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Generation of giant spatially localised Gaussian wave packets in active fibres with saturable inertial nonlinearity

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***Abstract*. The dynamics of wave packets in an active fibre with saturable inertial nonlinearity of logarithmic type is considered. The possibility of forming spatially localised Gaussian wave packets (TEM_{00} mode) with a large (significantly exceeding 100 μm^2) mode area and a large (above 1 TW) peak power in such structures is shown.**

Keywords: *active fibre, saturable logarithmic nonlinearity, three-dimensional soliton-like pulse, single-mode wave packet with giant peak power.*

1. Introduction

Significant interest in active fibres with a large effective mode area, relatively small nonlinear parameters and a high damage threshold is determined by the possibility of forming pulses with a large peak power (more than 1 MW), which makes them suitable for a number of practical applications [1–7]. In this case, a standard single-mode fibre, on the contrary, has a sufficiently small effective mode area, not exceeding 100 μm^2. Generation of an effective single-mode signal (for example, a Gaussian TEM_{00} mode) at the fibre output will obviously require a larger core area of the fibre, while for further effective compression (temporal and spatial focusing) of the beam, such a fibre should be as single-mode as possible.

In this paper we consider the dynamics of spatially localised wave packets in a medium with inertial saturable nonlinearity of logarithmic type. Media with a nonlinearity of this type have been previously considered in [8–14]. It is shown that the model of saturable logarithmic nonlinearity more accurately describes the dynamics of high-power pulses propagating in doped optical fibres than the classical two-level model of saturable nonlinearity.

A mechanism is proposed for generating single-mode Gaussian wave packets (pulses) with a mode area substantially exceeding 100 μm^2 and an energy exceeding 1 mJ in such fibres. Generation of wave packets with corresponding parameters makes it possible to further compress them by standard methods to wavelength scales (diameter ~1 μm) and intensities $I \gg 10^{16}$ W cm^{-2}. As a result, the implementation of laser facilities with such characteristics can contribute to solving a number of important technological problems, including the development of laser accelerators of charged particles and systems for controlled nuclear and thermonuclear fusion [3–8].

The most promising from this point of view is the use of optical disk amplifiers (with a transverse size much greater than that of single-mode fibres) based on semiconductors [15–19] or glasses heavily doped with rare-earth elements (Fig. 1). In this case, use can be made of a short (less than 1 m) large-diameter optical fibre (quartz disk) heavily doped with erbium, neodymium or ytterbium, which, if properly manufactured and appropriately pumped, can have a nonlinearity coefficient 5×10^5 times larger than that of a conventional fibre [20]. For semiconductor amplifiers, it is possible to use current pumping and to obtain giant values of the gain increment, much larger than 10 cm^{-1}. An important additional advantage of short (much less than 1 m long) high-power disk amplifiers can be the possibility of their effective cooling.

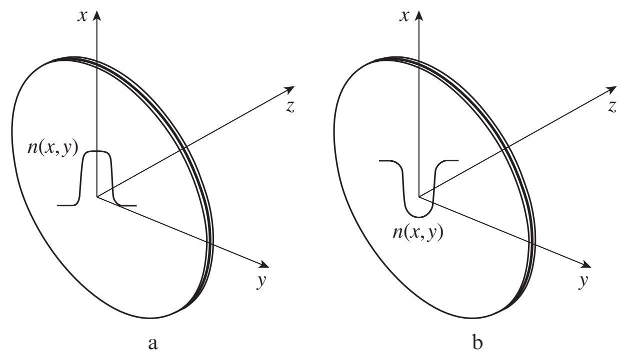

Figure 1. (a) 'Focusing' parabolic fibre with a refractive index $n(x,y,z,t) = n_0(1 - |m_x|x^2/x_0^2 - |m_y|y^2/y_0^2) + \Delta n(I)$ decreasing away from the centre and (b) 'defocusing' parabolic fibre with a refractive index $n(x,y,z,t) = n_0(1 + |m_x|x^2/x_0^2 + |m_y|y^2/y_0^2) + \Delta n(I)$ increasing away from the centre.

Another method for forming such high-energy Gaussian pulses (in the single-mode regime – TEM_{00} mode) with a large mode area can be heavily doped (as a rule, with ytterbium ions) tapered fibres (see, for example, Refs [21–26]) with an increasing diameter and a large concentration of active centres.

Note that the considered effects described by the logarithmic saturation nonlinearity model can probably be observed in standard optical fibres, if they are heavily doped, and the peak intensity of the propagating pulse is $I_0 \gg 10^{11}$ W cm^{-2}.

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They can be germanium-doped optical fibres, or phosphate glass core fibres [27–31]. Obviously, when considering the dynamics of high-intensity pulses (with $I_0 > 10^{11}$ W cm^{-2}) in heavily doped laser media, a question inevitably arises about the need to take into account saturation of nonlinearity and inertia of the nonlinear response [32–35].

In this paper, we demonstrate the possibility of generating a single-mode Gaussian pulse with an extremely high peak power (more than 1 TW) and a large mode area (i.e., a three-dimensional soliton with a large mode area and energy). Such a pulse can subsequently be easily compressed by classical techniques (using lenses and diffraction gratings) to peak intensities $I_0 \gg 10^{16}$ W cm^{-2}.

