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Factor of spontaneous emission into the optical dielectric waveguide mode

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Abstract. **Within the framework of a complete vector analysis, an expression for the factor of spontaneous emission into the transverse mode of a dielectric waveguide is found. The results are compared with those previously obtained in the framework of the scalar approximation. It is shown that their most significant quantitative difference is possible for waveguides used in diode lasers and amplifiers in which the transverse beam size is commensurable with the wavelength. The conditions for the applicability of the scalar approximation for calculating the spontaneous emission factor are obtained.**

Keywords: *spontaneous emission factor, diode laser, optical amplifier, waveguide.*

1. Introduction

When analysing the effect of spontaneous emission on the characteristics of a diode laser or amplifier, a question arises as to the intensity of this radiation. This problem is especially acute in those problems where the noise characteristics are considered and spontaneous emission is a fundamentally unavoidable source of noise. Since in most diode lasers and amplifiers the generation of radiation and its amplification occur in an optical waveguide, these problems are essentially reduced to finding the fraction of the total spontaneous emission of an elementary radiator that is launched (is captured) in a dielectric waveguide. Looking ahead, we note that this fraction is known to be much less than unity, i.e., spontaneous emission is coupled into those modes that are the free space modes, and only a small fraction of it is captured by the waveguide. The determination of this fraction is the problem of finding the factor of spontaneous emission into the waveguide mode, the solution of which, due to its importance, has a rather long history (see, for example, [1–12]). To calculate this factor, as a starting point we make use of the almost obvious fact that the intensity of spontaneous emission into the mode is proportional to the fraction of its phase volume in the total volume of all the modes in which spontaneous emission takes place.

This situation persisted until the appearance of Petermann's paper [3], where an additional dimensionless factor K was introduced, whose magnitude depends on the curvature of the wave front of the mode. For modes with a waveguide formed exclusively by an index-guided profile and a plane wave front, this factor is equal to unity. In lasers with a gain-guided (g-g) waveguide, whose modes have a significant curvature of the wave front, the value of this factor may exceed the order of magnitude. Work [3] received wide recognition, and this factor was called the Petermann's K factor in the literature. Then, however, critical remarks appeared about the validity of its use in the expression for the spontaneous emission into the mode [4, 9], and a discussion was even published on this subject [11, 12]. Critical remarks and disagreements were basically reduced to different approaches to the condition of mode orthogonality for gain-guided lasers. During the polemic, not only calculations were considered, but also qualitative and even heuristic considerations [8]. Perhaps this discussion would continue even further if g-g lasers, starting from some time, had not lost some of their relevance.

As for the question of the correct calculation of spontaneous emission into the mode of a diode laser or amplifier, it has remained controversial. However, the interest in solving this problem has been preserved to some extent, not only because of the lack of a correct solution of one of the fundamental problems concerning diode lasers, but also due to the recent appearance of new objects – integrated single-crystal multisection chips (see, for example, [13, 14]) based on laser heterostructures. Such chips can contain g-g regions of active waveguides in the form of optical amplifiers, optical branching devices and, in particular, power amplifiers considered, for example, in Refs [15–19]. Spontaneous emission in such devices is a 'natural' and unrecoverable source of noise, which requires its adequate consideration.

The foregoing discussion motivated this research. In the approach adopted in previous papers, two approximations can be singled out, the validity of which is not obvious. The first is the scalar approximation for the mode amplitude. It is difficult to adopt, especially for diode laser modes, in which the transverse sizes of the waveguide are comparable to the wavelength, and especially for the case of a curved wave front of the mode. It is clear that in this case there is an essential component of the Umov–Poynting vector in the direction transverse to the wave propagation axis. The presence of a gradient of the electric field amplitude leads to a redistribution of the amplitudes of the components of the field vectors. This must be taken into account when calculating the power flow along the optical axis. The second approximation (it has already been mentioned above) is the orthogonality conditions of the modes in a waveguide having significant optical losses and amplification in spatially separated regions, which is caused by a significant change in the imaginary part of the complex dielectric constant in the direction transverse to the waveguide axis. Although the orthogonality condition for modes of a lossy waveguide is rather definitely formulated in

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Vainshtein's book [20], its use with respect to the modes of a dielectric waveguide in the scalar approximation is not entirely obvious.

