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Delta-layer model for the boundary of a bubble excited by an electron bunch or laser pulse in a plasma channel

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Abstract. **We present an analytical model of a plasma bubble excited by a relativistic electron bunch or a high-power laser pulse moving in the plasma, with allowance for a surrounding thin electron layer (delta-layer). It is shown that in calculations of accelerating and focusing fields, the electron layer at the bubble boundary, provided the layer thickness is small enough, can be considered as two-dimensional and consisting of surface charges and currents. This model is compared with the bubble model having a finite thickness of the electron layer, and it is shown that the parameter region, within which our model is valid, is small compared to the region where the effect of finite thickness is significant. On the other hand, the proposed model, representing the simplest model of a bubble in a transversely inhomogeneous plasma, allows the transition to a model with a finite thickness of the electron layer by means of changing the scales of coordinates.**

Keywords: *laser wake-field acceleration, strongly nonlinear regime.*

1. Introduction

The main methods of electron acceleration in a plasma wake wave are laser wake-field acceleration (LWFA) [1] and plasma wake-field acceleration (PWFA) [2]. The most effective regime of laser-plasma acceleration is the so-called bubble regime [3]. In this regime, the laser intensity is high enough, which allows all plasma electrons to be expelled out of the first half period of the plasma wave. In using PWFA, the bubble regime is also possible, which is achieved in the case of a dense relativistic electron bunch [4]. In both methods, a wake wave, which is free of electrons and called a bubble, surrounds the driver (a bunch or a laser pulse exciting the wave) moving at a near-light speed through the plasma [5].

A key feature of the regime of plasma bubble acceleration is generation of quasi-monoenergetic electron bunches [6, 7], for which a correct driver configuration is required. In pursuing this goal, the main directions of the theoretical approach have been the development of analytical models [8] and derivation of similarity laws [9, 10], which have been widely tested in 3D modelling using the 'particle-in-cell' (PIC) method [11]. Until recently, only a homogeneous background plasma has been studied, but some recent publications have been dedicated to electron acceleration in a plasma with a hollow plasma channel [8, 12–15]. These papers consider analytical models for the fields inside and outside the bubble in a plasma with an arbitrary radial profile of particle densities. To date, these models predict the following advantages in the use of profiled plasma: 1) independent control over the focusing and accelerating forces [12]; 2) possibility of adjusting the length of laser pulse depletion and dephasing length, which can improve the electron energy increment [13]; 3) addition of new degrees of freedom that enable additional adjustment of the measured values, such as varying both the particle energy in the electron bunch and fraction of trapped particles [13]; 4) compensation for defocusing of a relativistic laser pulse [6, 16, 17]; 5) possibility of injection a pre-accelerated electron bunch that does not need a selection of parameters for matching with a strong focusing field [8]; and 6) possibility of injection into a plasma bubble of an electron bunch with such a density profile that the bunch would only be accelerated in a uniform longitudinal electric field, not experiencing any influence from the focusing force (this allows maintaining small emittance and energy spread in the case that the bunch does not reach the walls [15]).

In some of these works, which are based on the use of the Lu model [18] applied to the case of a plasma channel, first analytical models for the fields inside the bubble in a plasma channel with an arbitrary transverse density profile were developed. These fields depend on the envelope radius of the bubble (the distance between the electron layer and the driver axis), which is determined by a rather complicated ordinary differential equation that depends on the electron layer shape at the bubble boundary. The electron layer shape (rectangular [8, 18] or exponential [19]) defines the bubble-surrounding fields, and this information is required, in particular, when considering the external injection of electron bunches.

In many previous works (see, e.g., [20, 21]) dedicated to the case of a homogeneous plasma, the impact of the electron layer was not taken into account, while the bubble shape was assumed to be perfectly spherical. First models of the plasma bubble regime were obtained using this strong approximation. In a more detailed model for a transversely inhomogeneous plasma, developed by Golovanov et al. [14], it has been established under which conditions the choice of density distributions of currents and charges in an electron layer does not affect the plasma bubble shape. Work [14] also discusses the regions of parameters, within which the plasma bubble

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envelope equation can be represented in a simple form, and it is shown that these regions are similar for the similar parameters in significantly different electron layer profiles.