2. Basic equations for describing the dynamics of a wave packet in a medium with saturable nonlinearity of logarithmic type

Let us consider the dynamics of a wave packet propagating in a nonlinear medium with a parabolic distribution of the refractive index over the optical fibre cross-section. In this case, it can be described by the equation

$$\frac{\partial A}{\partial z} + \frac{\mathrm{i}}{2}\left(D_x\frac{\partial^2}{\partial x^2} + D_y\frac{\partial^2}{\partial y^2} + D_\tau\frac{\partial^2}{\partial \tau^2}\right)A$$
$$= \mathrm{i}k_0\delta n(z,x,y,t,I)A + g(z)A, \quad (1)$$

where A is the slowly varying amplitude of the wave packet; $k_0 = \omega_0/c$; δn is the disturbance of the refractive index of the optical fibre; $g(z)$ is the gain increment of the fibre, which we assume to be dependent only on the longitudinal coordinate z; $\tau = t - \int_0^z \mathrm{d}z/u_g(z)$ is the time in the 'running' coordinate system; $u_g = (\partial\beta/\partial\omega)^{-1}_{\omega=\omega_0}$ is the group velocity; β is the propagation constant of the wave packet; $D_\tau = (\partial^2\beta/\partial\omega^2)_{\omega=\omega_0}$ is the dispersion of group velocities; $D_x \cong D_y \cong 1/(n_0k_0) = 1/\beta_0$ are the corresponding diffraction parameters (assumed to be approximately equal, which is valid in the case of weak birefringence at the fibre axis); n_0 is the refractive index of the fibre core; and $\beta_0 \approx n_0k_0$.

In this case, for the 'disturbed' refractive index of the fibre with allowance for saturable nonlinearity (which we assume to be homogeneous over the optical fibre cross section) and the parabolic distribution of the linear component of the refractive index, we can write

$$\delta n(x,y,z,t) = n_0\left(m_x\frac{x^2}{x_0^2} + m_y\frac{y^2}{y_0^2}\right) - \Delta n(I(x,y,z,t)), \quad (2)$$

where $m_{x,y}$ are the modulation coefficients in the corresponding directions (below we assume that $|m_{x,y}| \ll 1$); and x_0 and y_0 are the effective transverse sizes of the fibre. In this case, we consider that intense radiation leads to a change in the refractive index, which can be estimated as $n(x=0, y=0, z) = n_0 + \sum\chi_i\Delta n_i(I)$, where χ_i is a function depending on the concentration of the corresponding ith component of the medium (a parameter varying from 0 to 1).

In this case, it is considered that each individual component makes its contribution $\Delta n_i(I)$ to the change in the refractive index. At $\chi = 1$, the material (an undoped silica fibre, a liquid or a semiconductor of certain type, etc.) is assumed to be completely homogeneous.

It should be noted that in media with large cubic (Kerr) nonlinearity, the response time is almost always large enough and increases almost linearly with increasing Kerr nonlinearity [32–35]. Taking into account the response time (inertia) and saturation of nonlinearity for each ith component of the medium, we can write the relation:

$$\Delta n_i(I) + \tau_{\mathrm{nl},i}\frac{\partial\Delta n_i(I)}{\partial t} = n_i^{(2)}I_{n,i}\ln(1 + I/I_{n,i}), \quad (3a)$$

where $\tau_{\mathrm{nl},i}$ is the time of the nonlinear response of the ith component of a medium; $n_i^{(2)}$ is a parameter characterising the cubic (Kerr) nonlinearity of the ith component; $I_{n,i}$ is the saturation intensity for the ith component; and $I = |A(z,\tau)|^2$ is the radiation intensity. Thus, in this paper we consider a fibre with saturable nonlinearity of logarithmic type. Models describing the dynamics of radiation in such media are presented in [9–14, 32, 33].

It seems to us that the logarithmic saturation nonlinearity given by equation (3a) more accurately describes the dynamics of a high-power wave packet than the saturation nonlinearity given by the expression [32–34]:

$$\Delta n_i(I) + \tau_{\mathrm{nl},i}\frac{\partial\Delta n_i(I)}{\partial t} = \frac{n_i^{(2)}I}{1 + I/I_{n,i}}. \quad (3b)$$

In contrast to (3a), expression (3b) does not virtually take into account the modulation (time dependence) of the refractive index in the approximation of high powers of the radiation propagating in the fibre, i.e. at $I/I_{n,i} \gg 1$.

