, when expression (26) holds true. Moreover, due to the strong Gaussian factor $\exp(-\Delta\omega^2\tau_{\rm i}^2/2)$ in (26), only one or two terms with not very large $\Delta\omega$ and $a + b$ can be significant for any particlualr value of the ratio ω_1/ω_0. At the ratio ω_1/ω_0 that is close to a fraction a/b with a not very large odd sum of the numerator and denominator, we actually can neglect all the terms, but one that meets these specific values of a and b, and assume that expression (26) describes the total value of the RCD. In other words, for those ratios of the frequencies at which the RCD is not too small because of the mentioned Gaussian factors, we can keep only one term in the sum. If the final degree of ionisation is not very large, we can set $t_0 = 0$ and $\sigma \approx (2\pi)^{1/2}\bar{w}(|F_0|)\tau_{\rm i}$, which turns (26) in a closed-form formula for the RCD under the given dependences $A_{0,1}(t)$ and $\bar{w}(E)$. This formula and the dependences of the RCD on the laser pulse parameters show that RCD excitation by a two-colour pulse can be interpreted as a wave mixing in the interaction of b quanta of the additional field and a large number (about $n_0 \gg 1$) of quanta of the main field. Below, we discuss these dependences in detail.

When $\omega_1/\omega_0 \approx a/b$ and $\boldsymbol{j}_{\rm RCD} \approx \boldsymbol{j}_{\rm RCD}^{(a,b)}$, the value of the RCD has a periodic dependence on the phase shift ϕ between the carriers of the main and additional fields with the period $2\pi/b$ that is determined by the denominator of the fraction a/b. The maximum absolute RCD value achieved at an optimum phase shift is given by the expression

$$\max_{\phi}|\boldsymbol{j}_{\rm RCD}^{(a,b)}| \approx g\,|j_{\rm osc}|\,\sigma \exp\left(-\frac{\Delta\omega^2\tau_{\rm i}^2}{2}\right)\left|\frac{n_0 F_1}{2F_0}\right|^b. \quad (28)$$

Because in the sum of (27) either one term dominates over all others, or all the terms are sufficiently small, we can assume that

$$\max_{\phi}|\boldsymbol{j}_{\rm RCD}| \approx \sum_{a,b\,\in\,N} \max_{\phi}|\boldsymbol{j}_{\rm RCD}^{(a,b)}| \text{ (with odd } a + b). \quad (29)$$

Formulas (28) and (29) give an idea about the dependence of the maximum RCD (corresponding to the optimal phase shift) on the parameters of a two-colour laser pulse, in particular on the frequency of the additional field at a fixed frequency ω_0. Each term in the sum of (29) corresponds to a resonance-like Gaussian peak near the frequency ω_1 corresponding to some rational fraction a/b, where a and b satisfy the above conditions. Figure 1 shows this dependence calculated for a two-colour pulse with Gaussian envelopes of the main and additional fields,

$$A_{0,1}(t) = \sqrt{\frac{8\pi I_{0,1}}{c}}\exp\left(-2\ln 2\frac{t^2}{\tau_{\rm p}^2}\right), \quad (30)$$

where I_0 and I_1 are the maximum intensities of the main and additional fields; $\tau_{\rm p} = 2(\ln 2)^{1/2}\tau$ is the laser pulse duration (intensity full width at half maximum), which is the same for the envelopes of the main and additional fields; and c is the speed of light. For the ionisation probability we use the Tong formula for the hydrogen atom [19],

$$w(E) = 4\Omega_{\rm a}\frac{E_{\rm a}}{E}\exp\left(-\frac{2}{3}\frac{E_{\rm a}}{E} - 12\frac{E}{E_{\rm a}}\right), \quad (31)$$

where $\Omega_{\rm a}$ and $E_{\rm a}$ are the atomic frequency and field. In contrast to the traditional formula for the probability of tunnelling ionisation, the Tong formula correctly describes an increase in the plasma concentration in the above-barrier ionisation regime, and its use to find the RCD leads to results that are in good agreement with the results of quantum-mechanical calculations [20]. One can see from Fig. 1 that in the range of frequency ratios from 0.3 to 2.7, most pronounced are the peaks near the ratios 1/2, 2/3, 2/5, 3/4 and their inverse values, i.e., with small enough a and b. Figure 1 shows both the results of analytical calculations by formulas (28) and (29) and the result of direct numerical integration by formulas (4) and (6). Despite the fact that these dependences were plotted for a sufficiently large value of the additional field intensity located on the border of applicability of the analytical model, the positions and widths of the peaks (Fig. 1) are very well described by the derived analytical expressions. The amplitudes of the peaks coincide fairly well in analytical and numerical calculations for not very large values of the denominator b, while for large b discrepancies are observed.