In this paper, in contrast to previous works, the analysis is performed in full vector representation. In addition, the calculation does not rely on the condition of the mode orthogonality (in explicit form).

2. Physical model

The approach we adopted is based on classical electrodynamics. Taking into account the fact that a consecutive calculation of spontaneous emission is possible only within the framework of quantum electrodynamics, we use the correspondence principle between classical and quantum theories. In this case, we assume that averaging over a micro-ensemble occurs using a statistical matrix. From our problem, we exclude the mechanism of spontaneous emission associated with the intrinsic motion of a charge of an elementary radiator, providing it with all the attributes of a classical dipole with previously known parameters. Analytical expressions for these parameters can be obtained within the framework of the quantum theory, and their numerical values are then found, for example, directly from the experiment, taking into account the analytical relations obtained. They include the dipole moment, lifetime, emission spectrum, etc. Thus, our problem is reduced to finding in a dielectric waveguide the amplitude of a wave generated by an emitting dipole, which is placed into this waveguide.

For a metal waveguide, such a solution is well known and is given, for example, in [20]. It was obtained using the field expansion in the modes of a waveguide. The approach adopted in the present paper is analogous to that presented in [20] with some modification for the case of a dielectric waveguide with a substantially complex dielectric constant ε.

We will further assume that the presence of a waveguide does not in any way affect the character of the internal motion of the dipole charge, as if this dipole radiated in a homogeneous transparent medium with a refractive index $n = \sqrt{\mathrm{Re}\,\varepsilon}$. For example, the average radiated power p_0 of a harmonically oscillating dipole at a frequency ω_0 in this case would be

$$p_0 = \frac{\omega_0^4 d_0^2 n}{3c^3} = \frac{\hbar\omega_0}{\tau}, \tag{1}$$

where d_0^2 is the square of the modulus of the dipole moment; $\hbar\omega_0$ is the quantum energy; and τ is the spontaneous lifetime. From the frequently used similar expression, relation (1) differs only by the factor n. Equating expression (1) to the expression for the average power of spontaneous emission of an excited elementary radiator in the form $\hbar\omega_0/\tau$, we find the value of d_0^2 for the oscillator in question. In this form, the part, related to quantum mechanics, is reduced to finding τ for spontaneous emission. In our case, the value of τ is well known both from the calculations and from the experiment, and for typical direct-gap semiconductors it is $\sim$1 ns. Within the framework of this approach, our problem becomes exclusively classical.

3. Calculation of the wave amplitude of a waveguide mode emitted by an elementary dipole

The scheme of the waveguide model and the oscillating dipole for our analysis is shown in Fig. 1. The complex dielectric constant $\varepsilon(x,y)$ depends only on two transverse coordinates, and the radiation wave with the electric field intensity amplitude $\mathcal{E}(x,y,z,t)$ propagates unlimitedly along the z axis. Let us consider the free propagation of a monochromatic wave in a waveguide. To this end, we present the vectors of the amplitudes of the electric field $\mathcal{E}(x,y,z,t)$ and magnetic field $\mathcal{H}$ (x,y,z,t) in the complex form as

$$\mathcal{E} = \frac{1}{2}\{\boldsymbol{E}(x,y,z)\exp(-\mathrm{i}\omega t) + \mathrm{c.c.}\},$$
$$\mathcal{H} = \frac{1}{2}\{\boldsymbol{H}(x,y,z)\exp(-\mathrm{i}\omega t) + \mathrm{c.c.}\}. \tag{2}$$

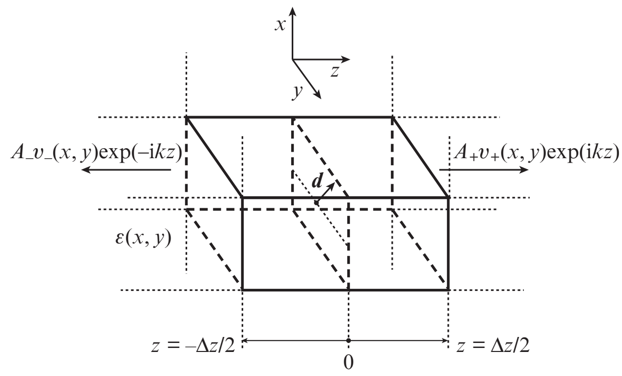

Figure 1. Scheme of an optical waveguide containing a radiating dipole.

