

INTEGRATED OPTICS

https://doi.org/10.1070/QEL16483

Modes of a nonlinear planar waveguide with a dielectric layer immersed in a hyperbolic medium

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Abstract. **The guided waves of a symmetric planar waveguide formed by an isotropic dielectric placed in a hyperbolic medium and having a cubic-nonlinear response are studied theoretically. The optical axis of the hyperbolic medium is directed along the normal to the interfaces between the media. If the permittivity of the waveguide core exceeds the main permittivity for an extraordinary wave in the hyperbolic medium, then each TM mode is characterised by two cut-off frequencies. Dispersion relations for these modes are found in the case of focusing and defocusing core media. The number of the modes possible at a given frequency depends on the radiation intensity. It is shown that zero values of the mode propagation constants are possible in the waveguide, which corresponds to the formation of a standing wave between the boundaries of the waveguide. In addition, in the case of a defocusing waveguide layer, such stopped modes can be obtained with increasing field intensity. The dependences of the propagation constant and the width of the transverse distribution of the mode field on the radiation intensity are found and analysed.**

Keywords: *metamaterials, hyperbolic dispersion, nonlinear dielectric, waveguide.*

1. Introduction

A hyperbolic material can be defined as a strongly anisotropic uniaxial medium, whose principal components of the permittivity tensor or of the magnetic susceptibility tensor have opposite signs [1, 2]. An extraordinary wave propagating in such a nonmagnetic medium is characterised by an isofrequency surface of the refractive index describing the hyperboloid in the phase space. This leads to a number of unusual optical properties of hyperbolic media (see [3–8], and also reviews [9–11]).

Plasmonic or photonic guiding devices, such as waveguides, are widely used in various communication networks and information processing systems. Many guiding structures, including a hyperbolic medium, have already been considered. Among them are the interface between a dielectric and a hyperbolic medium, supporting the propagation of surface plasmon polaritons [12, 13]; waveguides with a hyperbolic core [14–16], characterised by the coexistence of forward and backward modes, by the possibility of the light slowing down, and by negative refraction; and waveguides with a hyperbolic cladding [17, 18]. In all the papers mentioned, only the linear response of media to the applied radiation has been examined.

An optically linear planar waveguide with a transparent isotropic dielectric layer placed in a hyperbolic medium was studied in [18]. If the anisotropy axis of the hyperbolic medium is directed perpendicular to the interfaces between media, then the TM waves are extraordinary waves in this medium and are ordinary waves in the dielectric waveguide layer. Because of this, unlike conventional dielectric waveguides, in waveguides with a hyperbolic cladding each TM mode is characterised by two cut-off frequencies, i.e., each TM mode exists in the waveguide only in a certain frequency range at its fixed thickness [18].

Nonlinear guided modes in an asymmetric planar waveguide that was formed by an isotropic dielectric layer placed on a linear or nonlinear substrate and covered with a hyperbolic material were investigated in [10]. In this case, in addition to guided waveguide modes, additional modes appear that arise when a local maximum of the electric field is present in the nonlinear substrate. These modes are absent in the linear case and can be excited when the radiation power exceeds a certain threshold; the cut-off frequencies of each of them are determined by the permittivities of the media and the field intensity.

In this paper, we determine the modes of a guided wave in a symmetrical planar waveguide, which is formed by an isotropic nonlinear dielectric layer surrounded by a hyperbolic linear material (Fig. 1). We consider the cubic-nonlinear response of the medium from which the waveguide layer is made, and its effect on the dispersion characteristics of the TM modes. The distribution of the guided wave fields and the dispersion relation connecting the propagation constant and the radiation frequency are analytically obtained and analysed. The cases of focusing and defocusing Kerr media of the waveguide layer are considered. In all the cases in question, the TM modes are also characterised by two cut-off frequencies. For a focusing medium, the effective refractive index (or the propagation constant) of the mode increases monotonically with increasing field intensity. In the case of a defocusing medium, the propagation constant decreases monotonically, reaching zero in the limit. This situation corresponds to a zero energy flux along the waveguide, or 'stopped light', and arises precisely in a symmetric hyperbolic waveguide. It is shown that the mode density per unit of the reduced waveguide thickness depends monotonically on the radiation intensity.

