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Practical application of the adiabatic approximation and its simplified version for solving a nonintegrable system of nonlinear Schrödinger equations

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Abstract. **Using the example of the interaction of two cnoidal waves with essentially different periods, we compare the results of the application of the adiabatic approximation and its simplified version in the process of solving a nonintegrable system of nonlinear Schrödinger equations. The time boundary of the region of applicability of the simplified adiabatic-approximation version is estimated.**

Keywords: *cubic nonlinearity, gyrotropy, dispersion, cnoidal waves, adiabatic interaction, amplitude and frequency modulation.*

1. Introduction

Many problems of wave physics require an analysis of solutions to nonintegrable systems of nonlinear differential equations [1–8]. Its realisation is facilitated by the knowledge of particular and approximate analytical solutions to these systems. The adiabatic approximation [9–12] is very effective for constructing analytical solutions to the system of two coupled nonlinear Schrödinger equations (NSE) widely used in nonlinear optics and describing, in particular, the interaction of two cnoidal waves [13, 14], a cnoidal wave with bright [15] and dark [16] solitons, as well as a cnoidal wave with a rational soliton [17], a Kuznetsov–Ma soliton of [18] and an Ahmediev breather [19]. The simplified version of solutions in the adiabatic approximation was used in [13–17], essentially simplifying rather cumbersome analytical formulas [18–20] and making them visual and convenient for practical application at short interaction times. It was found that the slowly varying component of the electric field of the propagating wave causes amplitude and frequency modulation of the rapidly varying component localised in the region of intensity variation of the former.

In the present paper, using the example of the interaction of two cnoidal waves with essentially different periods, we compare the results of the application of the adiabatic approximation [18–20] and its simplified version [13–17] and estimate the time boundary of the region of applicability of the latter.

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2. Adiabatic interaction of elliptically polarised components of the electric field

We implement the adiabatic approximation in the problem of the propagation of a plane elliptically polarised electromagnetic wave through an isotropic nonlinear gyrotropic medium, which is described by a nonintegrable NSE system [13–20]

$$\frac{\partial A_\pm}{\partial z} - \mathrm{i}\frac{k_2}{2}\frac{\partial^2 A_\pm}{\partial t^2} + \mathrm{i}\Big[\mp\rho_0 + \Big(\frac{\sigma_1}{2}\mp\rho_1\Big)|A_\pm|^2 + \Big(\frac{\sigma_1}{2}+\sigma_2\Big)|A_\mp|^2\Big]A_\pm = 0 \qquad (1)$$

for slowly varying amplitudes $A_\pm(z,t)$ of orthogonal circular components of the electric field strength vector. Here, z is the propagation coordinate; t is the time in the running coordinate system; the parameter $k_2 = \partial^2 k/\partial\omega^2 \neq 0$ takes into account the second-order frequency dispersion; $k(\omega)$ is the wave number; ω is the carrier frequency; $\sigma_1 = 4\pi\omega^2\chi^{(3)}_{xyxy}/kc^2$ and $\sigma_2 = 2\pi\omega^2\chi^{(3)}_{xxyy}/kc^2$ are given by the independent components of the local cubic nonlinearity tensor $\hat{\chi}^{(3)}(\omega;-\omega,\omega,\omega)$; and $\rho_{0,1} = 2\pi\omega^2\gamma_{0,1}/c^2$ are determined through pseudoscalar constants $\gamma_{0,1}$ of linear and nonlinear gyration.

It is convenient to compare the results of the application of the adiabatic approximation and its simplified version by the example of the results obtained in Refs [20] and [13], respectively, where we found one-type periodic solutions $A_\pm(z,t) = r_\pm(t)\exp(\mathrm{i}k_\pm z)$ to system (1) in the class of functions with separable variables, whose amplitudes $r_\pm(t)$ satisfy a nonintegrable system of ordinary differential equations