It follows from Maxwell's equations that the complex amplitude $\boldsymbol{E}$ must satisfy the well-known equation

$$\mathrm{rotrot}\boldsymbol{E} - k_0^2\varepsilon(x,y)\boldsymbol{E} = 0, \tag{3}$$

where $k_0 = \omega/c$. The addition of equation (3) with the standard boundary conditions for waveguide modes leads to the fact that the solutions for $\boldsymbol{E}$ have the form

$$\boldsymbol{E} = \boldsymbol{v}_k(x,y)\exp(\mathrm{i}kz), \tag{4}$$

where $k = k' + \mathrm{i}k'' = \omega n_{\mathrm{eff}}/c + \mathrm{i}\alpha/2$, and the transverse distribution of the amplitude of the wave is characterised by the complex vector-function $\boldsymbol{v}_k(x,y)$. In this case, its square of the modulus – the scalar product $\boldsymbol{v}_k(x,y)\boldsymbol{v}_k^*(x,y)$ – rapidly enough tends to zero at an unbounded increase in $|x|$, $|y|$. The discrete set of complex propagation constants k, each corresponding to its function $\boldsymbol{v}_k(x,y)$ and therefore denoted by k, is a set of transverse waveguide modes. The real part k' determines the effective refractive index n_{eff}, and the imaginary part k'' corresponds to absorption $\alpha = 2k''$ or amplification $g = -2k''$ [if $k'' < 0$ for a mode with a transverse profile $\boldsymbol{v}_k(x,y)$].

Note that solutions of form (4) do not exist for all $\varepsilon(x,y)$ profiles, and if so, then the given profile $\varepsilon(x,y)$ does not support the waveguide propagation of radiation. We will consider only those waveguides, for which there are solutions of form (4) already known to us solutions of a separate problem outside the present study. It is obvious that the change of sign for k in (4) changes the propagation direction of the wave along the z axis to the opposite one. For definiteness we assume that the positive sign for k corresponds to the positive sign of its real part and characterises the wave moving in the positive direction along the z axis.

In what follows, we shall confine ourselves to considering only one, for example, a fundamental, transverse mode with its function $\boldsymbol{v}(x,y)$; therefore, the index k for this function is omitted hereafter. The amplitudes of the electric-field waves travelling in the positive and negative directions along the z axis have the form:

$$\boldsymbol{E}_{\pm} = \boldsymbol{v}_{\pm}(x,y)\exp(\pm\,\mathrm{i}kz),$$
$$\boldsymbol{H}_{\pm} = \frac{1}{\mathrm{i}k_0}\mathrm{rot}\boldsymbol{E}_{\pm} = \boldsymbol{u}_{\pm}(x,y)\exp(\pm\,\mathrm{i}kz). \qquad (5)$$

The components of the vector-functions $\boldsymbol{v}_{-}$ and $\boldsymbol{u}_{-}$ are connected with the components of the functions $\boldsymbol{v}(x,y) \equiv \boldsymbol{v}_{+}(x,y)$ and $\boldsymbol{u}(x,y) \equiv \boldsymbol{u}_{+}(x,y)$ for the positive direction of motion by the relations

$$\boldsymbol{v}_{-}(x,y) = [v_x(x,y);\, v_y(x,y);\, -v_z(x,y)],$$
$$\boldsymbol{u}_{-}(x,y) = [-u_x(x,y);\, -u_y(x,y);\, u_z(x,y)]. \qquad (6)$$

Without loss of generality, we place the point and harmonically oscillating dipole $\boldsymbol{d}$ in the plane $z = 0$ with transverse coordinates x_0 and y_0. For $\boldsymbol{d}$ we have:

$$\boldsymbol{d}(t,x,y,z) = \frac{1}{2}d_0\{\boldsymbol{\xi}\exp(-\mathrm{i}\omega t) + \mathrm{c.c.}\}$$
$$\times\delta(x-x_0)\,\delta(y-y_0)\,\delta(z), \qquad (7)$$

where $\boldsymbol{\xi}$ is the complex vector with a unit modulus. The amplitude $\boldsymbol{\mathcal{E}}_{\mathrm{m}}$ of the wave excited by this dipole can be written in the form:

$$\boldsymbol{\mathcal{E}}_{\mathrm{m}} = \frac{1}{2}\{\boldsymbol{E}_{\mathrm{m}}\exp(-\mathrm{i}\omega t) + \mathrm{c.c.}\},$$
$$\boldsymbol{E}_{\mathrm{m}} = \begin{cases} A_{+}\boldsymbol{v}_{+}(x,y)\exp(\mathrm{i}kz) & \text{for } z > 0, \\ A_{-}\boldsymbol{v}_{-}(x,y)\exp(-\mathrm{i}kz) & \text{for } z < 0, \end{cases} \qquad (8)$$

where $A_{\pm}$ are constant coefficients. Relations (8) represent two waves originating from the $z = 0$ plane in the negative and positive directions. The complex amplitude $\boldsymbol{E}_{\mathrm{m}}$ must satisfy the inhomogeneous equation

$$\mathrm{rotrot}\boldsymbol{E}_{\mathrm{m}} - k_0^2\varepsilon\boldsymbol{E}_{\mathrm{m}} = \frac{4\pi\mathrm{i}k_0}{c}\boldsymbol{j}$$
$$= 4\pi k_0^2 d_0\boldsymbol{\xi}\delta(x-x_0)\,\delta(y-y_0)\,\delta(z). \qquad (9)$$

The right-hand side of (9) contains a factor corresponding to the external current density $\boldsymbol{j}$, which is a time derivative of the dipole moment. Obviously, for $z > 0$ and $z < 0$, expressions (8) are one of the particular solutions of equation (3). At point $z = 0$ the solution has a singular point, which is determined by the presence of δ-functions on the right-hand side of (9). Note that solution (9) in form (8) cannot be considered complete, since the dipole is emitted not only in the waveguide mode, but also in waves radiated into the open space. However, because of the uniqueness of the solution for the accepted boundary conditions (the asymptotics $z \to \pm\infty$), we exclude the fields that represent the near field of the dipole and the free-space waves satisfying equation (9) but not satisfying the boundary conditions.

To find $A_{\pm}$, we use the identity known from vector analysis, which is a vector analogue of Green's formula:

$$\int_V(\boldsymbol{F}\mathrm{rotrot}\boldsymbol{Q} - \boldsymbol{Q}\mathrm{rotrot}\boldsymbol{F})\,\mathrm{d}V$$
$$= \int_S(\boldsymbol{Q}\mathrm{rot}\boldsymbol{F} - \boldsymbol{F}\mathrm{rot}\boldsymbol{Q})\boldsymbol{n}\mathrm{d}S, \qquad (10)$$

where $\boldsymbol{F}$ and $\boldsymbol{Q}$ are regular vector functions for which there exist integrands and integrals over the volume V and the surface S bounding this volume; and $\boldsymbol{n}$ is the outer normal to the surface S. As the function $\boldsymbol{F}$ we use the expression $\boldsymbol{E}_{\mathrm{m}}$ from (8), and as $\boldsymbol{Q}$ we first take the expression $\boldsymbol{E}_{-}$ defined by (5), i.e.,

$$\boldsymbol{Q} = \boldsymbol{E}_{-} = \boldsymbol{v}_{-}(x,y)\exp(-\mathrm{i}kz). \qquad (11)$$

By V we mean the volume of space enclosed between the planes $z = -\Delta z/2$ and $z = \Delta z/2$ (see Fig. 1). By omitting cumbersome but simple calculations of integrals (10), which do not require additional comments, and taking into account equalities (5), (7) and (9), we obtain

$$A_{+} = 2\pi\mathrm{i}k_0 d_0\frac{\boldsymbol{v}_{-}(x_0,y_0)\boldsymbol{\xi}}{S}, \quad S = \int[\boldsymbol{v}\times\boldsymbol{u}]\boldsymbol{e}\mathrm{d}x\mathrm{d}y, \qquad (12)$$

where $\boldsymbol{e}$ is the unit vector along the z axis. In calculating integrals (10), we assume that the integral over the remote part of the surface S ($|x|, |y| \to \infty$) tends to zero for all finite values of Δz. This is the requirement that the value of $|\boldsymbol{v}(x,y)|$ rapidly decreases with increasing $|x|$ and $|y|$, which has already been mentioned as one of the boundary conditions for finding $\boldsymbol{v}(x,y)$.