From the present work it follows that one of the parameter regions is equivalent to the original assumption that the electron layer has a zero thickness, and the current and charge densities in this layer can be represented as two-dimensional distributions of the surface densities. A comparison with the model of a finite-thickness electron layer shows that the parameter region for which our approximation is valid is rather small. However, changing the longitudinal axis scale enables a transition to the model corresponding to another limiting case of a plasma bubble, when electrons in the boundary layer are relativistic. Since the latter model often demonstrates good agreement with the PIC simulation, the use of our model in combination with a change in the scales of coordinates allows us to obtain the simplest model of a plasma bubble in a transversely inhomogeneous plasma.

In the proposed model, the electronic response to a given driver propagating in a plasma along the longitudinal axis z is considered in the quasi-stationary approximation, the use of which is stipulated by fact that all of the processes being of interest to us occur over the times that are much smaller compared to the characteristic time of the driver evolution. As a consequence, we assume that the fields only depend on $\xi = ct - z$ [22]. In our model, coordinates are normalised to the inverse plasma wave number $k_p^{-1} = c/\omega_p$, velocities – to the velocity of light in vacuum c, charges – to the elementary charge $e > 0$, fields – to $E_0 = m_e c\omega_p/e$, and time – to ω_p^{-1}. Here, $\omega_p^2 = 4\pi e^2 n_0/m_e$ is the electron plasma frequency, and n_0 is a certain density used for normalisation of the electron density $n_e(r)$ and ion charge density $\rho_{ion}(r)$. In particular, it is convenient to assume that a deep channel is in-built into the homogeneous plasma. In this case, the unperturbed density of a homogeneous plasma outside the channel can be chosen as n_0.

The following section presents a plasma bubble model with a delta-layer in the pre-formed plasma channel, and then the results obtained are compared with the results of paper [14].

2. Delta-layer model in a plasma channel

In the development of a delta-layer model excited by a bunch of particles in a deep plasma channel, following [8] we separate the regions with a high electron density (driver and accelerated bunch) and the regions inside the bubble, being free from electrons (Fig. 1). Regions I (red) and III (black dots) include the driver and accelerated electron bunch, respectively. The fields in these regions can be used to describe a reverse impact of the plasma bubble boundary on electron bunches and their possible self-modulation. Regions II (pale red) and IV (blue) are located outside the driver and accelerated bunch, respectively. In these regions, the current in these bunches affects both the magnetic field and radial electric field. It should be noted that our theory is also valid in the case when the laser driver is located in the regions where the laser pulse is absent, since the plasma bubble shape in a strongly nonlinear regime does not depend on the type of a driver.

Because the dimensionless longitudinal velocity of relativistic electron bunches is approximately equal to unity, their normalised electron charge density (current) in the plasma bubble can be written in the form

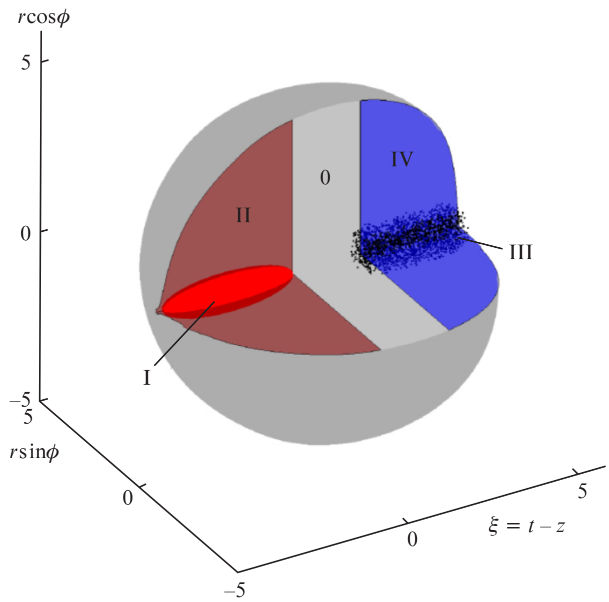

Figure 1. (Colour online) Partition of the plasma bubble into five regions: (0) region in which electron bunches are absent; (I) region inside the electron bunch exciting the bubble (driver) (I); (II) region outside the driver; (III) region (points) inside the accelerated electron bunch; (IV) region outside the accelerated electron bunch.

