

METAMATERIALS

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Parametric interaction of optical waves in metamaterials under low-frequency pumping

R.J. Kasumova, Sh.Sh. Amirov, Sh.A. Shamilova

Abstract. **The influence of phase effects under three-wave parametric interaction and low-frequency pumping in metamaterials is studied in the case of a negative refractive index at a signal-wave frequency. It is found that the efficiency of the backward signal-wave amplification is the higher, the greater the ratio of the intensities of the idler and signal waves at the input to the metamaterial. An increase in the idler wave intensity at the input by five times, as compared to the signal-wave intensity, leads to a nonlinear increase in the signal-wave amplification by almost 20 times. According to the analytic expressions obtained in the constant-intensity approximation, the choice of the optimal parameters for the pump intensity, total length of the metamaterial and phase detuning will facilitate the implementation of regimes of effective amplification and generation of the signal wave. A comparison is made with the results obtained in the constant-field approximation, and a numerical estimate of the expected efficiency of the frequency conversion is presented. Control of frequency and pump power is shown to make possible the smooth tuning of the parametric converter frequency. The developed method can be used to design frequency converters based on nonlinear metamaterials.**

Keywords: *parametric interaction, phase effects, metamaterial, negative refraction.*

1. Introduction

The progress in photonics is associated, in particular, with the development of metamaterials. In optical systems of information processing, photons are used as information carriers, which, however, causes a problem of their control. The discovery of metamaterials contributed to the emergence of the possibility of controlling light radiation by changing the optical properties of such artificial structures. As is known, a metamaterial can be made of a composite material forming a dielectric matrix, with inclusions providing its resonance properties. A similar approach was applied in the development of solid-state lasers, when activator ions were embedded into a matrix, for example, a crystal, which ultimately determined the physical properties of the laser medium. In such an inhomogeneous medium, the metamaterial is characterised by abrupt changes in material parameters of a medium [dielectric constant (ε) and magnetic permeability (μ)] and a negative value of the refractive index in a certain frequency range (negative refraction). As a consequence, in such a medium, an electromagnetic wave in this frequency range propagates with multidirectional phase and group velocities, and the incident electromagnetic wave undergoes unusual changes, which leads to new unconventional effects.

Resonant interactions, existence of backward waves and negative refraction were considered in the works of H. Lamb [1], L.I. Mandelstam [2], D.V. Sivukhin [3], V.N. Agranovich and V.L. Ginzburg [4], V.G. Veselago [5] and A.N. Lagarkov et al. [6].

The main problem in the development of metamaterials has been and remains the presence of large losses. In a metamaterial, the main effects appear near the frequency resonance, where ε and μ are simultaneously negative. In classical electrodynamics, according to the Kramers–Kronig dispersion relations, which determine the dependence of the behavior of the optical constant of a medium – the real (refractive index) and imaginary (absorption coefficient) parts of the dielectric constant – on frequency, the absorption coefficient at the resonant frequency increases sharply, which leads to significant energy losses of an electromagnetic wave. Thus, metamaterials are characterised by inevitable losses that weaken the electromagnetic wave. Metamaterials containing metal structural elements (wires, rings) absorb light energy and dissipate it in the form of heat.

At present, intensive research is being carried out to compensate for losses in metamaterials. It is proposed to introduce into the metamaterial a semiconductor, which will amplify a signal attenuated due to losses. Smalley et al. [7] have developed a metamaterial, which absorbs or re-emits light in the IR range of the spectrum, depending on the polarisation of incident radiation. In addition, studies of the electromagnetic field of localised plasmons accumulating large electric fields [8] are underway, which allows losses in such structures to be reduced [9].

The main features of the second harmonic generation in metamaterials in the interaction of the forward and backward waves were presented in [10], the preprint of which [11] preceded papers [12–14]. Important results of studies of this process in metamaterials were presented in papers [15–18], and the third harmonic generation was considered in [19–22], where the role of phase detuning was emphasised.