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Received 21 July 2017; revision received 28 September 2017
Kvantovaya Elektronika **47** (11) 1053–1063 (2017)
Translated by I.A. Ulitkin

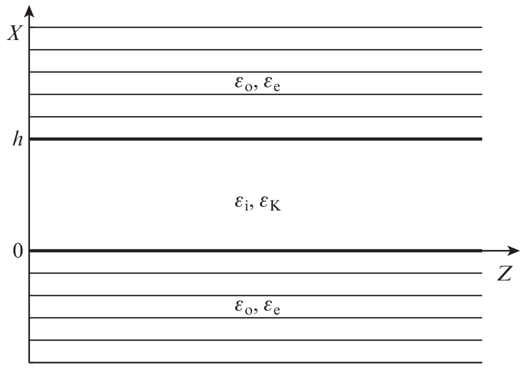

Figure 1. Scheme of a hyperbolic symmetric planar waveguide.

This makes it possible to control the number of modes excited in the waveguide by changing the radiation intensity. It is also shown that, in the case of a defocusing core, the mode width decreases with increasing field intensity.

2. Basic equations

We consider a planar waveguide whose dielectric core, or waveguide layer, has a thickness h and is made of a transparent nonmagnetic material. The substrate and the cover layer are uniaxial hyperbolic materials, the optical axis of which is perpendicular to the waveguide axis (Fig. 1). The electromagnetic properties of the hyperbolic material are determined by the main permittivities (eigenvalues of the permittivity tensor) ε_o and ε_e having different signs. The waveguide layer is a transparent cubic-nonlinear (Kerr) isotropic dielectric characterised by a linear permittivity ε_i and a Kerr constant ε_K. The coordinate axes are defined as follows. The X axis is normal to the medium interface plane, the Y and Z axes lie in the plane of the waveguide, with the Z axis coinciding with the direction of radiation propagation in the waveguide. In this geometry, the electromagnetic fields do not depend on the coordinate y.

In the general case, the system of equations describing the propagation of a wave with an electric field strength $\boldsymbol{E}$ in an anisotropic medium with an optical axis directed along the X axis has the form [18, 19]:

$$\left[\frac{\varepsilon_e(x)}{\varepsilon_o(x)}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+k_0^2\varepsilon_e(x)\right]E_x+4\pi k_0^2 P_{\mathrm{nl}\,x}=0,$$
$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+k_0^2\varepsilon_o(x)\right]E_y+4\pi k_0^2 P_{\mathrm{nl}\,y}=0, \qquad (1)$$
$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+k_0^2\varepsilon_o(x)\right]E_z+\frac{\Delta\varepsilon(x)}{\varepsilon_o(x)}\frac{\partial^2}{\partial x\partial z}E_x+4\pi k_0^2 P_{\mathrm{nl}\,z}=0.$$

Here, $\boldsymbol{P}_{\mathrm{nl}}$ is the nonlinear polarisation vector; $k_0=\omega/c$ is the wave number in vacuum; and $\Delta\varepsilon(x)=\varepsilon_e(x)-\varepsilon_o(x)$.