$$\rho_e = J_e = \begin{cases} 0, \ \xi < \xi_d;\ \xi_d + l_d < \xi < \xi_b;\ \xi > \xi_b + l_b, \\ J_d(\xi, r), \ \xi_d < \xi < \xi_d + l_d,\ r < R_d, \\ 0, \ \xi_d < \xi < \xi_d + l_d,\ r > R_d, \\ J_b(\xi, r), \ \xi_b < \xi < \xi_b + l_b,\ r < R_b, \\ 0, \ \xi_b < \xi < \xi_b + l_b,\ r > R_b, \end{cases} \tag{1}$$

where $\xi_{d,b}$ are the coordinates of leading edges of the driver and accelerated bunch, respectively; and $l_{d,b}$ and $R_{d,b}$ are their lengths and radii. A scheme of a plasma bubble which corresponds to distribution (1) is shown in Fig. 1. We assume that the symmetry of the problem under consideration is cylindrical. If the electron bunch exciting the bubble has a bi-Gaussian distribution with characteristic longitudinal and transverse sizes σ_ξ and σ_r, respectively, the limits correspond to $R_d \approx 3\sigma_r$ and $l_d \approx 3\sigma_\xi$.

Due to their large masses, ions remain motionless on the time scales of interest, while their density $\rho_{ion}(r)$ is homogeneous along the z axis. In the direction perpendicular to the z axis, the density $\rho_{ion}(r)$ only depends on the radial coordinate r (distance to the axis). The distributions of the current density J_z and charge density ρ are separated into three regions: the inner part of the plasma bubble, the boundary layer, and the region outside the bubble. Since these densities and fields vanish outside the bubble, we restrict ourselves to the case $r \leqslant r_b$, so that

$$S(\xi, r) = J_z - \rho = s_{ion}(r) + s_0(\xi)\delta(r - r_b). \tag{2}$$

This source inside the electron layer only depends on ξ, whilst inside the plasma cavity ($r < r_b$) $S = S(r) = s_{ion}(r) = -\rho_{ion}(r)$.

The continuity equation formulated in a cylindrical geometry in the quasi-stationary approximation appears as

$$\frac{r\partial(\rho - J_z)}{\partial\xi} + \frac{\partial(rJ_r)}{\partial r} = 0$$

and leads to a relationship between the sources located on the plasma bubble boundary and inside it:

$$s_0(\xi) = \frac{-c_0 - S_{\mathrm{I}}(r_{\mathrm{b}}(\xi))}{r_{\mathrm{b}}(\xi)}, \tag{3}$$

where c_0 is an arbitrary constant. Here we introduce the integral source $S_{\mathrm{I}}(r) = \int_0^r s_{\mathrm{ion}}(r')r'\mathrm{d}r'$ and observe the first difference from works [8, 14], where it was necessary to determine a relative thickness of the plasma bubble boundary.

If the wake potential is expressed in terms of the vector potential A and the wake potential $\Psi = \varphi - A_z$, the Lorenz gauge $\partial(rA_r)/\partial r = -r\partial\Psi/\partial\xi$ allows us to obtain the normalised Poisson equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = -J_z, \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) = -\rho + J_z. \tag{4}$$

From the second equation, we derive the most general wake potential:

$$\Psi(\xi, r) = \int_0^r \frac{\mathrm{d}y}{y}\int_0^y xS(\xi, x)\mathrm{d}x + \Psi_0(\xi) = I(\xi, r) + \Psi_0(\xi). \tag{5}$$

To determine I and Ψ_0 in each plasma bubble region, we start with the simplest one, namely the neutral plasma region. In that region $\Psi = 0$ and $r > r_{\mathrm{b}}$, i.e. $\Psi_0 = -I$. Since I is the radial integral over all internal regions, then $I = I_1 + I_2$, where

$$I_1(\xi, r) = \lim_{\varepsilon\to 0}\int_0^{r_{\mathrm{b}}-\varepsilon}\frac{\mathrm{d}y}{y}\int_0^y xS(\xi, x)\mathrm{d}x, \tag{6}$$

$$I_2(\xi, r) = \lim_{\varepsilon\to 0}\int_{r_{\mathrm{b}}-\varepsilon}^{r}\frac{\mathrm{d}y}{y}\int_0^y xS(\xi, x)\mathrm{d}x. \tag{7}$$