In the case of generation of an optical harmonic in a metamaterial, the intensity maximum of the harmonic is achieved at the input to a nonlinear medium rather than at its output. In this case, the metamaterial plays the role of a nonlinear mirror, as in the case of degenerate four-wave interaction

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when observing the phase conjugation effect of laser radiation [23–25]. The geometry of consecutive opposed interaction of nonlinear optical waves, observed in metamaterials due to negative refraction, was also considered by Volkov and Chirkin [26] for parametric interaction in conventional media. Slabko et al. [27] studied parametric interaction in a metamaterial for a practically important case of pulsed pumping. Comparing the results of the study by numerical simulation methods for two wave geometries, namely, in the case of counterpropagation and co-propagation of optical waves, Slabko et al. [27] observed for the first time and explained the quantitative differences of the transient processes in these two cases.

To date, thanks to the improvement of the technology of manufacturing metamaterials, their development is being conducted not only for the radio frequency range, but also for the region of shorter wavelengths. The authors of Refs [28, 29] reported the results of the developments for the near-IR and visible ranges of the spectrum.

In the constant-field approximation, the nonlinear optical interaction in such artificial structures has been theoretically studied in a number of papers, of which we note [9–15, 18, 21, 22, 30–35]. Using the constant-intensity approximation [36, 37], we have investigated the second and third harmonic generation, as well as self-action effects in the metamaterial [38].

The study of phase effects under three-wave parametric interaction and low-frequency pumping in metamaterials is the goal of this paper.

2. Theory

We assume here for definiteness that in the case of parametric three-wave interaction in a metamaterial, the medium is 'left-handed' only at a signal-wave frequency ω_1. We shall investigate the case of low-frequency pumping at a frequency ω_3, when $\omega_1 = \omega_2 + \omega_3$ (ω_2 is the frequency of the idler wave). When choosing a metamaterial, one should take into account that one of the basic conditions for an effective frequency conversion in the case of three-wave interaction is a high quadratic nonlinearity of the material at working frequencies.

In the case of negative values of the dielectric constant and magnetic permeability at the signal-wave frequency ω_1 and their positive values at the frequencies ω_2, ω_3, the truncated equations for the pump waves, as well as the signal and idler waves, are transformed [39] to the form

$$\frac{dA_1}{dz} + \delta_1 A_1 = i\gamma_1 A_2 A_3 e^{-i\Delta z},$$

$$\frac{dA_2}{dz} + \delta_2 A_2 = -i\gamma_2 A_1 A_3^* e^{i\Delta z}, \tag{1}$$

$$\frac{dA_3}{dz} + \delta_3 A_3 = -i\gamma_3 A_1 A_2^* e^{i\Delta z}.$$

Here, $A_{1,2,3}$ are the corresponding complex amplitudes of the interacting waves; δ_j are the absorption coefficients of the medium at frequencies ω_j ($j = 1 - 3$);

$$\gamma_1 = -\frac{8\pi\chi_{\rm eff}^{(2)}\omega_1^2\varepsilon_1}{k_1 c^2} = \frac{8\pi\chi_{\rm eff}^{(2)}\omega_1^2|\varepsilon_1|}{k_1 c^2},$$

$$\gamma_2 = \frac{8\pi\chi_{\rm eff}^{(2)}\omega_2^2\varepsilon_2}{k_2 c^2}, \quad \gamma_3 = \frac{8\pi\chi_{\rm eff}^{(2)}\omega_3^2\varepsilon_3}{k_3 c^2}$$

are the coefficients of nonlinear coupling of waves at the corresponding frequencies; $\chi_{\rm eff}^{(2)}$ is the effective quadratic susceptibility of the medium; $\Delta = k_1 - k_2 - k_3$ is the phase detuning between the interacting waves; and $\varepsilon_{1,2,3}$ are the dielectric constants at frequencies ω_1, ω_2, ω_3.

We assume that the energy fluxes $S_{2,3}$ of the idler wave and the pump waves fall along the normal to the left-hand side surface of the metamaterial of length l and propagate in the positive direction of the z axis. Consequently, the energy transfer of the signal wave, for which the medium is 'left-handed', occurs in the opposite direction. The boundary conditions corresponding to this geometry of the waves under study can be represented in the form

$$A_{2,3}(z=0) = A_{20,30}\exp(i\varphi_{20,30}),\ A_1(z=l) = A_{1l}\exp(i\varphi_{1l}). \tag{2}$$

Here, $z = 0$ corresponds to the input to the metamaterial on the left; $A_{20,30}$, $\varphi_{20,30}$ are the initial amplitudes and phases of the idler wave and the pump wave at the input to the nonlinear medium on the left; A_{1l}, φ_{1l} are the initial amplitude and phase of the signal wave at the input to the nonlinear medium on the right ($z = l$).

