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Some specific features of light propagation in a three-channel nonlinear directional coupler

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Abstract. **Exact analytical solutions have been obtained to the system of nonlinear differential equations for the intensities of the waves propagating in a three-channel nonlinear directional coupler with a Kerr nonlinearity and different coupling constants between the optical waveguides.**

Keywords: *three-channel nonlinear directional coupler, coupling constant, coupling length, self-trapping.*

1. Introduction

Among the artificial semiconductor structures with specific functional characteristics, aimed at storage, transmission and processing of optical data, a specific position is occupied by different spatially periodic structures, in particular, nonlinear directional couplers (NDCs). The principal condition for their functioning is the possibility of controlling the process of radiation propagation. To date, a satisfactory theory of laser radiation propagation in NDCs with a Kerr nonlinearity of the refractive index is developed. For this case the exact analytical solutions were obtained for the system of nonlinear equations, describing the intensities of propagating waves [1–4]. Maier [2] predicted the phenomenon of the self-switching of waves in a NDC, when a small variation in the input intensity of one of the waves causes sharp changes in the intensities of the rest two waves at the NDC output. However, analytical solutions are derived only for the NDC, consisting of two optical waveguides with Kerr nonlinearities. As to a NDC comprising three or more optical waveguides, the light propagation in them was studied only by means of numerical methods, applied to the system of nonlinear equations for the coupled waves [2–11]. Therefore, it is of great interest to derive analytical solutions to multi-channel NDC equations and to study the specific features of their functioning. In the present paper, we have derived the exact analytical solution to the system of nonlinear equations for the intensities of waves propagating in a symmetric three-channel NDC with a Kerr nonlinearity of the propagation constant and different coupling constants between the waveguides. The approach proposed in Refs [12, 13] is used.

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2. Formulation of the problem. Basic equations

Consider a symmetric NDC consisting of three identical parallel optical waveguides, located at the vertices of an isosceles triangle (Fig. 1). The coupling constants between the waveguides with the numbers $n = 1$, 2 and $n = 1$, 3 are equal to γ, and for the waveguides with the numbers $n = 2$, 3 the coupling constant is γ_1. Thus, in two limiting cases, $\gamma_1 = 0$ and $\gamma_1 = \gamma$, the present configuration is reduced to the systems considered previously [5–8].

We consider the optical waveguides with a Kerr nonlinearity of the propagation constant, $\beta = \beta_0 + \alpha I$, where β_0 is the linear part of the propagation constant; α is the Kerr coefficient; and I is the intensity of propagating wave. In this case, the nonlinear differential equations for the field amplitudes E_1, E_2 and E_3 of the coupled waves, propagating along the x axis in each of the waveguides, have the following form [2–11]:

$$\frac{dE_1}{dx} = -i(\beta_0 + \alpha I_1)E_1 + i\gamma(E_2 + E_3),$$

$$\frac{dE_2}{dx} = -i(\beta_0 + \alpha I_2)E_2 + i\gamma E_1 + i\gamma_1 E_3, \quad (1)$$

$$\frac{dE_3}{dx} = -i(\beta_0 + \alpha I_3)E_3 + i\gamma E_1 + i\gamma_1 E_2.$$

Here $I_n = (c/8\pi)|E_n|^2$, $n = 1, 2, 3$.

Consider the case when the first optical waveguide ($n = 1$) is pumped (Fig. 1). Let us complete Eqns (1) with the boundary conditions

$$E_{1|x=0} = E_0, \quad E_{2|x=0} = E_{3|x=0} = 0. \quad (2)$$

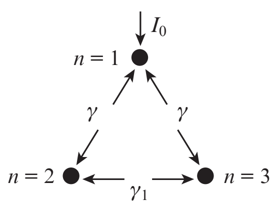

Figure 1. Schematic of the NDC cross section, perpendicular to the axis of waveguides.