Thus, it is well known that in a rough quasi-monochromatic approximation [32–34] (i.e., neglecting the effect of dispersion and nonlinear effects of higher orders) for the classical Kerr nonlinearity of form RI, where $I = |A|^2 = I_0\exp(-\tau^2/\tau_p^2)$, and R is the coefficient of the cubic (Kerr) nonlinearity [32, 33], the truncated equation describing the dynamics of the amplitude A can be written in the form

$$\frac{\partial A}{\partial z} + \mathrm{i}R|A|^2A = 0, \quad (4)$$

whence

$$A \approx A_0\exp(-\mathrm{i}R|A|^2z) \approx A_0\exp\{-\mathrm{i}RI_0[1 - (\tau^2/\tau_p^2)]z\}. \quad (5)$$

In this case, in the quasi-monochromatic approximation for a chirp (i.e., the rate of frequency modulation) of the wave packet, we can write down the approximate relation [33, 34]

$$\partial^2\varphi/\partial\tau^2 \approx 2RI_0z/\tau_p^2.$$

The most common model, taking into account saturation of nonlinearity, corresponds to a two-level atomic system, for which the nonlinearity parameter can be written as $RI/(1 + I/I_n)$ [32, 34]. In this case, the truncated equation (4) takes the form

$$\frac{\partial A}{\partial z} + \mathrm{i}\frac{RI}{1 + I/I_n}A = 0. \quad (6)$$

Thus, as above, in the rough quasi-monochromatic approximation

$$A \approx A_0\exp\left(-\mathrm{i}\frac{RIz}{1 + I/I_n}\right)$$

$$\approx A_0\exp\left\{-\mathrm{i}\frac{RI_0[1-(\tau^2/\tau_{\rm p}^2)]z}{1+I_0[1-(\tau^2/\tau_{\rm p}^2)]/I_n}\right\}$$

$$\approx A_0\exp(-\mathrm{i}RI_n z) \quad (7)$$

$$(\text{at } \tau^2 \ll \tau_{\rm p}^2 \text{ and } I_0 \gg I_n).$$

Consequently, for the maximum of the pulse at $\tau \ll \tau_{\rm p}$, the chirp value (the rate of frequency modulation, or the linear component of the rate of change in the carrier frequency) can be estimated as $\partial^2\varphi/(\partial\tau^2)_{\tau\to 0} \to 0$.

Obviously, for high-power beams propagating in a medium with inertial saturable nonlinearity, this approximation is not completely correct. It is difficult to imagine a nonlinear medium during the propagation of which the linear component of the rate of frequency modulation (chirp) disappears in a high-power laser pulse. In fact, a sufficiently long pulse in such a medium should become practically transform limited. Consequently, in our case of generation (formation) of a high-energy pulse, such a model does not take into account the most important factor – nonlinear phase modulation – and, therefore, is not completely correct.

Consider now a model that takes into account saturable nonlinearity of logarithmic type, $RI_n\ln(1 + I/I_n)$. For this case, valid is the quasi-monochromatic approximation of form

$$\frac{\partial A}{\partial z} + \mathrm{i}RI_n\ln\left(1+\frac{I}{I_n}\right)A = 0. \quad (8)$$

Thus, we can write for the amplitude

$$A \approx A_0\exp\left[-\mathrm{i}RI_n\ln\left(1+\frac{I}{I_n}\right)z\right]$$

$$\approx A_0\exp\left\{-\mathrm{i}RI_n\ln\left[1+\frac{I_0\exp(-\tau^2/\tau_{\rm p}^2)}{I_n}\right]z\right\}$$

$$\approx A_0\exp\left[-\mathrm{i}RI_n\left(-\frac{\tau^2}{\tau_{\rm p}^2}+\ln\frac{I_0}{I_n}\right)z\right] \quad (9)$$

$$(\text{at } I_0 \gg I_n \text{ and } \tau \ll \tau_{\rm p}).$$

From (9) we obtain the expression for the chirp of the carrier frequency: $\partial^2\varphi/(\partial\tau^2)_{\tau\to 0} \to 2RI_n z/\tau_{\rm p}^2$. It is obvious that for $I_0 \gg I_n$ the chirp grows much more slowly than in the case of the standard Kerr nonlinearity; however, it does not tend to zero, as in the case of the classical saturating model.

Note that for $I/I_n \ll 1$ and $\tau_{\rm nl} \to 0$, both types of saturable nonlinearity (two-level and logarithmic) reduce to the well-known classical nonlinearity of Kerr type [32–35].

The situation is usually realised when dopants with large and 'slow' nonlinearity (for example, dopants based on semiconductors or rare-earth elements) are introduced into the matrix with small and 'fast' Kerr nonlinearity (for example, in quartz glass). In this case, as a rule, $n_i^{(2)}(I) \gg n_l^{(2)}(I)$ and $I_{n,i} \ll I_{n,l}$ (subscripts l determine the parameters of the fibre matrix). In addition, it is well known that usually the nonlinearity depends strongly on the response time of the nonlinear medium, such that $\tau_{\rm nl} \sim n^{(2)}$ [33–35], which implies the condition $\tau_{{\rm nl},i} \gg \tau_{{\rm nl},l}$. Thus, for media with large saturable nonlinearity, as a rule, the nonlinear response time is significantly more than several femtoseconds. Thus, for standard silica fibres, $\tau_{\rm nl} < 10^{-14}$ s, but even for fibres with a high degree of doping (for example, with erbium and ytterbium ions), the response time $\tau_{\rm nl}$ can be much larger, $\sim 10^{-13}$ s, and for heavily doped glasses it is $\sim 10^{-11}$ s.

Let us consider in detail the situation with dopants based on semiconductors and rare-earth elements, for which $\tau_{\rm nl}$ varies in the range $10^{-11}-10^{-13}$ s.