It is known that under such boundary conditions, the wave vectors k_j of all three interacting waves in a metamaterial propagate in the positive direction of the z axis.

Thus, the five wave vectors considered (i.e., three wave vectors $k_{1,2,3}$ and two Poynting vectors $S_{2,3}$) are opposite in direction to the signal-wave vector S_1, which is the backward wave with respect to other two interacting waves.

We use the standard method for solving system (1) with respect to the signal-wave amplitude, by applying the constant-intensity approximation. Within the boundary conditions (2), we obtain an expression for the complex amplitude $A_1(z)$ of the backward wave as it propagates from right to left in a nonlinear medium ($\delta_j = 0$):

$$A_1(z) = \exp(-i\Delta z/2)$$

$$\times\left\{C\cos\lambda z + \left[\frac{i\gamma_1 A_{20}A_{30}}{\lambda}\exp[i(\varphi_{20}+\varphi_{30})] - iC\frac{\Delta}{2\lambda}\right]\sin\lambda z\right\}, \tag{3}$$

where

$$C = \frac{A_{1l}\exp[i(\varphi_{1l}+\Delta l/2)] - i(\gamma_1/\lambda)A_{20}A_{30}\exp[i(\varphi_{20}+\varphi_{30})]\sin\lambda l}{\cos\lambda l + i(\Delta/2\lambda)\sin\lambda l};$$

$$\lambda = \sqrt{\frac{\Delta^2}{4} - \Gamma_3^2 - \Gamma_2^2};\ \Gamma_3^2 = \gamma_1\gamma_2 I_{30};\ \Gamma_2^2 = \gamma_1\gamma_3 I_{20};\ I_j = A_j A_j^*.$$

The obtained analytical expression (3) makes it possible to analyse the process of nonlinear interaction in a metamaterial in the general case when all three waves are present at the input as well as to the right and to the left of the medium.

As can be seen from the expression obtained, since $\Gamma_3 > \Gamma_2$, in the case of low-frequency pumping under consideration, the parameter $\lambda_{\rm low}^{\rm CIA} = \sqrt{\Delta^2/4 - \Gamma_3^2 - \Gamma_2^2}$ in the constant-intensity approximation (for comparison: in the constant-field approximation, $\lambda_{\rm low}^{\rm CFA} = \sqrt{\Delta^2/4 - \Gamma_3^2}$). In this connection, there is a minimum admissible phase detuning of the interacting waves, when the radicand is positive and, as a consequence, the parameter $\lambda_{\rm low}^{\rm CIA}$ becomes real:

$$\Delta^{\rm CIA} \geqslant 2\sqrt{\Gamma_3^2 + \Gamma_2^2}. \tag{4}$$

In the case of high-frequency pumping, this expression in the constant-intensity approximation has the form $\lambda_{\rm high}^{\rm CIA} = \sqrt{\Delta^2/4 + \Gamma_3^2 - \Gamma_2^2}$ (in the constant-field approximation, $\lambda_{\rm high}^{\rm CFA} = \sqrt{\Delta^2/4 + \Gamma_3^2}$) and there is no such restriction, since $\Gamma_3 > \Gamma_2$ by definition. As a consequence, under high-frequency pumping, the radicand remains positive for any values of the phase detuning up to zero.

At a small phase detuning, when $\Delta^{\rm CIA} < 2\sqrt{\Gamma_3^2 + \Gamma_2^2}$, the parameter $\lambda_{\rm low}^{\rm CIA}$ becomes complex; as a result, we need to go over to the hyperbolic cosine and sine functions in (3).

As can be seen from (4), for low-frequency pumping it is necessary to work away from the phase-matching condition so that the condition of the positive radicand is satisfied. It should be noted that the regime of strong phase detunings requires, in this case, considerable amplification at the frequency of the wave in question. This minimum allowable value of phase detuning [see (4)] depends both on the input value of the pump intensity I_{30} and on the idler wave intensity I_{20} through the parameters $\Gamma_{2,3}$. By changing the pump intensity, it is possible to vary the value of the minimum phase detuning, and, consequently, the range of frequency tuning of the idler wave. As is known, the generation frequency can also be tuned by changing the pump frequency.