It follows from Eqns (2) and (3) that

$$I_1 = \int_0^{r_b}\frac{S_{\mathrm{I}}(y)}{y}\mathrm{d}y, \quad I_2(\xi, r) = -c_0\int_{r_b}^{r}\frac{\mathrm{d}y}{y} = 0, \tag{8}$$

then,

$$\Psi_0(\xi) = -I_1 = \int_0^{r_b}\frac{S_{\mathrm{I}}(y)}{y}\mathrm{d}y, \tag{9}$$

Inside the plasma bubble, $I = \int_0^r S_{\mathrm{I}}(\xi, y)y^{-1}\mathrm{d}y$. Summarising, we have

$$\Psi(\xi, r) = -\int_r^{r_b}\frac{S_{\mathrm{I}}(y)}{y}\mathrm{d}y \tag{10}$$

for $r \leqslant r_{\mathrm{b}}$ and $\Psi(\xi, r) = 0$ for $r > r_{\mathrm{b}}$. If we compare expression (10) for the wake potential with that found in [8], we can see that they are virtually identical, except that the function β in [8], which depends on the electron layer thickness, is absent in Eqn (10). Since the radial component of the vector potential in our theory

$$A_{\mathrm{r}}(\xi, r) = r\sigma(\xi) = -\frac{r}{2}\frac{\mathrm{d}\Psi_0(\xi)}{\mathrm{d}\xi} \tag{11}$$

is also a derivative of Ψ_0, the calculations are greatly simplified. Next,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\xi} = \frac{S_{\mathrm{I}}(r_{\mathrm{b}})}{2r_{\mathrm{b}}}r_{\mathrm{b}}'' + \frac{s_{\mathrm{ion}}(r_{\mathrm{b}})}{2}(r_{\mathrm{b}}')^2 - \frac{S_{\mathrm{I}}(r_{\mathrm{b}})}{2r_{\mathrm{b}}^2}(r_{\mathrm{b}}')^2, \tag{12}$$

which bring us to the differential equation for the plasma bubble envelope:

$$A(r_{\mathrm{b}})r_{\mathrm{b}}'' + B(r_{\mathrm{b}})(r_{\mathrm{b}}')^2 + C(r_{\mathrm{b}}) = \frac{\Lambda(\xi)}{r_{\mathrm{b}}}. \tag{13}$$

Here $r_{\mathrm{b}}'(\xi) = \mathrm{d}r_{\mathrm{b}}(\xi)/\mathrm{d}\xi$; $r_{\mathrm{b}}''(\xi) = \mathrm{d}^2 r_{\mathrm{b}}(\xi)/\mathrm{d}\xi^2$;

$$A = 1 - \frac{S_{\mathrm{I}}}{2}; \quad B = -\frac{s_{\mathrm{ion}}(r_{\mathrm{b}})r_{\mathrm{b}}}{2}; \quad C = -\frac{S_{\mathrm{I}}}{r_{\mathrm{b}}}; \tag{14}$$

$$\Lambda(\xi) = -\int_0^{r_{\mathrm{b}}} J_z(\xi, r')r'\mathrm{d}r' \tag{15}$$

is the integral of the longitudinal current density formed by electron bunches.

A more general theory by Golovanov et al. [14] considers a simplified differential equation for a finite-size electron layer with a source $S(r > r_{\mathrm{b}}) = s_0(\xi)g[(r - r_{\mathrm{b}})/\Delta]$. It is shown that in the general case, two approximations exist, in the frame of which the solution to this differential equation does not depend on the thickness Δ and profile g of the electron layer. The fields of applicability of these approximations turn out similar for different profiles. It can be shown that when use is made of a rectangular profile of the electron layer: $g(x) = \theta(1 - x)$, where $\theta(x)$ is the Heaviside step function, the differences between the exact (numerically obtained) values of the coefficients A, B, C for small Δ and their approximate values in the infinitely thin layer approximation are small (Fig. 2).