For the planar waveguide in question, the linear principal permittivities can be represented by a piecewise continuous function of the transverse coordinate

$$\varepsilon_{o,e}(x)=\begin{cases}\varepsilon_{o,e}, & x<0,\\ \varepsilon_i, & 0\leqslant x\leqslant h,\\ \varepsilon_{o,e}, & x>h.\end{cases} \qquad (2)$$

In the planar geometry (Fig. 1), TE and TM waves can be treated independently. The present work is devoted to the study of directed TM waves, which are extraordinary and described by the vectors $\boldsymbol{E}=(E_x,0,E_z)$ and $\boldsymbol{H}=(0,H_y,0)$. The components of these vectors are related to each other by the equations

$$H_y=\frac{\mathrm{i}}{k_0}\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right),$$
$$\frac{\partial H_y}{\partial z}=\mathrm{i}k_0\varepsilon_e(x)E_x+\mathrm{i}4\pi k_0 P_{\mathrm{nl}\,x}, \qquad (3)$$
$$\frac{\partial H_y}{\partial x}=-\mathrm{i}k_0\varepsilon_o(x)E_z-\mathrm{i}4\pi k_0 P_{\mathrm{nl}\,z}.$$

It is assumed that parametric processes (generation of harmonics or addition of frequencies) are absent. The nonlinear polarisation of a TM wave in an isotropic dielectric can be represented by the expressions

$$P_{\mathrm{nl}\,x}=(\gamma_{xx}|E_x|^2+\gamma_{xz}|E_z|^2)E_x,$$
$$P_{\mathrm{nl}\,z}=(\gamma_{zz}|E_z|^2+\gamma_{zx}|E_x|^2)E_z,$$

where the parameters γ_{ij} depend on the components of the fourth-rank nonlinear susceptibility tensor χ_{mnkl}. In this analysis, we assume the approximation

$$\boldsymbol{P}_{\mathrm{nl}}=\frac{\varepsilon_K}{4\pi}|E_z|^2E_z\boldsymbol{e}_z \qquad (4)$$

to be valid. This approximation was used in a number of studies (e.g., [20–24]), where the influence of the nonlinear response of a medium on guided waves was investigated. Such an approximation was sufficient to reveal the main features of the guiding structures.

3. Field distributions for TM waves

The waveguide under study is uniform in the direction of the Z axis, so that the fields of the guided wave can be represented as $\boldsymbol{E}=(\tilde{E}_x,0,\tilde{E}_z)\exp(\mathrm{i}\beta z)$ and $\boldsymbol{H}=(0,\tilde{H}_y,0)\exp(\mathrm{i}\beta z)$, where β is the propagation constant. Under this assumption, system (1) yields the equations for the component $\tilde{E}_z$:

$$\frac{\partial^2}{\partial x^2}\tilde{E}_z+\frac{\varepsilon_o}{\varepsilon_e}(k_0^2\varepsilon_e-\beta^2)\tilde{E}_z=0, \quad x<0,$$
$$\frac{\partial^2}{\partial x^2}\tilde{E}_z+(k_0^2\varepsilon_i-\beta^2)\tilde{E}_z+k_0^2\varepsilon_K\tilde{E}_z^3=0, \quad 0\leqslant x\leqslant h, \qquad (5)$$
$$\frac{\partial^2}{\partial x^2}\tilde{E}_z+\frac{\varepsilon_o}{\varepsilon_e}(k_0^2\varepsilon_e-\beta^2)\tilde{E}_z=0, \quad x>h.$$

Next, we introduce the parameters $q^2=(\varepsilon_o/\varepsilon_e)(\beta^2-k_0^2\varepsilon_e)$ and $p^2=k_0^2\varepsilon_i-\beta^2$. For the guided waves, the condition $q^2>0$ must be satisfied. In the case of surrounding hyperbolic media, this inequality is met only for $\varepsilon_o<0$ and $\varepsilon_e>0$. This is the so-called hyperbolic medium of metallic type [11].

There is no restriction on the sign of p^2. The case $p^2<0$ corresponds to a pair of surface waves (plasmon polaritons [12, 13]) propagating along the waveguide boundaries. The case $p^2>0$ corresponds to the guided waves, which are called waveguide modes. In what follows, only the case $p^2>0$ is considered.