In the constant-intensity approximation, the input intensity of the idler wave increases with increasing minimum permissible phase detuning. If at $I_{20}/I_{30} = 0.1$ and $\Gamma_3 = 1$ cm^{-1}, according to formula (4), the minimum $\Delta^{\rm CIA} = 2\Gamma_3\sqrt{1 + I_{20}/I_{30}} = 2.097768$, then at $I_{20}/I_{30} = 0.5$ ($\Gamma_3 = 1$ cm^{-1}) the corresponding phase detuning is already 2.44949. The growth of the minimum phase detuning is explained by the nonzero parameter γ_3 entering into the radicand through Γ_2 and taking into account the inverse action of the pump wave on the signal wave. In the constant-field approximation, $\gamma_3 = 0$; therefore, the minimum phase detuning of $\Delta^{\rm CFA}$ is constant, does not depend on the input intensity of the idler wave (see the expression for $\lambda_{\rm low}^{\rm CFA}$) and is equal to 2.0.

The analysis of the interaction dynamics in a 'left-handed' medium shows the following. The main difference between the behaviour of a signal wave and its behaviour in an ordinary quadratic nonlinear medium is due to the counterpropagating direction of the energy transfer velocity of the signal wave relative to its phase velocity. From (3) follows the dependence of the signal wave field (through the boundary conditions) on the total length l of the metamaterial. Due to wave propagation in a nonlinear medium, the nonlinear interaction results in an energy exchange between the counterpropagating wave packets of two types of waves: direct (pump and idler waves) and backward (signal wave) waves. As a result, the energy of the pump wave and the idler wave is transferred to the energy of the signal wave. The efficiency of this process depends on the phase relationship between the interacting waves.

We introduce two important parameters that determine the dynamics of the amplification process and conversion efficiency in a metamaterial under low-frequency pumping. This is the efficiency of frequency conversion of a signal wave in a medium of length z,

$$\eta_1 = \frac{I_1(z)}{I_{20}},$$

and the amplification factor of the signal wave (at $A_{20} = 0$)

$$\eta_{\rm ampl} = \frac{I_1^{\rm output}(z=0)}{I_1^{\rm input}(z=l)} = \frac{I_1(z=0)}{I_{1l}}.$$

First, we determine the conversion efficiency in the case when a signal at a high frequency on the right is not fed to the metamaterial, i.e., at $A_{1l} = 0$. Then the complex amplitude of the signal wave is determined by formula (3), and the parameter C is transformed to the form

$$C' = -\frac{{\rm i}(\gamma_1/\lambda)A_{20}A_{30}\exp[{\rm i}(\varphi_{20}+\varphi_{30})]\sin\lambda l}{\cos\lambda l + {\rm i}(\Delta/2\lambda)\sin\lambda l}.$$

For this case, we find such a practically important parameter as the efficiency of the signal wave at the output from the metamaterial on the left, $\eta_1 = I_1(z=0)/I_{20}$, which has the meaning of the reflection coefficient of the mirror, whose role is played by the metamaterial:

$$\eta_1(z=0) = \frac{\Gamma_3^2\sin^2\lambda l}{\lambda^2\cos^2\lambda l + (\Delta^2/4)\sin^2\lambda l}. \quad (5)$$

Expression (5) is analogous to the corresponding expression for the case of high-frequency pumping. The only difference is in the value of the parameter λ, which essentially distinguishes the dynamics of the processes with low-frequency ($\lambda_{\rm low}^{\rm CIA}$) and high-frequency ($\lambda_{\rm high}^{\rm CIA}$) pumping. If, at high-frequency pumping, the process is analogous to the generation of a backward wave at a difference frequency, then at low-frequency pumping it coincides with the generation of the backward wave at the sum frequency. It follows from (5) that there are optimal values of both the intensity of the fundamental radiation and the phase detuning at which the conversion efficiency at the output from the metamaterial is maximal.