The system of equations (1) describes the steady-state propagation of light in optical waveguides. The factors $\beta_0 + \alpha I_j$ ($j = 1, 2, 3$) describe the variation in the effective propagation constant in each waveguide as a function of the intensity of propagating light (Kerr effect). As to the coupling constants between the waveguides, in Eqns (1) they are assumed constant and independent of the intensity. Equations (1) are valid under the condition that no new modes are generated during the nonlinear propagation of the light in the waveguides. They are applied in the cases of weak nonlinear perturbations and for the optical waveguides, essentially separated from each other [1–8].

Consider the solution in the form $E_n(x) = f_n(x)\exp(-\mathrm{i}\beta_0 x)$. For the functions $f_n(x)$ we obtain

$$\frac{\mathrm{d}f_1}{\mathrm{d}x} = -\mathrm{i}\alpha I_1 f_1 + \mathrm{i}\gamma(f_2 + f_3),$$

$$\frac{\mathrm{d}f_2}{\mathrm{d}x} = -\mathrm{i}\alpha I_2 f_2 + \mathrm{i}\gamma f_1 + \mathrm{i}\gamma_1 f_3, \qquad (3)$$

$$\frac{\mathrm{d}f_3}{\mathrm{d}x} = -\mathrm{i}\alpha I_3 f_3 + \mathrm{i}\gamma f_1 + \mathrm{i}\gamma_1 f_2.$$

The boundary conditions for the functions $f_n(x)$ are similar to (2): $f_{1|x=0} = f_0, f_{2|x=0} = f_{3|x=0} = 0$. Let us show that under these conditions the field amplitudes (intensities) in the waveguides $n = 2$ and 3 will be similar at any point x. To this end, we introduce the function $p = f_2 - f_3$ with the boundary condition $p_{|x=0} = 0$. Then from Eqn (3) we obtain the differential equation for the function $p(x)$:

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\mathrm{i}\alpha\frac{c}{8\pi}\left[|p|^2 p + p^2 f_3^* + 2|p|^2 f_3 + 2p|f_3|^2 + f_3^2 p^*\right] - \mathrm{i}\gamma_1 p. \qquad (4)$$

Let us present p and f_3 as $p = Q\exp(\mathrm{i}\varphi)$, $f_3 = F\exp(\mathrm{i}\psi)$, where Q and F are the amplitudes, and φ and ψ are the phases of the functions p and f_3. Substituting these expressions into Eqn (4) and separating the real and imaginary parts in Eqn (4), we obtain

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -\alpha Q\left[QF\sin(\varphi - \psi) + F^2\sin(2\varphi - 2\psi)\right]. \qquad (5)$$

Since all functions depend on x, the factor in square brackets in the right-hand side of Eqn (4) is also a certain function depending on x:

$$f(x) = QF\sin(\varphi - \psi) + F^2\sin(2\varphi - 2\psi).$$

Then, Eqn (5) can be rewritten in the form

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -\alpha Q f(x).$$

Its formal solution has the form

$$Q = A\exp\left(-\int_0^x f(\xi)\,\mathrm{d}\xi\right),$$

where A is the integration constant. Satisfying the boundary condition $Q_{|x=0} = 0$, we obtain $A = 0$. Thus, at any point of the NDC the following relation holds:

$$Q(x) = |f_2(x) - f_3(x)| = 0, \qquad (6)$$

i.e., the absolute value of the difference of the complex field amplitudes f_2 and f_3 is equal to zero. Presenting them in the form $f_2 = F_2\exp(\mathrm{i}\varphi_2)$ and $f_3 = F_3\exp(\mathrm{i}\varphi_3)$ and substituting into Eqn (6), we arrive at the relation

$$F_2^2 + F_3^2 - 2F_2F_3\cos(\varphi_2 - \varphi_3) = 0.$$

According to the law of cosines, this relation is valid for $F_2 = F_3$ and $\varphi_2 = \varphi_3 + 2\pi m$, $m = 0, 1, 2, \ldots$. From this fact it finally follows that $f_2 = f_3$ at any point x of the NDC. This fact allows essential simplification of Eqns (3) and its reduction to the form