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}r_\pm - \frac{2}{k_2}\Big[\Delta\kappa_\pm + \Big(\frac{\sigma_1}{2}\mp\rho_1\Big)r_\pm^2 + \Big(\frac{\sigma_1}{2}+\sigma_2\Big)r_\mp^2\Big]r_\pm = 0, \qquad (2)$$

where $\kappa_\pm = \kappa_\pm \mp \rho_0$. For definiteness, let $r_+(t)$ change slowly in comparison with $r_-(t)$. This makes it possible to write one of the periodic solutions to the equation for the fast component $r_-(t)$ in the form of Jacobi's elliptic function [20]:

$$r_-(t) = C_-(t)\,\mathrm{dn}[\varphi_-(t), \mu_-], \qquad (3)$$

where the amplitude $C_-(t)$ and the instantaneous frequency $v_-(t) = \mathrm{d}\varphi_-(t)/\mathrm{d}t$ are the functions slowly varying with the same speed; and μ_- is a free parameter. Substituting (3) into the second equation of system (2) and neglecting the derivatives $\mathrm{d}C_-(t)/\mathrm{d}t$, $\mathrm{d}^2C_-/\mathrm{d}t^2$ and $\mathrm{d}^2\varphi_-/\mathrm{d}t^2$ from slowly varying functions, we find:

$$C_-(t) = \left(\frac{2[2\Delta\kappa_- + (\sigma_1 + 2\sigma_2) r_+^2(t)]}{(\mu_-^2 - 2)(\sigma_1 + 2\rho_1)}\right)^{1/2}, \quad (4)$$

$$\varphi_-(t) = \int_0^t v_-(t_1)\,\mathrm{d}t_1 = \int_0^t \left(\frac{2\Delta\kappa_- + (\sigma_1 + 2\sigma_2) r_+^2(t_1)}{k_2(2 - \mu_-^2)}\right)^{1/2} \mathrm{d}t_1. \quad (5)$$

In the equation for the slow component $r_+(t)$, the adiabatic approximation allows us to average the fast function $r_-^2(t)$ over its period $T_- = 2\mathrm{K}(\mu_-)/(\mathrm{d}\varphi_-/\mathrm{d}t)$. Taking into account the slowness of the functions $C_-(t)$ and $\mathrm{d}\varphi_-(t)/\mathrm{d}t$, we obtain

$$\langle r_-^2 \rangle_t = \frac{1}{T_-}\int_t^{t+T_-} \{C_-(t_1)\,\mathrm{dn}[\varphi_-(t_1, \mu_-)]\}^2 \mathrm{d}t_1$$

$$= \frac{C_-^2(t)}{T_-}\left[\frac{\mathrm{d}\varphi_-(t)}{\mathrm{d}t}\right]^{-1} \int_{\varphi_-}^{\varphi_- + 2\mathrm{K}(\mu_-)} \mathrm{dn}^2(\varphi_-, \mu_-)\,\mathrm{d}\varphi_-$$

$$= \frac{2\mathrm{E}(\mu_-)[2\Delta\kappa_- + (\sigma_1 + 2\sigma_2) r_+^2(t)]}{\mathrm{K}(\mu_-)(\mu_-^2 - 2)(\sigma_1 + 2\rho_1)}. \quad (6)$$

Here, $\mathrm{K}(\mu_-)$ and $\mathrm{E}(\mu_-)$ are complete elliptic integrals of the first and second kind, respectively. After substituting (6) into the first equation of system (2), it becomes independent. Similarly to [20], we obtain one of its solutions in the form

$$r_+(t) = C_+ \mathrm{cn}(v_+ t, \mu_+), \quad (7)$$

where

$$C_+ = 2\left(\frac{\mu_+^2[2\mathrm{E}(\mu_-)(\sigma_1 + 2\sigma_2)\Delta\kappa_- - \mathrm{K}(\mu_-)(2 - \mu_-^2)(\sigma_1 + 2\rho_1)\Delta\kappa_+]}{(2\mu_+^2 - 1)[\mathrm{K}(\mu_-)(2 - \mu_-^2)(\sigma_1^2 - 4\rho_1^2) - 2\mathrm{E}(\mu_-)(\sigma_1 + 2\sigma_2)^2]}\right)^{1/2}; \quad (8)$$

$$v_+ = \left(\frac{2}{k_2(2\mu_+^2 - 1)}\left[\Delta\kappa_+ - \frac{2\mathrm{E}(\mu_-)(\sigma_1 + 2\sigma_2)\Delta\kappa_-}{\mathrm{K}(\mu_-)(2 - \mu_-^2)(\sigma_1 + 2\rho_1)}\right]\right)^{1/2}; \quad (9)$$

and $\mathrm{cn}(v_+ t, \mu_+)$ is the Jacobi elliptic cosine.