In the absence of the idler wave at the input to the metamaterial on the left, i.e., at $A_{20} = 0$, we find the second practically important parameter – the amplification factor of the signal wave at the output of the metamaterial, which has the meaning of transmittance. From (3) for the amplification factor of the signal wave at the output from the metamaterial, we obtain

$$\eta_{\rm ampl} = \frac{1}{\cos^2\lambda l + (\Delta/2\lambda)^2\sin^2\lambda l}. \quad (6)$$

From (6), we can obtain an equation whose numerical solution will allow one to determine the optimal value of the phase detuning for given pump intensities and idler waves:

$$\tan\lambda l = \lambda l\left[1 - \left(\frac{\Delta}{2\lambda}\right)^2\right]. \quad (7)$$

Let us now consider the general case when all three waves are present at the input to the metamaterial on the right and left. Then from (3) for $\eta'_{\rm ampl} = I_1^{\rm output}(z)/I_1^{\rm input}(z=l)$ we obtain (at $\varphi_{1l,20,30} = 0$)

$$\eta'_{\rm ampl}(z) = \frac{[\cos(\Delta l/2)\cos\lambda z - (\Delta/2\lambda)\sin(\Delta l/2)\sin\lambda z]^2}{\cos^2\lambda l + (\Delta/2\lambda)^2\sin^2\lambda l}$$

$$+ \frac{[\sin(\Delta l/2)\cos\lambda z + (\Delta/2\lambda)\cos(\Delta l/2)\sin\lambda z - a\sin\lambda(l-z)]^2}{\cos^2\lambda l + (\Delta/2\lambda)^2\sin^2\lambda l}, \quad (8)$$

where

$$a = \frac{\gamma_1}{\lambda}\frac{A_{20}A_{30}}{A_{1l}}.$$

3. Discussion of the results

Below we present the results of a numerical analysis of the obtained analytical expressions. First, let us analyse, according to (5), the efficiency of the energy conversion of two direct waves – the pump and idler waves – into the energy of the backward signal wave.

Figure 1 shows the behaviour of η_1 as a function of the total length of the metamaterial for different phase detunings and intensities of the idler wave at the input to the metamaterial in the constant-intensity approximation. Here, the dotted curves illustrate the results of the calculation in the constant-field approximation. The difference in the behaviour of the dependences in both approximations is explained by the nonzero parameter γ_3 in the constant-intensity approximation, taking into account the inverse action of the pump wave on the signal wave.

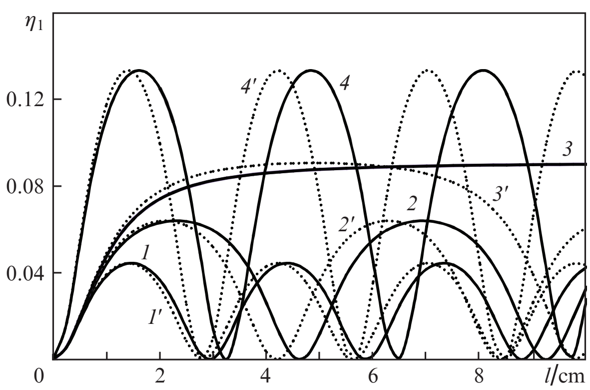

Figure 1. Dependences of the conversion efficiency into the signal wave $\eta_1 = I_1(z)/I_{20}$ on the total length l of the metamaterial, obtained in the constant-intensity approximation for $I_{1l} = 0$, $\Gamma_3 = 1$ cm^{-1} at Δ = (*1*, *4*) 3, (*2*) 2.5 and (*3*) 2.097688 cm^{-1}, and I_{20}/I_{30} = (*1*–*3*) 0.1 and (*4*) 0.3. The corresponding dependences obtained in the constant-field approximation (*1′*–*4′*) are also shown.

In the case of nonzero $\lambda_{\text{low}}^{\text{CIA}}$ values [see (4)], one can observe oscillations of the conversion efficiency into the signal wave [curves (*1*), (*2*) and (*4*)]. For each phase detuning value, there is an optimal value of the total length of the metamaterial, at which the conversion efficiency is maximal. In this case, the optimal value of the total length of the metamaterial plays the role of the so-called coherent total length of a nonlinear medium, a concept that in a typical medium is characteristic of the current length of the material. As the phase detuning increases, the amplitude of the oscillations decreases, and their frequency increases [curves (*1*) and (*2*)].