$$\frac{\mathrm{d}f_1}{\mathrm{d}x} = -\mathrm{i}\alpha I_1 f_1 + 2\mathrm{i}\gamma f_2,$$

$$\frac{\mathrm{d}f_2}{\mathrm{d}x} = -\mathrm{i}\alpha I_2 f_2 + \mathrm{i}\gamma f_1 + \mathrm{i}\gamma_1 f_2. \qquad (7)$$

From Eqns (7), one can see that when the NDC with three identical optical waveguides, located at the vertices of an isosceles triangle, is pumped via the apical waveguide, it is equivalent to the NDC with two different waveguides. These waveguides possess different propagation constants, and the coupling constant of the first waveguide to the second one is by two times greater than the coupling constant of the second waveguide to the first one.

Let us introduce the functions [12]

$$I_{1,2} = \frac{c}{8\pi}|f_{1,2}|^2, \quad Q = \frac{\mathrm{i}c}{8\pi}(f_2^* f_1 - f_1^* f_2),$$

$$R = \frac{c}{8\pi}(f_2^* f_1 + f_1^* f_2). \qquad (8)$$

Using Eqns (7) and the system of conjugate equations, we obtain the system of coupled nonlinear differential equations for the new functions:

$$\frac{\mathrm{d}I_1}{\mathrm{d}x} = -2\gamma Q, \quad \frac{\mathrm{d}I_2}{\mathrm{d}x} = \gamma Q, \qquad (9)$$

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = \left[\alpha(I_1 - I_2) + \gamma_1\right]R + 2\gamma(I_1 - 2I_2), \qquad (10)$$

$$\frac{\mathrm{d}R}{\mathrm{d}x} = -\left[\alpha(I_1 - I_2) + \gamma_1\right]Q. \qquad (11)$$

According to (2), the boundary conditions for the system of equations (9)–(11) have the form

$$I_{1|x=0} = I_0, \quad I_{2|x=0} = 0, \quad Q_{|x=0} = R_{|x=0} = 0. \qquad (12)$$

From Eqn (9) it is easy to obtain the first integral of motion

$$I_1 + 2I_2 = I_0, \qquad (13)$$

which represents the law of conservation of energy in the system. Using Eqn (9) and Eqn (13), we obtain the second integral of motion from Eqn (11)

$$R = \frac{\alpha}{2\gamma}I_2(3I_2 - 2I_0) - \frac{\gamma_1}{\gamma}I_2. \qquad (14)$$

Finally, from Eqn (10) we derive the third integral of motion

$$Q^2 = I_2\left[4I_0 - \left(8 + \frac{\gamma_1^2}{\gamma^2}\right)I_2 - 2\frac{\gamma_1}{\gamma}\frac{\alpha}{\gamma}I_2\left(I_0 - \frac{3}{2}I_2\right) - \frac{\alpha^2}{\gamma^2}\left(I_0 - \frac{3}{2}I_2\right)^2\right]. \quad (15)$$

It is easy to show that there is one more integral of motion, relating all the functions,

$$Q^2 + R^2 = 4I_1I_2, \quad (16)$$

which is actually a consequence of expressions (13)–(15). Substituting Eqn (15) into Eqn (9) it is easy to obtain the nonlinear differential equation describing the spatial variation of the light intensity I_2 in the second optical waveguides of the NDC

$$\frac{dI_2}{dx} = \gamma\left\{I_2\left[4I_0 - \left(8 + \frac{\gamma_1^2}{\gamma^2}\right)I_2 - 2\frac{\gamma_1}{\gamma}\frac{\alpha}{\gamma}I_2\left(I_0 - \frac{3}{2}I_2\right)^2\right]\right\}^{1/2}. \quad (17)$$