3. Formation of three-dimensional soliton-like pulses in a medium with saturable nonlinearity of logarithmic type

In the case of saturable nonlinearity of logarithmic type and parabolic inhomogeneity of the refractive index over the optical fibre cross section, equation (1) can be transformed to the form

$$\frac{\partial}{\partial z}A + \frac{\mathrm{i}}{2}\left(D_x\frac{\partial^2}{\partial x^2} + D_y\frac{\partial^2}{\partial y^2} - D_\tau\frac{\partial^2}{\partial\tau^2}\right)A$$

$$+\mathrm{i}\left(\sum_i \chi_i R_i I_{n,i}\left[\ln\left(1+\frac{I}{I_{n,i}}\right)\right]\right)A$$

$$-\mathrm{i}\left\{\sum_i \chi_i\tau_{{\rm nl},i}R_i I_{n,i}\frac{\partial}{\partial\tau}\left[\ln\left(1+\frac{I}{I_{n,i}}\right)\right]\right\}A$$

$$= \mathrm{i}\beta_0\left[\frac{m_x(z)x^2}{x_0^2(z)} + \frac{m_y(z)y^2}{y_0^2(z)}\right]A + g(z)A. \quad (10)$$

Below, we will consider the dynamics of transform limited Gaussian pulses that at the input to the medium have the form

$$A(z=0) = A_0\exp\left[-\frac{x^2}{2\Delta x^2(0)} - \frac{y^2}{2\Delta y^2(0)} - \frac{\tau^2}{2\tau_{\rm p}^2(0)}\right] \quad (11)$$

in the case when fast Kerr nonlinearity can be considered small in comparison with the inertial 'slow' nonlinearity. This is logical, since we will consider the possibility of generating three-dimensional solitons with a giant transverse size (including considerably larger than 1 mm), when the radiation intensity does not exceed 10^{10} W cm^{-2} even at peak powers much greater than 1 MW. In this case, for 'slow' nonlinearity, we can assume that $I/I_n^{\rm (slow)} \gg 1$ and, as a consequence, $\ln(1 + I/I_n^{\rm (slow)}) \approx \ln(I/I_n^{\rm (slow)})$. On the other hand, for 'fast' nonlinearity [for example, for the cubic (Kerr) nonlinearity typical for glass dielectric matrices], with a good degree of accuracy we can assume that $I/I_n^{\rm (fast)} \ll 1$ and, as a consequence, $\ln(1 + I/I_n^{\rm (fast)}) \approx I/I_n^{\rm (fast)}$.

Let us consider the case of a single dopant in the matrix, when only one component with inertial saturable nonlinearity is taken into account. It is this situation, as a rule, that is realised in practice. For example, it can be glass heavily doped with rare-earth elements (erbium, neodymium or ytterbium), or glass doped with semiconductors.

In this case, on the length of the formed wave packet, i.e., for $z \gg u_{\rm g}\tau_{\rm nl}$, we can, by substituting

$$A = \bar{A}\exp\int_0^z\left[g(z) + \mathrm{i}\chi_i R_i I_{n,i}\left(F(z) - \frac{\tau_{\rm nl}^2}{\tau_{\rm p}^2(z)}\right)\right]\mathrm{d}z,$$

where

$$F(z) = \ln\left[\frac{I_0}{I_{n,i}}\frac{\tau_{\rm p}(0)}{\tau_{\rm p}(z)}\frac{\Delta x(0)}{\Delta x(z)}\frac{\Delta y(0)}{\Delta y(z)}\exp\left(2\int_0^z g(z)\mathrm{d}z\right)\right],$$

pass from the nonlinear equation (8) to the linear equation for the amplitude A:

$$\frac{\partial \bar{A}}{\partial z}+\frac{\mathrm{i}}{2}\left(D_x\frac{\partial^2}{\partial x^2}+D_y\frac{\partial^2}{\partial y^2}-D_\tau\frac{\partial^2}{\partial^2\bar{\tau}}\right)\bar{A}$$
$$=\mathrm{i}(\Omega_x(z)x^2+\Omega_y(z)y^2+\Omega_\tau(z)\bar{\tau}^2)\bar{A}. \qquad (12)$$

Here, $\bar{\tau}=\tau-\tau_{\rm nl}$ is the nonlinear response time of the medium. In (12), we introduced parabolic potentials determined by the value of nonlinearity saturation and the distribution of the change in the refractive index over the fibre cross section:

$$\Omega_x(z)=\frac{\chi_i R_i I_{n,i}}{\Delta x^2(z)}+\frac{m_x\beta_0}{x_0^2},$$

$$\Omega_y(z)=\frac{\chi_i R_i I_{n,i}}{\Delta y^2(z)}+\frac{m_y\beta_0}{y_0^2},$$

$$\Omega_\tau(z)=\frac{\chi_i R_i I_{n,i}}{\tau_{\rm p}^2(z)}.$$

For simplicity, but not limiting the generality of the problem in question, we assume that the parameters D_x, D_y, D_τ are independent of z. The corresponding problem, taking into account the initial conditions, has localised soliton-like solutions:

$$\bar{A}(z,x,y,\tau)=A_0\psi^{(\tau)}(z,\tau)\psi^{(x)}(z,x)\psi^{(y)}(z,y). \qquad (13)$$