According to (4), at $I_{20}/I_{30} = 0.1$, the expression $\lambda_{\text{low}}^{\text{CIA}} = \sqrt{\Delta^2/4 - \Gamma_3^2 - \Gamma_2^2} = 0$ for $\Delta = 2.097768$ cm^{-1}, and the dependence $\eta(l)$ becomes horizontal [curve (*3*)]. A similar behaviour of the dependence obtained in the constant-intensity approximation agrees with the result of a rigorous analysis of the conversion efficiency described by the elliptic Jacobi function, which in the case of the zero value of the radicand is equal to the hyperbolic tangent [37, 40].

With an increase in the input intensity of the idler wave, one can observe an increase in the conversion efficiency [curves (*1*) and (*4*)] and a shift of the maxima and minima of the oscillations for solid curves obtained in the constant-intensity approximation at $\Gamma_2 \neq 0$. The distance between two neighbouring minima, i.e., the oscillation period, can easily be determined from (3). In the constant-field approximation, when the parameter $\Gamma_2 = 0$, the shift is absent and the maxima and minima of the dotted curves (*1′*) and (*4′*) coincide. When the input intensity of the idler wave is increased by three times, the conversion efficiency also increases almost threefold, from 4.4% to 13%.

Figure 2 shows the dynamics of the amplification process of a signal wave propagating in the metamaterial, $\eta_{\text{ampl}} = I_1^{\text{output}}(z)/I_1^{\text{input}}(z = l)$, in the case of four phase detunings, two of which are the solutions of equation (7): $\Delta_{\text{opt}\,1} = 5.9484$ cm^{-1} [curve (*2*)], $\Delta_{\text{opt}\,2} = 12.42$ cm^{-1} [curve (*4*)] at $\Gamma_3 = 1$ cm^{-1} and l = 1 cm. The comparison of curves shows that at optimal values of the phase detuning, parametric amplification of the backward wave, i.e. $\eta_{\text{ampl}} > 1$, is observed.

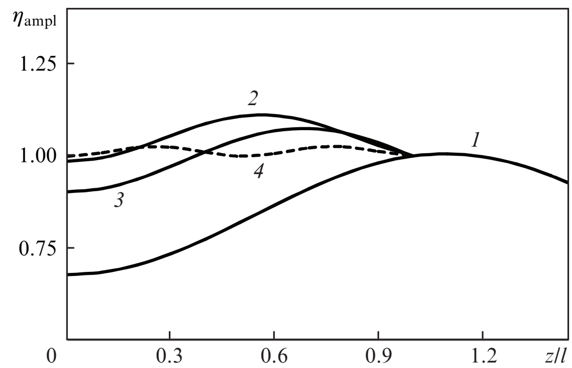

Figure 2. Dependences of the amplification factor $\eta_{\text{ampl}} = I_1(z)/I_{1l}$ on the reduced length of the metamaterial for $I_{20} = 0$, $l = 1$ cm, $\Gamma_3 = 1$ cm^{-1} at Δ = (*1*) 3.5, (*3*) 5 cm^{-1}, (*2*) $\Delta_{\text{opt1}} = 5.9484$ cm^{-1} and (*4*) $\Delta_{\text{opt2}} = 12.42$ cm^{-1}.

Figure 3 shows the dependence of the amplification factor of the signal wave at the output of the metamaterial, $\eta_{\text{ampl}} = I_1^{\text{output}}(z = 0)/I_1^{\text{input}}(z = l)$, on the total length of the metamaterial, l, in the absence of the idler wave at the input. There are two groups of dependences. When the phase detuning between the interacting waves is greater than the minimum allowable value $\Delta^{\text{CIA}} \geqslant 2\sqrt{\Gamma_3^2 + \Gamma_2^2}$, the curves oscillate [curves (*3*–*5*)]. Otherwise, the dependences are determined by the behaviour of the hyperbolic sine and cosine functions, and there are no