Let us introduce the normalised quantities

$$y = \frac{I_2}{I_0}, \quad y_1 = \frac{I_1}{I_0}, \quad z = 2\gamma x, \quad a = \frac{\alpha I_0}{2\gamma}, \quad s = \frac{\gamma_1}{\gamma}. \quad (18)$$

Then the solution of Eqn (17) in quadratures for the function $y(z)$ can be presented as

$$\int_0^y dy\left(y\left\{1 - 2y - y\left[\frac{s}{2} + a\left(1 - \frac{3}{2}y\right)\right]^2\right\}\right)^{-1/2} = z. \quad (19)$$

From Eqn (19) one can see that the behaviour of the solutions is determined by two parameters, the nonlinearity a and the ratio of coupling constants s. Moreover, from Eqn (19) it follows that the intensity y of light propagating through the second optical waveguide periodically varies from zero to the maximal value y_{max} determined by the equation

$$1 - 2y_{max} - y_{max}\left[\frac{s}{2} + a\left(1 - \frac{3}{2}y_{max}\right)\right]^2 = 0. \quad (20)$$

The positive root closest to zero is chosen for the solution. When the level of excitation grows, no new roots of Eqn (20) appear. The appearance of a pair of real roots of Eqn (20) manifests the occurrence of the self-trapping effect. Hence, the above facts allow the conclusion that in the considered system pumped via one of the waveguides the self-trapping is absent. The latter takes the place in the case of the NDC consisting of two waveguides.

Note that the light self-trapping phenomenon was predicted in atomic systems [14–16], in atomic-molecular systems [17], in exciton-polariton systems in microcavities [18, 19], and was observed experimentally [20, 21]. It consists in a sharp (practically stepwise) change of the oscillation amplitude under the variation of the system parameters or the excitation level.

In the linear limit ($a = 0$) the intensity of light in the optical waveguides of the coupler is determined by the expressions:

$$y = \frac{I_2}{I_0} = \frac{4}{\kappa^2}\sin^2\frac{\kappa}{2}z, \quad y_1 = \frac{I_1}{I_0} = 1 - \frac{8}{\kappa^2}\sin^2\frac{\kappa}{2}z, \quad (21)$$

where $\kappa = \sqrt{8 + s^2}$. Thus, the light intensity is periodically transferred from the first optical waveguide to the second and the third one, $y_{max} = 4/\kappa^2$, $y_{1\,min} = 1 - 8/\kappa^2$, and the coupling length is $L_0 = \pi/(2\gamma\kappa)$. From these results, it is seen that the coupling length in the three-channel linear coupler is smaller than in the coupler consisting of two similar waveguides.

The solution of Eqn (20) shows that the maximal light intensity y_{max} in the second (third) waveguide monotonically decreases from the value $y_{max} = 4/\kappa^2$ to zero with the growth of the nonlinearity parameter a (with the growth of the excitation level at the input face of the first optical waveguide). Two other roots of Eqn (20) are complex conjugate at any combination of the parameters s and a. Assuming that y_{max} is known, these roots can be presented as $y_{2,3} = \mu \pm i\nu$, where

$$\mu = \frac{a_1 - y_{max}}{2}; \quad \nu = \frac{1}{2}\sqrt{4a_2 + (y_{max} - a_1)(3y_{max} + a_1)};$$
$$a_1 = \frac{4}{3}\left(1 + \frac{s}{2a}\right); \quad a_2 = \frac{4}{9}\left[\left(1 + \frac{s}{2a}\right)^2 + \frac{2}{a^2}\right]. \quad (22)$$