Then, choosing $\boldsymbol{Q} = \boldsymbol{E}_{+} = \boldsymbol{v}_{+}(x,y)\exp(\mathrm{i}kz)$ and performing similar integration of identity (10), we obtain

$$A_{-} = 2\pi\mathrm{i}k_0 d_0\frac{\boldsymbol{v}_{+}(x_0,y_0)\boldsymbol{\xi}}{S}. \qquad (13)$$

For definiteness, we choose the coordinate system x, y and the normalisation of the functions $\boldsymbol{v}(x,y)$ such that the maximum of $\mathrm{Re}([\boldsymbol{v}(0,0)\times\boldsymbol{u}^*(0,0)]\boldsymbol{e})$ is at the origin and equal to unity. In view of the foregoing, the total power P_{+} emitted by the dipole in the positive direction will be as follows:

$$P_{+} = \frac{c}{8\pi}\mathrm{Re}\Big(\int[\boldsymbol{E}_{+}\times\boldsymbol{H}_{+}^*]\boldsymbol{e}\Big)\mathrm{d}x\mathrm{d}y = \frac{c}{8\pi}|A_{+}|^2 n_{\mathrm{eff}}S_0, \qquad (14)$$

where

$$S_0 = \frac{1}{n_{\mathrm{eff}}}\mathrm{Re}\Big(\int[\boldsymbol{v}\times\boldsymbol{u}^*]\boldsymbol{e}\Big)\mathrm{d}x\mathrm{d}y.$$

Taking into account the normalisation $\boldsymbol{v}(x,y)$, we can understand by S_0 some effective transverse area of the optical beam. In both directions, the total power has the form

$$P = \frac{c}{8\pi}(|A_{+}|^2 + |A_{-}|^2)n_{\mathrm{eff}}S_0. \qquad (15)$$

Now we can find the total power $\Delta P_{\rm m}$ in the mode from the micro-ensemble of ΔN mutually incoherently radiating dipoles located in some neighbourhood of the point with the coordinates x, y, z. Using equations (12)–(14), we obtain

$$\Delta P_{\rm m} = \pi n k_0^2 \frac{S_0 n_{\rm eff}}{|S|^2} d_0^2 \langle |v(x,y)|^2 \rangle \Delta N, \qquad (16)$$

$$\langle |v(x,y)|^2 \rangle = |v_x|^2 \zeta_x + |v_y|^2 \zeta_y + |v_z|^2 \zeta_z. \qquad (17)$$

Here, $\Delta N = N(x,y,z)\Delta x \Delta y \Delta z$; $\zeta_x = \langle |\xi_x|^2 \rangle$; $\zeta_y = \langle |\xi_y|^2 \rangle$; $\zeta_z = \langle |\xi_z|^2 \rangle$; $N(x,y,z)$ is the concentration of dipoles (electrons); and $\langle |v(x,y)|^2 \rangle$ is the square of the modulus of the vector function $\boldsymbol{v}(x,y)$ averaged over the micro-ensemble with the weight vector of the squares of the moduli of the dipole moment projections; the angular brackets in (16), (17) denote averaging over the micro-ensemble. To find $\langle |v(x,y)|^2 \rangle$, one should know the values of ζ_x, ζ_y, ζ_z determined in (17), from which it follows that $\zeta_x + \zeta_y + \zeta_z = 1$. It can be seen that if the modulus of the dipole moment is independent of the orientation direction (an isotropic medium, for example, 'volume active region'), then it is obvious that $\zeta_x = \zeta_y = \zeta_z = 1/3$. In the case of a quantum-well active region, it is necessary in general to introduce for $\zeta_{i=x,y,z}$ some positive definite correction factor γ, i.e., $\zeta = \gamma(1/3)$. It is obvious that the value of γ is bounded and cannot differ much from unity.