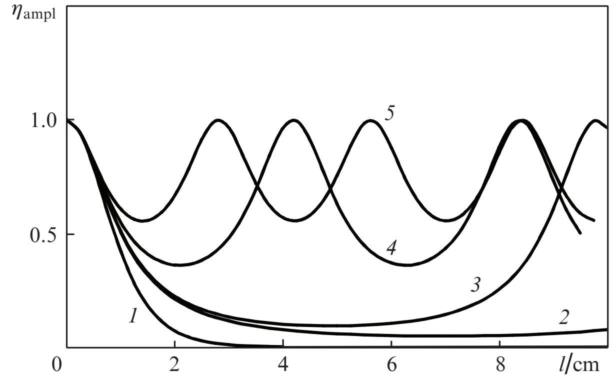

Figure 3. Dependences of the amplification factor $\eta_{\text{ampl}} = I_1(z)/I_{1l}$ on the total length l of the metamaterial for $I_{20} = 0$ and $\Gamma_3 = 1$ cm^{-1} at Δ = (*1*) 0, (*2*) 2.05, (*3*) 2.1, (*4*) 2.5 and (*5*) 3 cm^{-1}.

oscillations [curves (*1*) and (*2*)]. With increasing phase detuning, the frequency of oscillations increases, and the depth of modulation decreases.

Figure 4 shows the results of a numerical analysis of expression (8) in the case of nonzero values of the input intensities of the pump, idler and signal waves. Comparison of the curves shows that a substantial amplification of the signal wave can be achieved by varying the input intensity of both the pump wave and the idler wave. A fivefold increase in the input intensity of the idler wave, compared with the intensity of the signal wave, leads to a significant (almost 20-fold) nonlinear increase in the amplification of the signal wave [curves (*1*) and (*6*)]. At the same time, a change in the input intensity of the idler wave relative to the pump at the input by a factor of five allows a slight, by only 1.5 times, increase in the amplification factor $\eta'_{\rm ampl}$ [curves (*6*) and (*7*)]. Thus, the growth of the signal-wave amplification factor primarily depends on the ratio of the idler and signal wave intensities at the input to the metamaterial. Apparently, this is explained by the fact that the transfer of energy from the pump wave and a stronger idler wave to the energy of the signal wave is more efficient. In addition, a comparison of the behaviour of the group of curves (*1*), (*3*), (*4*), (*6*) and curves (*2*), (*5*) shows a shift in the maxima and minima of spatial oscillations. The shift amounts to a half-cycle with a change in the ratio I_{20}/I_{1l} from 0.1 to 0.3 [curves (*5*) and (*6*) or (*2*) and (*4*)]. Further analysis showed that such a shift is not observed in the constant-field approximation when $\Gamma_2 = 0$ (for more details, see above).

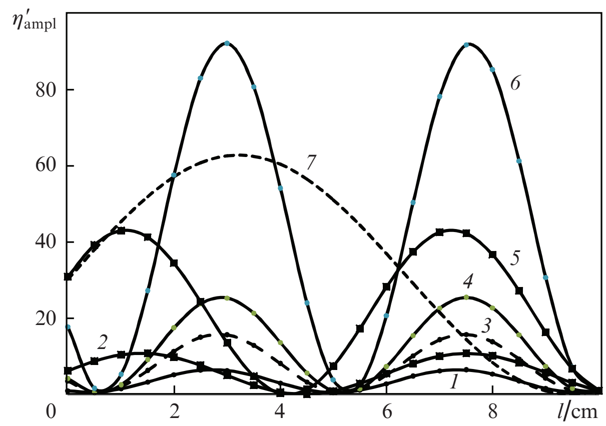

Figure 4. Dependences of the amplification factor $\eta'_{\rm ampl} = I_1(z)/I_{1l}$ on the metamaterial length for $\Gamma_3 = 1$ cm^{-1}, $l = 10$ cm, $\Delta = 2.5$ cm^{-1} at I_{20}/I_{30} = (*1, 3, 4, 6*) 0.1, (*2, 5*) 0.3 and (*7*) 0.5, I_{20}/I_{1l} = (*1*) 1, (*3*) 5, (*2, 4*) 10 and (*5–7*) 50.

As is known, for applications it is interesting to develop an efficient tunable parametric frequency converter. Under low-frequency pumping conditions in a metamaterial, this can be done at a sufficient intensity of the pump wave and the idler wave. In this case, the tuning of the signal-wave frequency occurs in a small frequency range within which the refractive index is negative. This range is determined by the existing technology for manufacturing metamaterials. Thus, for example, to date, for the THz range, it has been experimentally obtained that at a signal-wave frequency ω_1 = 1 THz, such a frequency interval is hundreds of MHz, i.e. 0.01% of ω_1 [33–35, 41]. If we also take into account the fact that the absorption coefficient increases sharply at resonance, it is necessary to exclude the frequency interval corresponding to the absorption maximum. This will lead to an even narrower working range of the frequencies of the signal wave, in which the conditions for the existence of negative refraction are satisfied.