Then, expression (15) can be easily integrated, and we obtain

$$y = y_{max}\sqrt{\mu^2 + \nu^2}\,\frac{1 - \mathrm{cn}(3maz/2)}{(n_1 - n_2)\mathrm{cn}(3maz/2) + n_1 + n_2}, \quad (23)$$

where $\mathrm{cn}(x)$ is the elliptic cosine [22, 23] with the modulus k equal to

$$k^2 = \frac{1}{2}\left(1 - \frac{\mu(\mu - y_{max}) + \nu^2}{\sqrt{[\mu(\mu - y_{max}) + \nu^2]^2 + \nu^2 y_{max}^2}}\right); \quad (24)$$

$$m = \{[\mu(\mu - y_{max}) + \nu^2]^2 + \nu^2 y_{max}^2\}^{1/2};$$
$$n_1 = \sqrt{(\mu - y_{max})^2 + \nu^2}; \quad n_2 = \sqrt{\mu^2 + \nu^2}. \quad (25)$$

From Eqn (23) it follows that the intensity of the light in the second optical waveguide $y(z)$ periodically changes from zero to y_{max}. The coupling length $L_z = 2\gamma L$ is expressed by the formula

$$L_z = \frac{4}{3ma}K(k), \quad (26)$$

where $K(k)$ is the complete elliptic integral of the first kind with the modulus k [22, 23].

3. Discussion of the results

Figure 2 presents the dependence of the maximal intensity y_{max}, transferred from the first (pumped) optical waveguide of the NDC into the second one, on the parameter of nonlinearity a for a number of values of the ratio of coupling constants s. One can see that with the growth of a, the value of y_{max} rapidly decreases, tending to zero at $a \gg 1$.

An increase in the coupling constant ratio affects the values of y_{max} at $a = 0$, but does not affect the general character of its further behaviour. Figure 2 shows that in the three-channel system of waveguides the phenomenon of self-trapping is absent.

Figure 3 presents the spatial dependence of intensity $y(z)$, transferred from the first (pumped) waveguide of the NDC into the second one, on the parameter a for a few values of

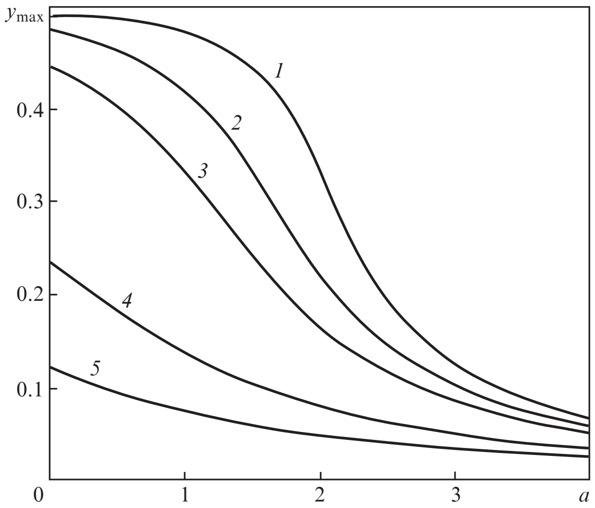

Figure 2. Dependence of the maximal intensity y_{max} transferred from the first (pumped) waveguide of the NDC into the second one on the nonlinearity parameters a for different values of the ratio of coupling constants s = (*1*) 0, (*2*) 0.5, (*3*) 1, (*4*) 3 and (*5*) 5.