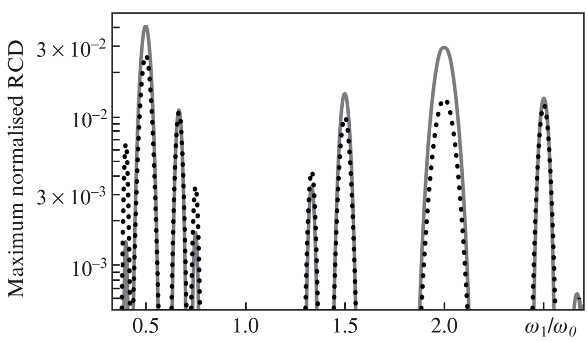

Figure 1. Dependences of the maximum normalised RCD on the ratio of frequencies ω_1 and ω_0 of the additional and main fields of an ionising two-colour laser pulse. Hereinafter, the RCD values are normalised to $j_{\rm a} = N_{\rm g}e^2E_{\rm a}/(m\Omega_{\rm a})$ and correspond to the optimal (at which the RCD reaches its maximum in absolute value) value of the phase shift ϕ between the components of the main and additional fields. The parameters of the laser pulse are as follows: the main field intensity is $I_0 = 10^{14}$ W cm^{-2}, the additional field intensity is $I_1 = 10^{13}$ W cm^{-2}, the pulse duration is $\tau_{\rm p} = 50$ fs, and the frequency of the main field ω_0 corresponds to a wavelength of 800 nm. Points were obtained using analytical formulas (28) and (29). The solid curve is the result of semi-classical calculations using formulas (4) and (6).

For peaks at ratios of the frequencies with the same denominators, the amplitudes of the peak maxima decrease with increasing a due to a Gaussian factor $\exp[-a^2/(2n_0)]$ in expression (25) for g at large a. At small a, the dependence of the peak width on a can be nonmonotonic. For example, at $n_0 > 7$, the amplitude of the peak at $a/b = 4$ is greater than at $a/b = 2$, i.e. the addition of the fourth harmonic leads to a greater residual current and a greater yield of THz radiation than the addition of the second harmonic with the same intensity. The value of the RCD is dependent on the maximum amplitude of the additional field as a power law with an exponent b; therefore, the ratio of amplitudes of the peak maxima at frequency ratios with different denominators essentially depends on the ratio of the intensities of the additional and main fields or, more accurately, on the value $n_0F_1/(2F_0)$. This is illustrated in Fig. 2, which shows the dependences of the maximum RCD on the additional field intensity. It is seen that at not very high intensities of the additional field, the dependences are given by a power law with high accuracy, and analytical expressions provide a very good approximation for the value of the residual current. In this case, the amplitudes of the peak maxima decrease rapidly with increasing b. At large ratios of the frequencies, near the boundary of applicability of the theoretical approach based on the perturbation theory with respect to the additional field, the peaks corresponding to larger b may be higher than the peaks corresponding to smaller b. For example, the peak at a ratio $a/b = 1/2$ is higher than that at $a/b = 2$ if the additional field is not very small [17].

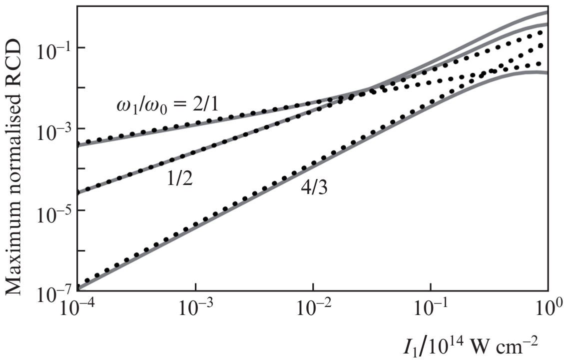

Figure 2. Dependences of the maximum normalised RCD on the additional field intensity of a two-colour laser pulse. The parameters of the laser pulse are as follows: the main field intensity is $I_0 = 10^{14}$ W cm^{-2}, the pulse duration is $\tau_p = 50$ fs, and the frequency of the main field ω_0 corresponds to a wavelength of 800 nm. Points were obtained using the analytical formula (28). Solid curves are the results of semi-classical calculations using formulas (4) and (6).