Since there is no specific experiment on the parametric interaction of nonlinear optical waves in metamaterials, we will perform a numerical estimate of the expected efficiency of the frequency conversion for low-frequency pumping, for example, for dielectric waveguides that have a high quadratic nonlinearity. We consider a metamaterial of length $l = 2$ cm, which is pumped by laser radiation with a power of several watts; the phase detuning Δ is equal to 5–6 cm^{-1}, assuming the coefficients of nonlinear coupling to be $\gamma_{1,2} \approx 1$ cm^{-1} W$^{-1/2}$ [33, 41]. The results of the corresponding calculation for $\eta_1(z)$ using expression (5) for the input signal-wave intensities I_{20}/I_{30} = 0.1 and 0.2 are shown in Fig. 5 [curves (*1–4*)]. As follows from the behaviour of the dependences, the amplification factor at the output on the left of the metamaterial $I_1(z = 0)$ doubles [curves (*2*), (*4*)] with increasing signal-wave intensity I_{1l} at the input (on the right) by two times at an optimum pump power of 2.85 W. Hence, by choosing a higher intensity of the backward wave at the input of the metamaterial, it is possible to realise a more intense signal wave at the output from it.

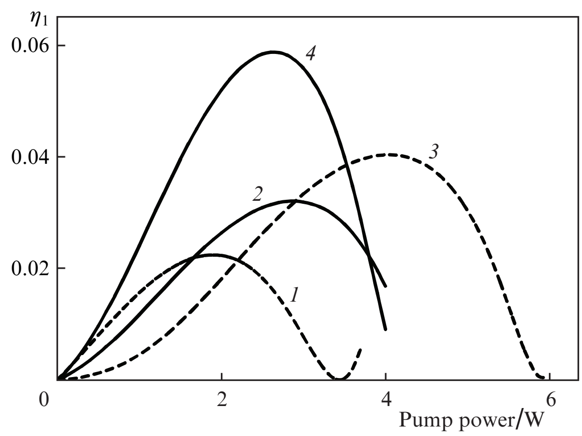

Figure 5. Dependence of the conversion efficiency into the signal wave $\eta_1 = I_1(z = 0)/I_{20}$ on the pump power for $I_{1l} = 0$ at I_{20}/I_{30} = (*1–3*) 0.1 and (*4*) 0.2 for Δ = (*3*) 5, (*2, 4*) 5.5 and (*3*) 6 cm^{-1}.

Thus, according to the analytic expressions obtained in the constant-intensity approximation, it is possible to calculate the expected values of the amplification factor and conversion efficiency in a metamaterial at low-frequency pumping for each particular experimental condition. Choosing the optimal parameters for the pump intensity, the total length of the metamaterial and the phase detuning will facilitate the implementation of regimes of effective amplification and generation of the signal wave. The developed method can be essential in the development of frequency converters based on nonlinear metamaterials.

4. Conclusions

With allowance for the phase effects, we have considered parametric interaction of waves under low-frequency pumping in a quadratic medium, which is 'left-handed' for the signal wave. An analytical expression has been derived for the signal-wave intensity for the general case of three-wave interaction in a metamaterial. The influence of various parameters on the signal-wave amplification factor and the conversion efficiency in the signal wave has been considered. The features of the process have been analysed in this case. It has been shown that the efficiency of the amplification process of the signal wave is the higher, the greater is the ratio of intensities of the idler and signal waves at the input to a metamaterial. The increase in the intensity of the idler wave at the input, in comparison with the intensity of the signal wave, by a factor of five leads to a nonlinear increase in the amplification of the latter by almost 20 times. The optimal values of the pump intensity, the total length of the metamaterial and the phase detuning have been determined, which make it possible to obtain a maximum of the conversion at the required frequency. In addition, it is possible to realise smooth frequency tuning of the parametric converter at considerable pump- and idler-wave intensities.

The next stage of the investigation will involve the consideration of the nonstationary problem of parametric interaction of counterpropagating optical waves in the constant-intensity approximation.

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