Expression (7) determines the amplitude and frequency modulation of the rapidly varying component $r_-(t)$ [see (3)–(5)].

In the simplified version of the adiabatic approximation [13–17], the component $r_+(t)$ at the first stage was considered to be completely 'frozen' rather than slowly changing. In this case, instead of (3), we have

$$r_-(t) = \left(\frac{2[2\Delta\kappa_- + (\sigma_1 + 2\sigma_2) r_+^2]}{(\mu_-^2 - 2)(\sigma_1 + 2\rho_1)}\right)^{1/2}$$

$$\times\,\mathrm{dn}\left(\left(\frac{2\Delta\kappa_- + (\sigma_1 + 2\sigma_2) r_+^2}{k_2(2 - \mu_-^2)}\right)^{1/2} t, \mu_-\right). \quad (10)$$

Substituting (10) into (6), of course, gives a result that coincides with (7)–(9). Thus, the solutions obtained in the course of the implementation of the adiabatic approximation [20] and its simplified version [13] equally predict the modulation of the amplitude and instantaneous frequency of the fast component of the electric field localised in the region of intensity variation of the slowly varying component. However, the use of a simplified algorithm for long wave interaction times can lead to the appearance of phase distortions (outwardly resembling chaotic changes) due to the fact that phase (5) in obtaining (10) is found approximately:

$$\varphi_-(t) = \int_0^t \left(\frac{2\Delta\kappa_- + (\sigma_1 + 2\sigma_2) r_+^2(t_1)]}{k_2(2 - \mu_-^2)}\right)^{1/2} \mathrm{d}t_1$$

$$\approx \left(\frac{2\Delta\kappa_- + (\sigma_1 + 2\sigma_2) r_+^2(t)}{k_2(2 - \mu_-^2)}\right)^{1/2} t. \quad (11)$$

Both adiabatic approximation algorithms are completely equivalent at times

$$t \ll t_0 \approx \left|\frac{\mathrm{d}\varphi_-(t)}{\mathrm{d}t}\left[\frac{\mathrm{d}^2\varphi_-(t)}{\mathrm{d}t^2}\right]^{-1}\right|. \quad (12)$$

Thus, for $t < t_0$ the regular behaviour of the fast component of the electric field was observed in [13–17], and for $t > t_0$ a nonregular variation of $r_-(t)$ occurred.

The appearance of increasing distortions with time, with approximate calculation of $\varphi_-(t)$, is illustrated in Fig. 1. For $t \leqslant t_0/8$, the $r_-(t)$ curves calculated in the adiabatic approximation (solid curves) and in the implementation of its simplified version (dashed curves) almost coincide, and in the region $t \approx t_0/2$ their difference does not exceed several percent. We note that, because of the periodicity of the fast component, the dependence of its phase (5) on the parameters of the slow component during the entire interaction time is realised only within the last period of the change in the fast component. Thus, for long interaction times, the adiabatic approximation and its simplified version give the same results at the very beginning of each period of variation of the fast component. In fact, the origin of the time count is reset for each next period of the fast component to zero, with recalculation of the initial conditions for the slow component.