In this case, equation (12) splits into three independent (autonomous) equations [36]:

$$\frac{\partial\psi^{(x)}}{\partial z}+\frac{\mathrm{i}}{2}D_x\frac{\partial^2}{\partial x^2}\psi^{(x)}=\mathrm{i}\Omega_x x^2\psi^{(x)},$$
$$\frac{\partial\psi^{(y)}}{\partial z}+\frac{\mathrm{i}}{2}D_y\frac{\partial^2}{\partial y^2}\psi^{(y)}=\mathrm{i}\Omega_y y^2\psi^{(y)}, \qquad (14)$$
$$\frac{\partial\psi^{(\tau)}}{\partial z}-\frac{\mathrm{i}}{2}D_\tau\frac{\partial^2}{\partial\bar{\tau}^2}\psi^{(\tau)}=\mathrm{i}\Omega_\tau\bar{\tau}^2\psi^{(\tau)}.$$

The system of equations (14), as is well known, describing the oscillations of a harmonic oscillator, has spatially localised soliton-like solutions:

$$\psi^{(j)}(j,z)=U^{(j)}(j)\exp\left(\mathrm{i}\int_0^z\lambda_j(z)\mathrm{d}z\right);\quad j=x,y,\tau. \qquad (15)$$

Here, $\lambda_j=\eta_j/\sqrt{\pm 2D_j/\Omega_j}$; and η_j is a nonnegative integer.

At $\Omega_{x,y}/D_{x,y}>0$ (for transverse components) or $\Omega_\tau/D_\tau<0$ (for the longitudinal time component), the system of equations (14) has soliton-like solutions in the form of known functions of a harmonic oscillator. In addition, the function $U_n^{(j)}$ will have the form [37, 38]

$$U_n^{(j)}(j)=\exp\left(-\frac{\xi_j^2}{2}\right)H_n^{(j)}(j), \qquad (16)$$

where $H_n^{(j)}(j)$ is a Chebyshev–Hermite polynomial of order n, defined by the relation [37, 38]

$$H_n^{(j)}(j)=\frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}}}\exp(\xi_j^2)\frac{\mathrm{d}^n\exp(-\xi_j^2)}{\mathrm{d}\xi_j^2}; \qquad (17)$$

$\xi_j^2=j^2\sqrt{\pm 2\Omega_j/D_j}$; for $j=x,y$ the sign '+' is true, and for $j=\tau$ – the sign '–'.

The functions that are solutions of equations (14) will be continuous and finite for $\eta_j=2n+1$ ($n=0,1,2,3,\ldots$).

If a Gaussian pulse is launched into the fibre (TEM_{00} mode), then the simplest zero-order polynomial ($n=0$, $\lambda_{x,y}=\sqrt{\Omega_{x,y}D_{x,y}/2}$ и $\lambda_\tau=\sqrt{-\Omega_\tau D_\tau/2}$) corresponds to the solution describing the dynamics of the wave packet. In this case, it becomes possible to form a soliton-like (spatially localised) pulse with a duration and a transverse size determined by the relations

$$\tau_{\rm s}\approx\left[-\frac{D_\tau}{2\chi_i R I_{n,i}}\right]^{1/2}, \qquad (18a)$$

$$\Delta j\approx\left[\frac{D_j}{2\Omega_j}\right]^{1/4},\ j=x,y. \qquad (18b)$$

Because we can assume that for a radially symmetric fibre $D_x\sim D_y\sim 1/\beta_0$, expression (18b) in this case is reduced to the form

$$\Delta j=\left\{-\frac{\chi R I_n j_0^2}{2m_j\beta_0}+\left[\left(\frac{\chi R I_n j_0^2}{2m_j\beta_0}\right)^2+\frac{j_0^2}{2m_j\beta_0^2}\right]^{1/2}\right\}^{1/2} \qquad (18c)$$

for $m_j>0$ and to the form

$$\Delta j=\left\{-\frac{\chi R I_n j_0^2}{2m_j\beta_0}\pm\left[\left(\frac{\chi R I_n j_0^2}{2m_j\beta_0}\right)^2+\frac{j_0^2}{2m_j\beta_0^2}\right]^{1/2}\right\}^{1/2} \qquad (18d)$$

for $m_j<0$.

Note that a stable soliton-like pulse can be formed only if the condition $\tau_{\rm nl}\leqslant\tau_{\rm s}$ is satisfied. Taking into account the fact that in the cases of optical fibres with large saturable nonlinearity we have $\tau_{\rm nl}\geqslant 10^{-13}$ s, the duration of the localised wave packet must satisfy the inequality $\tau_{\rm s}>\tau_{\rm nl}>10^{-13}$ s.

It is interesting that for $m_j<0$ (i.e., when a defocusing lens compensated for by nonlinear focusing arises in a fibre), the parameters Ω_x, Ω_y, which determine the effective mode area, can take practically any values that are arbitrarily close to zero. As a consequence, the conditions for generating a two-dimensional (over the cross section) soliton with a transverse size Δx, $\Delta y\gg 1$ mm can be easily realised.