The dependence of the magnitude of the peak on the intensity of the main field at a fixed ratio of the intensities of the main and additional fields is to a large extent determined by the factor $j_{osc}\sigma$. The dependence grows rapidly at small intensities of the main field when the degree of ionisation σ remains low. In the case of depletion of neutral particles and saturation of the increase in the ionisation degree, the residual current growth also slows down. Note that the dependence of the intensity can also be affected by other factors in formula (26) because n_0 and τ_i also depend on the main field intensity. Despite the fact that n_0 and τ_i usually depend on the intensity of the main field slower than σ, their dependences may lead to a substantial modification of the dependence of the RCD on the intensity of the main field (for example, nonmonotonicity) at large values of a, b or $\Delta\omega\tau$.

The shape of the peaks is almost completely described by a Gaussian factor $\exp(-\Delta\omega^2\tau_i^2/2)$ in (26). The width of the peaks is greater than the width of the main field spectrum and decreases with increasing denominator b. In the absence of ionisation saturation, the peak width is greater than the width of the main field spectrum by $n_0^{1/2}/b$ times. In the general case (even when the conditions of applicability of the analytical model are violated), this width is on the order of $1/(\tau_i b)$. Figure 3 gives an idea how the steepness n_0 of the ionisation

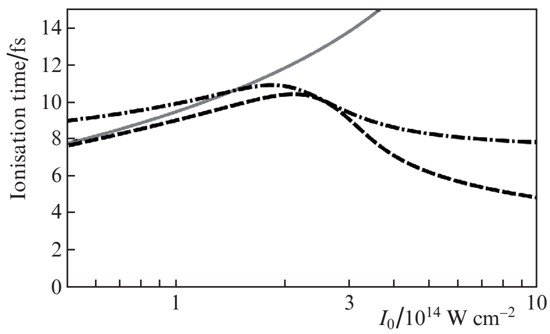

Figure 3. Ionisation time τ_i (solid and dashed curves) and inverse peak width of the RCD dependence on the additional field frequency (dot-and-dash curve) as a function of the intensity I_0 of the main field of a two-colour laser pulse. The parameters of the laser pulse are as follows: the pulse duration is $\tau_p = 50$ fs; the main field frequency ω_0 corresponds to a wavelength of 800 nm; the additional field frequency ω_1 corresponds to a wavelength of 1600 nm; and the intensity ratio of the main and additional fields is fixed, $I_1/I_0 = 0.01$. The dashed and dot-and-dash curves are the results of numerical calculations. The solid curve was obtained by the analytical formula for τ_i.

probability, characteristic ionisation time τ_i and peak width depend on the main field intensity and are related to each other. Figure 3 shows the dependence of the ionisation time, inverse peak width at $\omega_1 \approx \omega_0/2$ and value of $\tau/n_0^{1/2}$ at the pulse maximum on the intensity of the main field. The ionisation time is calculated by the formula

$$\tau_i = \sqrt{\frac{1}{N(+\infty)}\int_{-\infty}^{\infty}(t - \langle t\rangle)^2 \frac{dN}{dt}dt}, \quad (32)$$

where

$$\langle t\rangle = \frac{1}{N(+\infty)}\int_{-\infty}^{\infty} t\frac{dN}{dt}dt \approx t_0. \quad (33)$$

This definition of τ_i is practically equivalent to the previously given for the case when the shape of the function $d\bar{N}/dt$ is close to Gaussian. The peak width is calculated by the formula

$$\delta = \sqrt{\frac{\int_{0.45\omega_0}^{0.55\omega_0}(2\omega_1 - \omega_0)^2 \max_{\phi}|\boldsymbol{j}_{RCD}|d\omega_1}{\int_{0.45\omega_0}^{0.55\omega_0}\max_{\phi}|\boldsymbol{j}_{RCD}|d\omega_1}}. \quad (34)$$

One can see that in the entire range of intensities, the ionisation time and the inverse peak width are approximately equal; at low intensities when neutral particles are not depleted, both values coincide with $\tau/n_0^{1/2}$. This property of the residual current can be interesting in connection with the possibility of determining the values of τ_i and n_0 in the experiments on the generation of THz radiation by ionising frequency-tunable two-colour pulses through measuring tuning ranges in which the THz radiation is effectively generated. Such measurements are, for example, performed in [13].