In general, Golovanov et al. [14] single out two classes of thin layer approximations: the infinitely thin layer approximation and the relativistic approximation. In the first case $\Delta \to 0$, so that the relation $\Delta \ll |r_{\mathrm{b}}/(S_{\mathrm{I}}(r_{\mathrm{b}})M_1(0))|$ is valid, where

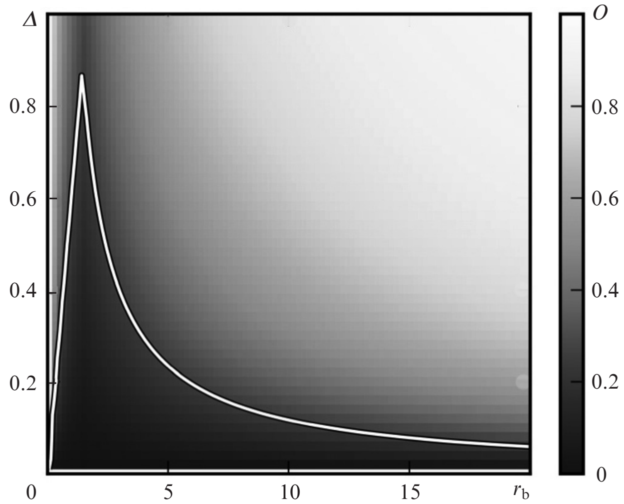

Figure 2. Maximum relative difference O between the coefficients A, B, and C calculated in the infinitely thin layer approximation and their exact values obtained numerically, in the space of parameters r_{b} and Δ for an electron layer of rectangular profile and in the case of a homogeneous plasma. The solid line corresponds to the level of 0.25.

$$M_1(x) = \int_x^\infty y g(y)\,\mathrm{d}y. \tag{16}$$

In the second case, the relation $\Delta \gg |r_b/(S_I(r_b)M_1(0))|$ holds true, which physically corresponds to the fact that electrons in the bubble boundary layer are relativistic (this is why this approximation is called relativistic).

If we compare the expressions derived in this paper for coefficients (14) with those obtained in [14], we can see that our assumption of two-dimensionality of the electron layer corresponds to the first approximation (infinitely thin layer approximation). Our assumption is also equivalent to the choice $g(x) = \delta(x)$ for which $M_1(0) = 0$, and therefore the criterion $\Delta \ll |r_b/(S_I(r_b)M_1(0))|$ is always fulfilled.

The relativistic approximation $\Delta \gg |r_b/(S_I(r_b)M_1(0))|$, $\Delta < r_b$ in which the coefficients $A = -S_I/2$ and $C = -S_I/(2r_b)$, cannot be obtained in the framework of our consideration. Thus, the applicability area of the delta-layer approximation is limited to the parameter region marked by dark colour in Fig. 2. However, since for any given value of Δ we can find a sufficiently large size of the plasma bubble r_b, for which the relativistic approximation is satisfied, this approximation is often in good agreement with the results of PIC simulations. Therefore, it makes sense to find a way to transform Eqn (13) with coefficients (14) into the equation in the relativistic approximation, for which $C = -S_I/(2r_b)$.

Such a transformation is possible by changing the scale of the longitudinal coordinate $\xi \to \tilde{\xi}/\sqrt{2}$ and substitutions $r_b(\xi) \to r_b(\tilde{\xi})$ and $\Lambda(\xi) \to 2\tilde{\Lambda}(\tilde{\xi})$. Then $r_b'(\xi) \to \sqrt{2}\,r_b'(\tilde{\xi})$, $r''(\xi) \to 2r_b''(\tilde{\xi})$, and Eqn (13) for electron layer takes the form

$$\left[1 - \frac{S_I(\tilde{\xi})}{2}\right] r_b''(\tilde{\xi}) - \frac{s_{\mathrm{ion}}(\tilde{\xi}) r_b(\tilde{\xi})}{2}(r_b'(\tilde{\xi}))^2 - \frac{S_I(\tilde{\xi})}{2r_b(\tilde{\xi})} = \frac{\tilde{\Lambda}(\tilde{\xi})}{r_b(\tilde{\xi})}, \tag{17}$$

which coincides with the result for the relativistic approximation in the case $|S_I| \gg 1$.

3. Conclusions

It is shown that the assumption about the existence of an infinitely thin two-dimensional electron layer on the plasma bubble boundary corresponds to the nonrelativistic limit of the model with a finite-size electron layer. Since a large part of the PIC simulation results corresponds to the relativistic approximation for a finite-size electron layer [14], our approach allows a transition to a more appropriate model by changing the ξ-axis scale. The quantitative differences between the results obtained using our model and the model with a finite-size layer show that the parameter region, for which our model is valid, is small. At the same time, it is the simplest model of a plasma bubble in a transversely inhomogeneous plasma.

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