In the approximation of large-diameter beams for a defocusing fibre ($m_j<0$), when valid are the inequalities

$$\chi_i R_i I_{n,i}\Delta x^2,\ \chi_i R_i I_{n,i}\Delta y^2\gg 1/\beta_0,$$

$$|m_x|\Delta x^4/x_0^2,\ |m_y|\Delta y^4/y_0^2\gg 1/\beta_0^2,$$

for beams with large dimensions (Δx, $\Delta y>100$ μm), there are solutions [solutions in (18d) with the sign '+'] approximated by the relations:

$$\Delta x\approx x_0\left[\frac{\chi R I_n}{|m_x|\beta_0}\right]^{1/2}, \qquad (19a)$$

$$\Delta y\approx y_0\left[\frac{\chi R I_n}{|m_y|\beta_0}\right]^{1/2}. \qquad (19b)$$

It can be seen that with the use of gradient defocusing fibres (with a refractive index increasing away from the the axis) one can ensure the generation of quasi-single-mode (with a Gaussian distribution of the field over the fibre cross section – TEM_{00} mode) wave packets a giant (over 1 mm^2) mode area. Note that the saturable character of nonlinearity leads to the fact that the beam in the corresponding medium is described by a well-known equation of a harmonic quantum oscillator [39]. As a consequence, the corresponding linear equations obviously give solutions that are stable to perturbations. Therefore, such beams are stable with respect to the appearance of a transverse modulation instability and decay of a single beam into individual small-scale beams, i.e. filaments [32–34, 40–47].

The energy is effectively accumulated when the shape of the wave packet is preserved until the effect of fast nonlinearity becomes comparable with the effect of slow nonlinearity with low saturation energy. When the influence of fast nonlinearity becomes comparable with that of slow nonlinearity, expression (10) is transformed into a three-dimensional Gross–Pitaevskii equation [20, 32, 34], and in the general case it can be analysed only by numerical methods.

It can be seen from relation (18a) that in a medium with inertial saturable nonlinearity in the region of anomalous group velocity dispersion (GVD), it is possible to form a soliton-like pulse with high energy and peak power. Because of the potential instability of such a wave packet (primarily because of the possible development of modulation instability) in media with large saturable nonlinearity, a stable three-dimensional soliton can be formed when $\chi_i R_i I_{n,i} < 10$ m^{-1}. For 'solid' highly nonlinear materials (for example, for photorefractive media [32, 34, 35]), as a rule, $R_i I_{n,i} > 10^3$ m^{-1} and, therefore, at $\chi_i \approx 1$, nonlinearity is too large to form 'saturated' stable soliton-like pulses, because in this case (in the optical and near-IR spectral ranges) the condition $\tau_{nl} \leqslant \tau_s$ does not hold. However, at certain dopant concentrations (when $\chi_i \ll 1$), a stable soliton-like wave packet can be realised at the fibre length, $z \gg u_g\tau_s$. Thus, for $D_\tau \approx -10^{25}$ s^2 m^{-1} and $\chi_i R_i I_{i,n} \approx 5$ m^{-1}, the duration of the soliton-like pulse is $\tau_s \approx 10^{-13}$ s, and its transverse size, determined by relation (18b), can take practically any values. The energy of a soliton-like amplified pulse will vary as

$$W(z) = W(0)\exp\left(2\int_0^z g(z)\mathrm{d}z\right)$$

and can achieve (at an appropriate pump power) greater values, higher than 1 J.

Apparently, for the experimental realisation of the effects under consideration, the concentration of erbium ions in optical fibres (glass matrices) should exceed 10^{20} см$^{-3}$ [27], up to $\sim 10^{21}$ cm^{-3}, which can correspond to $\chi \sim 0.01$. This will allow one to use a short, less than 1 m, segment of a heavily erbium-doped fibre-cone [21–23] or a properly doped (neodymium, ytterbium or erbium) quartz disk. Such elements, when properly manufactured, can be characterised by a nonlinearity factor that is 5×10^5 times greater than in a conventional fibre [34, 35], i.e. reach the values of $n^{(2)} \approx 10^{-8}$ cm^2 kW^{-1}. Of course, heavy doping with rare-earth elements will lead to a significant increase in losses (for example, due to clustering). However, the corresponding losses can be compensated for by using more high-power pumping.

4. Soliton-like pulses in a medium with a travelling wave of a change in the refractive index

A whole set of interesting solutions appears in the case when the refractive index is additionally modulated in time. Then, relation (2) can be rewritten as follows [48–52]:

$$\delta n(x,y,z,t) = n_0\left(m_x\frac{x^2}{x_0^2} + m_y\frac{y^2}{y_0^2} - \Delta n(I(x,y,z,t))\right)$$

$$+ m_\tau n_0\cos[\omega_\tau(\tau - \delta\tau(z))], \quad (20)$$

where

$$\delta\tau = \int_0^z (u_g^{-1} - u_m^{-1})\mathrm{d}z$$

is the parameter determined by the difference between the velocity u_m of a travelling wave of a change in the refractive index (TWCRI) and the group velocity of a soliton-like wave packet.