Note also an interesting feature of the considered phenomenon of excitation of the residual current, associated with the asymmetrical shape of the peak. This asymmetry is demonstrated by formula (25) for the factor g, which is explicitly dependent on the sign of frequency detuning $\Delta\omega$. The presence of this dependence leads to the fact that the peak maximum is shifted with respect to the exact value $\omega_1 = a\omega_0/b$. Finding the maximum of expression (28), we obtain that the peak maximum corresponds to the frequency detuning

$$\Delta\omega_{\rm opt} \approx \frac{2n_0 z141}{\omega_0\tau_{\rm i}^2(b^2-a^2)}. \tag{35}$$

4. Comparison of the results of the semi-classical and quantum-mechanical approaches

The quantum-mechanical approach to the calculation of residual current density is based on the numerical solution of the three-dimensional time-dependent Schrödinger equation for the electron wave function in the hydrogen atom in a given field $\boldsymbol{E}(t)$ of a two-colour laser pulse. The details of the approach and the methods of solution are described in [16]. Figure 4 shows the dependences of the maximum RCD on the frequency of one of the quasi-monochromatic components of a two-colour pulse (at equal intensities of the components of a two-colour pulse when it is impossible to separate the main and additional fields). These dependences are calculated using the semi-classical and quantum mechanical approaches. It can be seen that the above-described peaks at the rational ratios of the frequencies in a two-colour laser pulse appear both in semi-classical and quantum-mechanical calculations. For high intensities, in the tunnelling ionisation regime, the results obtained using the semi-classical approach are consistent both qualitatively and quantitatively with the results of quantum-mechanical calculations (Fig. 4a). Thus, under conditions of tunnelling ionisation, the semi-classical approach can adequately describe both the amplitudes of the peak maxima and their width.

At lower intensities corresponding to the multiphoton ionisation regime, when the conditions of applicability of the semi-classical approach are not fulfilled, the semi-classical approach understates the amplitudes of peak maxima and overstates their widths. It can also be seen in Fig. 5, which shows the dependence of the maximum RCD on the intensities of the components of a two-colour pulse at different frequency ratios. For ratios $\omega_1/\omega_0 = a/b$ with not very large a and b (such as 1/2 and 2/3), the dependence of the RCD on the intensity saturates at high intensities and is accurately described by the semi-classical model (Fig. 5a). For large a and b (with a/b = 2/5, 3/4), the dependences turn out to be more complex and do not saturate even at very high intensities, which is due to the proximity of a stronger peak corresponding to smaller denominators: the peak at a ratio a/b = 2/5 is close to the peak at a/b = 1/2, and the peak at a/b = 3/4 is close to the peak at a/b = 2/3 (Fig. 5b). Presumably, it also leads to the fact that the results of the semi-classical calculations less accurately describe the dependence of the RCD on the intensity. However, the semi-classical model retains all the features of this dependence,