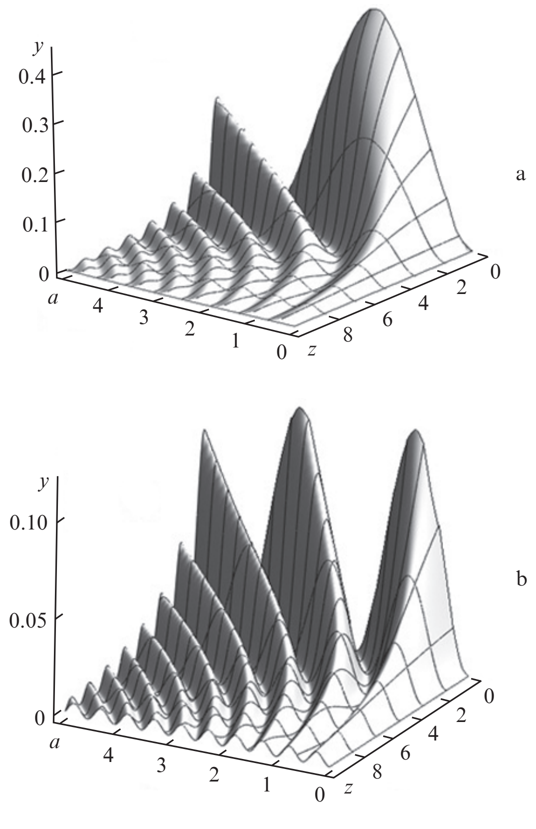

Figure 3. Spatial dependence of the intensity $y(z)$ transferred from the first optical waveguide into the second one on the nonlinearity parameter a for the ratio of coupling constants s = (a) 1 and (b) 5.

the parameter s. One can see that the energy of propagating light is periodically transferred from the first waveguide into other two waveguides and back. For a fixed nonlinearity parameter a the amplitudes of oscillations of the function $y(z)$ monotonically decrease with increasing z. However, with an increase in the nonlinearity parameter a the amplitude of oscillations of the function $y(z)$ increases. The positions of the maxima are determined by the coupling length L_z, which, as seen from Figs 2 and 4, depend on the excitation level.

Figure 4 presents the dependence of the coupling length L_z on the nonlinearity parameter a for several values of the coupling constant ratio s. With the growth of a the function $L_z(a)$ increases due to the increasing modulus k of the complete elliptic integral of the first kind $K(k)$, while for $a \gg 1$ the coupling length, as follows from (26), decreases inversely proportional to a. From Fig. 4 it also follows that in a three-channel NDC under the condition of a strong excitation level the self-trapping phenomenon is absent.

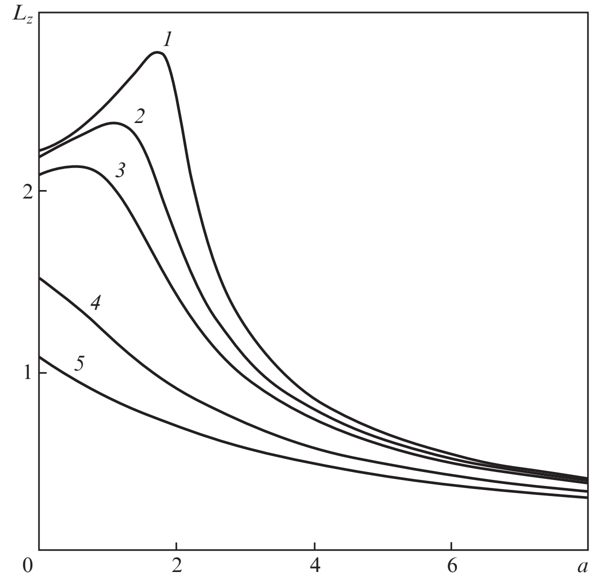

Figure 4. Dependence of the coupling length L_z on the nonlinearity parameter a for the values of the ratio of coupling constants s = (*1*) 0, (*2*) 0.5, (*3*) 1, (*4*) 3 and (*5*) 5.

Thus, let us summarise the basic results. The exact analytical solutions are obtained for the system of nonlinear equations, describing the intensities of the propagating waves in a symmetric three-channel NDC with a Kerr nonlinearity of the propagation constant and different coupling constants between the waveguides. It is demonstrated that the radiation is periodically transferred from the pumped waveguide into two other ones and back. The maximal intensity of the light, transferred into the adjacent waveguides, and the coupling length rapidly decrease with increasing excitation level. At high excitation levels, there is no self-trapping phenomenon that occurs in two-channel NDC.

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