In this case, starting from the results of [36, 52], we can conclude that in the expression for the duration and size of the soliton-like pulse being formed, relations (18a) and (18b) will remain valid if we write for the parabolic potential

$$\tilde{\Omega}_\tau \approx \chi_i R_i I_{n,i}/\tau_p^2(z) + m_\tau\beta_0\omega_\tau^2, \quad (21)$$

for the delay time of a soliton-like wave packet

$$\bar{\tau} = \tau - \tau_{nl} - \delta\tau, \quad (22)$$

and for the phase shift

$$\lambda_\tau = \sqrt{-\Omega_\tau D/2} + m_\tau\beta_0\omega_\tau^2/2. \quad (23)$$

Then, the duration of the soliton-like pulse is determined by the relation

$$\tau_s = \left\{-\frac{\chi_i R I_{n,i}}{2m_\tau\beta_0\omega_\tau^2} + \theta(m_\tau, D)\left[\left(\frac{\chi_i R I_{n,i}}{2m_\tau\beta_0\omega_\tau^2}\right)^2 - \frac{D}{2m_\tau\beta_0\omega_\tau^2}\right]^{1/2}\right\}^{1/2}, \quad (24)$$

where $\theta(m_\tau, D) = 1$ for $m_\tau > 0$ and $D < 0$, and also for $m_\tau < 0$ and $D > 0$; $\theta(m_\tau, D) = \pm 1$ (bistable solution) for $m_\tau < 0$ and $D < 0$. In the case $m_\tau > 0$ and $D > 0$, a soliton-like wave packet is not formed.

Figures 2–4 show the dependences of the normalised duration of the soliton-like pulse τ_s/τ_{nl} от χ_0/χ. In this case we used the normalisation parameters $\tau_{nl} = 10^{-13}$ с, $\chi_0 = 0.1$, and the parameters $RI_n = 10^4$ m^{-1}, $D = -10^{-26}$ s^2m^{-1} (Figs 2 and 3), $D = 10^{-26}$ s^2 m^{-1} (Fig. 4). Note that for $\tau_s \leqslant \tau_0$ the model under consideration becomes not completely correct. In this case, it is necessary to additionally take into account the influence of higher order nonlinear and dispersion effects.

Note some interesting features of the solutions presented in Figs 2–4. First, special attention should be paid to the fact that in synchronising a soliton-like wave packet with a TWCRI minimum (for $m_\tau < 0$) in the region of anomalous GVD, two stable soliton solutions are possible: a 'long' soliton $\theta(m_\tau, D) = +1$ and a 'short' soliton $\theta(m_\tau, D) = -1$ (Fig. 3). In this case, the bistability regime is implemented, including for small values of m_τ and ω_τ. It seems that the corresponding property of non-

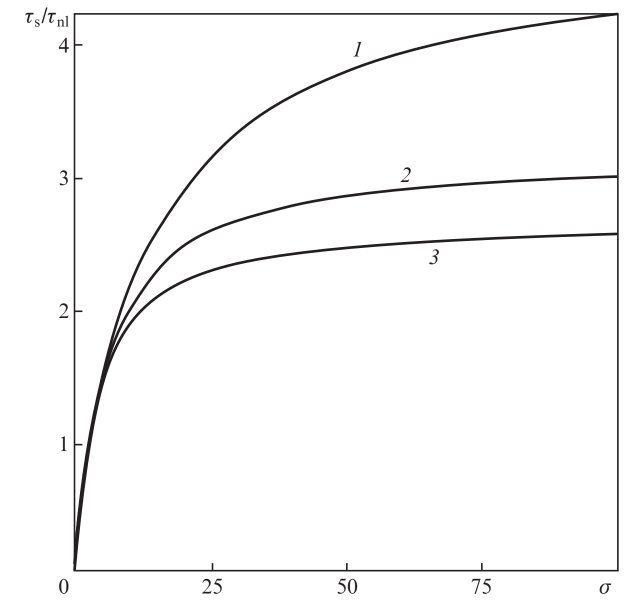

Figure 2. Dependences of the normalised duration of the soliton-like pulse τ_s/τ_{nl} on the normalised function $\sigma = 10^{-3}(\chi_0/\chi)$ in the case $D = -10^{-26}$ s^2 m^{-1} and $m_\tau > 0$ at $|m_\tau\beta_0\omega_\tau^2| =$ (*1*) 10^{23}, (*2*) 5×10^{23}, and (*3*) 10^{24} m^{-1} s^{-2}. Here and in Figs 3 and 4 the sign of $m_\tau\beta_0\omega_\tau^2$ depends on the sign of the modulation depth m_τ.