Note that, since averaging over the micro-ensembles yields $\langle \xi_z \rangle = 0$, the powers radiated in the positive and negative directions are equal, and this has already been taken into account in (16). Next we represent $N(x,y,z)$ in the factorized form:

$$N(x,y,z) = N(z) f(x,y), \; f(0,0) = 1,$$

$$\int f(x,y)\,{\rm d}x{\rm d}y = S_{\rm e}, \qquad (18)$$

where $S_{\rm e}$ is the effective area occupied by electrons. Passing to the limit $\Delta x, \Delta y \to 0$, we write the power $P_{\rm m}(z)$ of spontaneous emission into the mode from a layer of thickness Δz occupied by electrons, as

$$P_{\rm m} = \frac{\pi \gamma c k_0^2 n_{\rm eff} S_0}{3|S|^2} d_0^2 N(z) \Delta z S_{\rm a}. \qquad (19)$$

Here

$$S_{\rm a} = \frac{3}{\gamma} \int \langle |v(x,y)|^2 \rangle f(x,y)\,{\rm d}x{\rm d}y$$

is the effective area of the active region. Replacing d_0^2 by its value determined by equality (1), we finally obtain:

$$P_{\rm m} = \frac{h\omega_0}{\tau} N(z) S_{\rm e} \Delta z \beta, \quad \beta = \frac{\gamma \pi n_{\rm eff}}{k_0^2 n} \frac{S_0}{|S|^2} \frac{S_{\rm a}}{S_{\rm e}}. \qquad (20)$$

Equation (20) has a transparent physical meaning. The expression for $P_{\rm m}$ consists of the co-factors: $h\omega/\tau$ is the power radiated by a single dipole into the whole space; $N(z)S_{\rm e}\Delta z$ is the number of dipoles in the layer; and β is a dimensionless quantity that can be interpreted as a spontaneous emission factor, which is the average fraction of the power, launched into the waveguide from each dipole, with respect to the entire power radiated by this dipole. Knowing the vector function $\boldsymbol{v}(x,y)$, the $f(x,y)$ value and using equalities (12), (14), (18) and (19), we find the values of S, S_0, $S_{\rm a}$ and $S_{\rm e}$ entering into (20). In particular, it is not difficult to find the explicit form of S and S_0:

$$S = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \frac{k}{k_0} [v_x^2(x,y) + v_y^2(x,y)] + \frac{\rm i}{k_0}\left(v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} \right) \right\} {\rm d}x{\rm d}y, \qquad (21)$$

$$S_0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ |v_x(x,y)|^2 + |v_y(x,y)|^2 - {\rm Re}\left[\frac{\rm i}{k_0}\left(v_x \frac{\partial v_z^*}{\partial x} + v_y \frac{\partial v_z^*}{\partial y} \right) \right] \right\} {\rm d}x{\rm d}y. \qquad (22)$$

It was assumed above that the oscillations of the dipole are strictly harmonic. However, it is clear that if the dynamics of a dipole is expressed by quasi-monochromatic oscillations described by a random time function, then equation (7) can be considered as a definition of the Fourier components of this random function. Then, the dynamics of the dipole is characterised by its correlation function or spectral density $H(\omega)$. It follows that instead of $h\omega_0/\tau$ in (20) it is necessary to use the spectral density $h\omega_0 H(\omega)/(\Delta\omega_{\rm sp}\tau)$ corresponding to this value. In this case, the power $P_{\rm m}$ will correspond to its spectral density

$$P(\omega) = \frac{h\omega_0 H(\omega)}{\tau \Delta\omega_{\rm sp}} N(z) S_{\rm e} \Delta z \beta,, \qquad (23)$$

where $H(\omega)$ is the form factor of the spontaneous emission line; $H(\omega_0) = 1$; $\int H(\omega){\rm d}\omega = \Delta\omega_{\rm sp}$; and $\Delta\omega_{\rm sp}$ is the spectral line width. The function $H(\omega)$ is the result of summing the spectral densities of individual independent elementary radiators in accordance with the sum (16) and the integral (19). Then the spontaneous emission power entering the spectral interval $\delta\omega$ has the form

$$\delta P(\omega) = \frac{h\omega_0 H(\omega)}{\tau \Delta\omega_{\rm sp}} N(z) S_{\rm e} \Delta z \beta \delta\omega. \qquad (24)$$