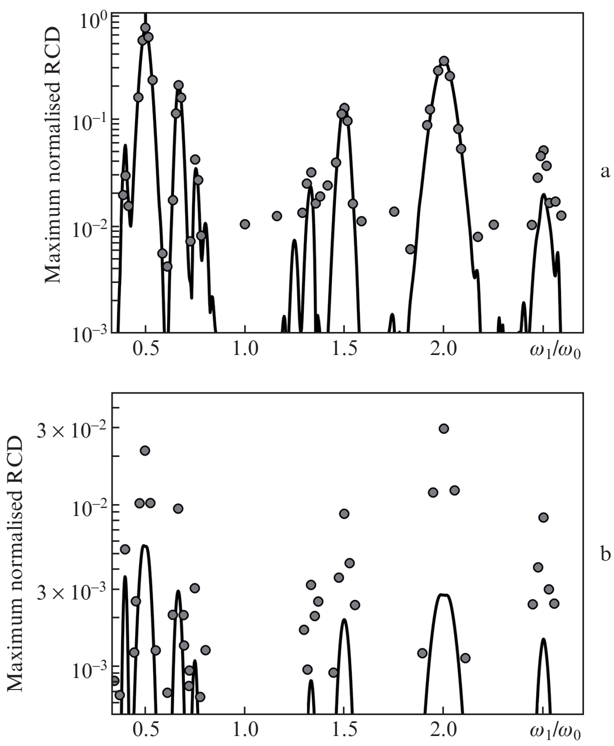

Figure 4. Dependences of the maximum normalised RCD on the ratio of the frequencies of the quasi-monochromatic components of a two-colour laser pulse. The parameters of the laser pulse are as follows: the pulse duration is $\tau_{\rm p}$ = 50 fs, the main field frequency ω_0 corresponds to a wavelength of 800 nm, and pulse component intensities are $I_0 = I_1$ = (a) 10^{14} and (b) 3×10^{13} W cm^{-2}. Solid curves are the results of semi-classical calculations, and points are the results of quantum-mechanical calculations.

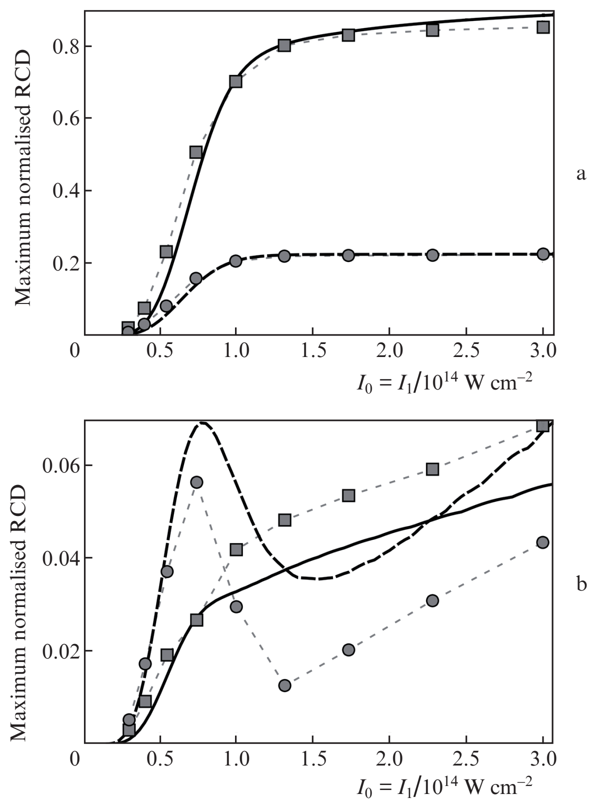

Figure 5. Dependences of the maximum normalised RCD on the intensities $I_1 = I_0$ of the quasi-monochromatic components of a two-colour laser pulse at frequency ratios (a) ω_1/ω_0 = 1/2 (solid curve, ■) and 2/3 (dashed curve, ●), as well as (b) ω_1/ω_0 = 3/4 (solid curve, ■) and 2/5 (dashed curve, ●). The pulse duration is $\tau_{\rm p}$ = 50 fs, and the main field frequency ω_0 corresponds to a wavelength of 800 nm. Curves are the results of semi-classical calculations, and points are the results of quantum-mechanical calculations.

such as its nonmonotonicity and position of the minimum and maximum at $a/b = 2/5$.

5. Conclusions

We have studied the generation of a quasi-constant RCD in a plasma produced by a two-colour laser pulse with an arbitrary ratio of the frequencies of the quasi-monochromatic components of the pulse. Based on the semi-classical approach, we have found a fairly simple analytical expression that describes the dependence of the RCD on the laser pulse parameters in the case when the field of the two-colour pulse represents a superposition of a strong quasi-monochromatic field at the fundamental frequency and a weak additional field at a different frequency. The obtained analytical expressions show that the dependence of the RCD on the frequency of the additional field is a set of resonance-like peaks near the frequencies for which the ratio of the frequencies of the main and additional fields is equal to the rational fraction with a not very big odd sum of the numerator and denominator. The widths of the peaks are greater than the widths of the spectrum of the main ionising field and are determined by the inverse characteristic ionisation time divided by the denominator of the fraction which expresses the ratio of the frequencies in a two-colour laser pulse. We have found the dependences of the amplitudes of the peak maxima on the intensities of the main and additional fields. The presented numerical calculations show that the obtained analytical formulas describe well the RCD in a fairly wide range of laser pulse parameters. The results of calculations based on the numerical solution of the three-dimensional time-dependent Schrödinger equation are in good agreement with the results of semi-classical calculations for the tunnelling ionisation regime.

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