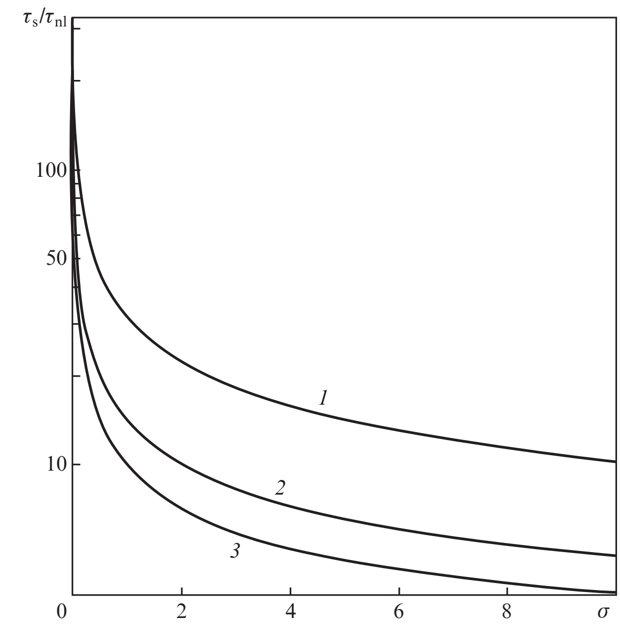

Figure 4. Dependences of the normalised duration of the soliton-like pulse τ_s/τ_{nl} on the normalised function $\sigma = 10^{-3}(\chi_0/\chi)$ in the case $D = 10^{-26}$ s^2 m^{-1} and $m_\tau < 0$ at $|m_\tau\beta_0\omega_\tau^2| =$ (*1*) 10^{23}, (*2*) 5×10^{23}, and (*3*) 10^{24} m^{-1} s^{-2}.

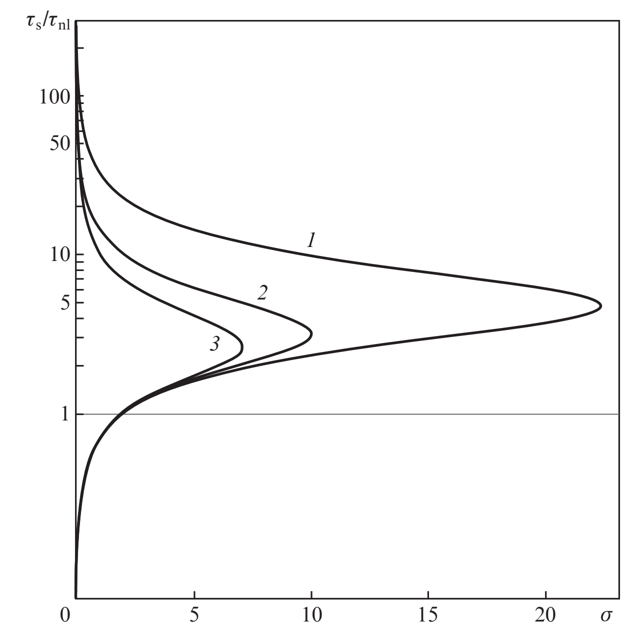

Figure 3. Dependences of the normalised duration of the soliton-like pulse τ_s/τ_{nl} on the normalised function $\sigma = 10^{-3}(\chi_0/\chi)$ in the case $D = -10^{-26}$ s^2 m^{-1} and $m_\tau < 0$ at $|m_\tau\beta_0\omega_\tau^2| =$ (*1*) 10^{23}, (*2*) 5×10^{23}, and (*3*) 10^{24} m^{-1} s^{-2}.

linear media with saturable nonlinearity can be used to design all-optical logic elements on their basis.

Second, one can see from Fig. 3 that in the linear approximation, i.e., at $\chi_0/\chi \to \infty$, the duration of the localised wave packet, τ_s, tends asymptotically to $(|D|/(2|m_\tau|\beta_0\omega_\tau^2))^{1/4}$.

Third, when a soliton-like wave packet is synchronised with the TWCRI minimum (for $m_\tau < 0$), it becomes possible to form a three-dimensional soliton in a medium with a normal material dispersion when $D > 0$ (Fig. 4). The duration of such a soliton can greatly exceed 1 ns; therefore, remaining stable, it will be able to accumulate soliton-like pulses with giant energy – much greater than 1 J. Under normal GVD conditions, the realisation of a soliton-like pulse substantially increases its resistance to perturbation by preventing the development of modulation instability, which occurs in a medium with anomalous GVD [32–34].

Note that the above-described scheme for the formation of high-energy spatially localised wave packets under conditions of interaction with the TWCRI resembles the mechanism of the formation of ball lightning proposed in Refs [53, 54]. The problem of experimental realisation of such a scheme requires separate consideration, but the development of appropriate acousto- or electro-optic modulators does not seem to be particularly difficult at present. In particular, the scheme used in conical fibres and bottle cavities can be promising, which makes it possible to implement the effective acousto-optical interaction of the TWCRI and the whispering-gallery mode waves [49, 55–59].

5. Conclusions

We have considered the conditions for the formation of spatially localised wave packets in doped fibres with large inertial saturable nonlinearity of logarithmic type. It is shown that soliton-like wave packets (TEM_{00} modes) with a large cross section and high energy (more than 1 J) and peak power (over 1 TW) can be formed in such fibres.

It is shown that large saturable (inertial) nonlinearity contributes to the formation of a soliton-like wave packet, and a large mode area reduces the effect of fast Kerr nonlinearity of the matrix (for example, a silica fibre). Thus, it is possible to generate a single-mode pulse with a giant mode area (well exceeding 100 μm^2) and large (above 1 J) energy.

To obtain wave packets with similar energy characteristics, it seems optimal to use conical fibres (doped with ytterbium, neodymium or erbium) with increasing diameter. Further amplification of the corresponding solitons can be realised in disk amplifiers made on the basis of rare-earth-doped quartz glasses.

It is shown that additional possibilities for generating high-energy wave packets in the single-mode regime can be implemented under conditions of their interaction with a TWCRI realised in a fibre with saturable inertial nonlinearity.

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