In a number of cases, it is necessary to know the power $P_{\rm lon}$ of spontaneous emission into one longitudinal mode of a diode laser. Then, as $\delta\omega$ it is necessary to use in (24) the intermode interval $\delta\omega_{\rm m} = \pi c/(L\tilde{n}_{\rm eff})$, where $\tilde{n}_{\rm eff}$ is the effective group refractive index and L is the resonator length. Besides, we should set, for example, $\Delta z = L$ in the case of a uniform distribution of the electron concentration throughout the resonator length [$N(z) = N = {\rm const}$]. As a result, we obtain

$$P_{\rm lon} = \frac{h\omega_0}{\tau} N S_{\rm e} L \beta_{\rm sp}, \quad \beta_{\rm sp} = \beta \frac{\pi c}{\tilde{n}_{\rm eff} L} \frac{H(\omega)}{\Delta\omega_{\rm sp}}. \qquad (25)$$

Expression (20) together with equations (12), (14), (18) and (19) allows us to find β in the most general variants, starting from the known solutions of the waveguide problem [the known vector function $\boldsymbol{v}(x,y)$ with its derivatives], known electronic properties of the active region (known values of ζ_x, ζ_y, ζ_z) and known spatial distribution of the electron density $N(z)f(x,y)$.

4. Comparison of the result obtained with the result of Petermann's work [3]

Let us find a simplified expression β for for one particular case, namely, for a plane dielectric waveguide. Such a waveguide is formed, for example, by a profile $\varepsilon(x,y)$ with a transverse size D along the x axis, which is substantially larger than that along the y axis and is much longer than the wavelength; in this case, $\zeta_x = \zeta_y = \zeta_z = 1/3$. Then

$$\frac{1}{k_0}\left| v_x\frac{\partial v_z^*}{\partial x} + v_y\frac{\partial v_z^*}{\partial y}\right| \approx \frac{|v_{x,y}(x,y)|^2}{k_0 D} \ll |v_{x,y}(x,y)|^2,$$

and the amplitude of the waveguide mode is determined by one scalar function $v(x,y)$. This case corresponds to the physical model adopted in [3]. Then we obtain

$$S_0 \approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v(x,y)|^2\mathrm{d}x\mathrm{d}y,\ \ S \approx n_{\mathrm{eff}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v^2(x,y)\,\mathrm{d}x\mathrm{d}y,$$

$$S_{\mathrm{a}} \approx \int|v(x,y)|^2 f(x,y)\,\mathrm{d}x\mathrm{d}y \qquad (26)$$

and find the factor of spontaneous emission into the longitudinal laser mode

$$\beta_{\mathrm{sp}} \approx \qquad (27)$$

$$\frac{\pi^2 c^3 H(\omega)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v(x,y)|^2\mathrm{d}x\mathrm{d}y\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|v(x,y)|^2 f(x,y)\,\mathrm{d}x\mathrm{d}y}{\Delta\omega_{\mathrm{sp}} n n_{\mathrm{eff}}\tilde{n}_{\mathrm{eff}}\omega^2\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v^2(x,y)\,\mathrm{d}x\mathrm{d}y\right|^2 L\int(x,y)\,\mathrm{d}x\mathrm{d}y}.$$

This expression completely coincides with the analogous expression derived by Petermann ([equality (19) in [3]) if we use identical notations:

$$E_0(x,y) = E_0(0,0)v(x,y),\ i_{\mathrm{sp}} = i_{\mathrm{sp}}(0,0)f(x,y),\ h(\omega) = \frac{H(\omega)}{\Delta\omega_{\mathrm{sp}}}.$$

5. Discussion and conclusions

A comparison of the results obtained in this paper with the results of [3] shows that the validity of formula (25) is based on the dominance of one vector component of the electric (magnetic) field over other vector components of the field. This follows from the necessity of satisfying relations (26), which implies a quantitative condition in the form of the inequality

$$\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v_x^2(x,y)+v_y^2(x,y)\,\mathrm{d}x\mathrm{d}y\right|$$
$$\gg\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{k}\left(v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}\right)\mathrm{d}x\mathrm{d}y\right|, \qquad (28)$$

that is equivalent in turn to the inequality

$$\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v_x^2(x,y)+v_y^2(x,y)\,\mathrm{d}x\mathrm{d}y\right|$$
$$>\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{k}\left(v_z\frac{\partial v_x}{\partial x}+v_z\frac{\partial v_y}{\partial y}\right)\mathrm{d}x\mathrm{d}y\right|. \qquad (29)$$