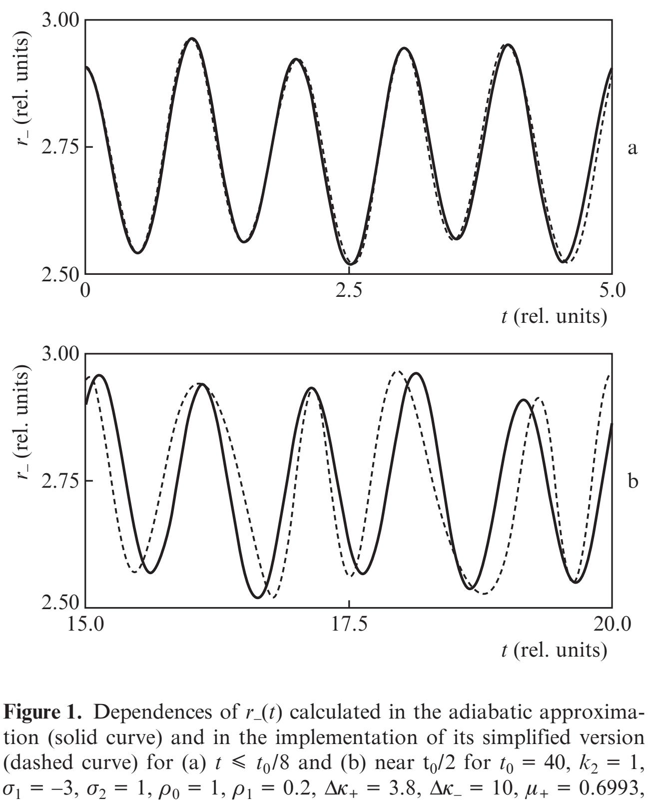

Figure 1. Dependences of $r_-(t)$ calculated in the adiabatic approximation (solid curve) and in the implementation of its simplified version (dashed curve) for (a) $t \leqslant t_0/8$ and (b) near $t_0/2$ for $t_0 = 40$, $k_2 = 1$, $\sigma_1 = -3$, $\sigma_2 = 1$, $\rho_0 = 1$, $\rho_1 = 0.2$, $\Delta\kappa_+ = 3.8$, $\Delta\kappa_- = 10$, $\mu_+ = 0.6993$, $\mu_- = 0.5$.

The dependence $r_-(r_+)$ calculated in the adiabatic approximation for large interaction times (actual for the determination of nonlinear couplings in the system) has, in contrast to [13], a regular form (Fig. 2) and does not go beyond the isoline of the potential well corresponding to the initial values of $r_\pm(0)$.

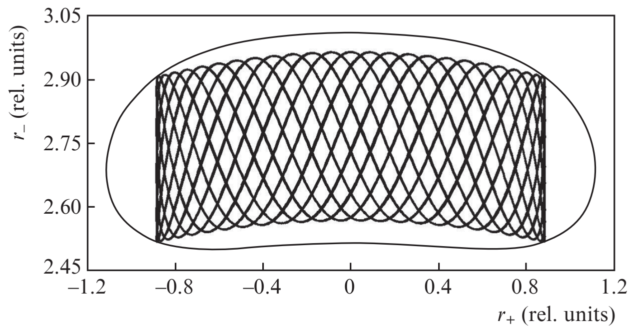

Figure 2. Dependence of $r_-(r_+)$ (thick line) and the isoline of the potential energy corresponding to the initial values of $r_\pm(0)$ (thin line) for $k_2 = 1, \sigma_1 = -3, \sigma_2 = 1, \rho_0 = 1, \rho_1 = 0.2, \Delta\kappa_+ = 3.8, \Delta\kappa_- = 10, \mu_+ = 0.6993, \mu_- = 0.5$.

3. Conclusions

The adiabatic approximation and its simplified version give the same expressions for the slow component of the field and predict the modulation of the amplitude and instantaneous frequency of the fast component localised in the region of the change in the intensity of the slow component. The simplified algorithm allows one to obtain correct quantitative results for the phase of the fast component at times $t \ll t_0$, determined by formula (12), and qualitative regularities – up to $t \leqslant t_0$. The main advantage of the simplified algorithm is its clarity and ease of use. Moreover, a rigorous version of the adiabatic approximation is fairly easy to implement in comparison with the method of multiple scales [21–23], traditionally used to solve linear and nonlinear equations with known slowly varying parameters. If in the equation for the fast component $r_-(t)$ [the second equation of system (2)] the factor containing the slow component is considered a given function, then it will turn into the well-known Duffing equation. Thus, formulas (7)–(9) turn out to be an approximate solution of the first order, obtained in [23] by the method of multiple (two) scales.

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