It is pertinent to note here that the essential difference of the K-factor from unity in [3, 8] is due to a decrease in $|S^2|$ with decreasing width of the region occupied by the field, but in this case the validity of the scalar approximation becomes not fully justified. With the transverse optical beam size comparable with the wavelength, the derivatives of the function $\boldsymbol{v}(x,y)$ increase so much that the values of the right- and left-hand sides of inequalities (28), (29) become commensurable. In any case, for the so-called fast axis, normal to the plane of the structure layers, this is almost always true. If we also take into account a small beam size (comparable to the wavelength) along the other axis, it is difficult to hope for a satisfactory fulfilment of inequalities (28), (29). In addition, it can be seen from (22) that simultaneously with the deterioration in the accuracy of the scalar approximation, the accuracy of S_0 calculation also deteriorates in the calculation of S. These remarks can also be fully attributed to [8]; therefore, the difference in the results of the above papers may lie within the accuracy range due to the approximations adopted.

In view of what was said above, in order to adequately calculate the spontaneous emission factor, it is necessary to know all the components of the vector function $\boldsymbol{v}(x,y)$ and its derivatives appearing in (21) and (22). Nevertheless, there is a version of the waveguide for which the expressions for S and S_0 are simplified. This is a waveguide formed mainly by the refractive index, i.e., in the case when $\varepsilon(x,y)$ is an exclusively real function. It is easy to show that in this case k is also a real quantity, and $\boldsymbol{v}(x,y)$ can be chosen so that $v_x(x,y)$ and $v_y(x,y)$ are real quantities, and $v_z(x,y) = \pm\mathrm{i}|v_z(x,y)|$ is an imaginary quantity. The sign for $v_z(x,y)$ is chosen in accordance with the signs for $v_x(x,y)$ and $v_y(x,y)$. For such values of $\boldsymbol{v}(x,y)$, $S = n_{\mathrm{eff}}S_0$, and then the spontaneous emission factor has the form

$$\beta = \frac{\gamma\pi}{k_0^2 n n_{\mathrm{eff}}}\frac{S_{\mathrm{a}}}{S_0 S_{\mathrm{e}}} \approx \frac{\gamma\pi}{k_0^2 S_0 n n_{\mathrm{eff}}}. \qquad (30)$$

The approximate value of β on the right-hand side of (30) is obtained due to the fact that for the vast majority of laser diodes $S_{\mathrm{e}} \approx S_{\mathrm{a}}$, since the active region, as a rule, is inside the optical beam. Expressions analogous to formula (30) were used in numerous papers earlier, beginning with [2]. The same formula appears in [18], with the only difference that it was used there for unidirectional radiation (therefore, it contains the factor 1/2) and with an optical confinement factor Γ, which is erroneously present in formula (9) of this work. It can be concluded from (30) that the spontaneous emission factor is practically independent of the thickness of the active layer of the laser diode. Note nevertheless that despite the similarity of expression (30) with the expressions for β used previously, we should bear in mind that while the quantity S_0 in (30) has a physical meaning of the cross-sectional area of the optical beam, its quantitative expression is not identical to that used previously. A particularly significant discrepancy is expected for small (commensurate with the wavelength) cross sections of optical beams, and this is the result of taking into account the vector nature of the optical wave field. Therefore, even for a waveguide formed due to the profile of the refractive index, the result of the vector analysis is different from the result obtained in the scalar approximation.

In conclusion, we note that although we have not used the orthogonality condition for the modes of the waveguide explicitly, it is nevertheless used implicitly in integrating equation (8) in combination with the uniqueness property of the solution (see [20]).

Thus, in the present paper, in the framework of a complete vector analysis, an expression for the factor of spontaneous emission into the waveguide mode is obtained in the form of equality (20). This expression is valid for dielectric waveguides, regardless of their parameters. For a particular case of a waveguide with a refractive index profile, a simplified expression for β [see (30)] is found that contains essentially only quantity, S_0, depending on the transverse distribution of the field amplitude. It is shown how the previously used approximate formulas follow from the derived expression and what are the conditions for the validity of the approximations.

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