The first and last equations of system (5) are easily integrated:

$$\tilde{E}_z^{(1)} = A_0 \exp(qx), \quad \tilde{E}_z^{(3)} = A_h \exp(-qx). \tag{6}$$

Superscripts '1' and '3' denote a substrate and a coating layer, respectively.

The solution of the second equation in (5) can be expressed in terms of elliptic Jacobi functions [25]. To do this, we must integrate equation (5) once, which will give

$$\left(\frac{\partial}{\partial x}\tilde{E}_z\right)^2 + p^2\tilde{E}_z^2 + \frac{1}{2}k_0^2\varepsilon_{\mathrm{K}}\tilde{E}_z^4 = G, \tag{7}$$

where G is the integration constant.

In the following derivation, we confine ourselves to the case of the defocusing medium of the waveguide core ($\varepsilon_{\mathrm{K}} < 0$). The case of the focusing medium ($\varepsilon_{\mathrm{K}} > 0$) can be treated similarly.

Next, we introduce the dependent variable $v = \sqrt{\alpha}\,\tilde{E}_z$, where $\alpha = \sqrt{k_0^2|\varepsilon_{\mathrm{K}}|/2}$ is the nonlinearity parameter. From (7) follows the equation

$$\left(\frac{\partial v}{\partial x}\right)^2 = \alpha(v^2 - \bar{v}_1^2)(v^2 - \bar{v}_2^2). \tag{8}$$

Here,

$$\bar{v}_1^2 = \frac{p^2}{2\alpha} - \sqrt{\left(\frac{p^2}{2\alpha}\right) - G}\,; \quad \bar{v}_2^2 = \frac{p^2}{2\alpha} + \sqrt{\left(\frac{p^2}{2\alpha}\right) - G}\,.$$

Replacing $w(x)$ by $v(x)/\bar{v}_1$ in (8) gives the equation

$$\left(\frac{\partial w}{\partial x}\right)^2 = \alpha\bar{v}_2^2(1 - w^2)(1 - \xi^2 w^2), \tag{9}$$

where $\xi^2 = \bar{v}_1^2/\bar{v}_2^2$; $0 \leqslant \xi^2 \leqslant 1$; and ξ monotonically increases with G. Consequently, the parameter ξ can be regarded as an alternative G to the integration constant. The parameters in (8) are expressed in terms of ξ as follows:

$$\bar{v}_1^2 = \frac{p^2}{\alpha}\frac{\xi^2}{1+\xi^2}, \quad \bar{v}_2^2 = \frac{p^2}{\alpha}\frac{1}{1+\xi^2}.$$

The integration in (9) leads to the expression

$$\pm\frac{p(x - x_0)}{\sqrt{1+\xi^2}} = \int_0^w \frac{\mathrm{d}\tilde{w}}{\sqrt{(1-\tilde{w}^2)(1-\xi^2\tilde{w}^2)}}, \tag{10}$$

where x_0 is the second integration constant. The integral on the right-hand side of this equation is an elliptic integral of the first kind [25]. Consequently, $w(x)$ can be expressed in terms of the Jacobi function, i.e. elliptic sine:

$$w(x) = \pm\,\mathrm{sn}\left[\frac{p(x-x_0)}{\sqrt{1+\xi^2}};\xi\right].$$

Then the z-component of the electric field in the waveguide core (subscript 'z') is given by

$$\tilde{E}_z^{(2)} = \frac{\bar{v}_1}{\sqrt{\alpha}}w(x) = \pm\frac{p}{\alpha}\frac{\xi}{\sqrt{1+\xi^2}}\,\mathrm{sn}\left[\frac{p(x-x_0)}{\sqrt{1+\xi^2}};\xi\right].$$

The maximum value of the z-component of the electric field E_z is

$$A_{\max} = \frac{p}{\alpha}\frac{\xi}{\sqrt{1+\xi^2}}. \tag{11}$$

Taking into account expressions (6) for the electric field in the substrate and the coating layer, we can write the system

$$\begin{aligned}
&\tilde{E}_z^{(1)}(x) = \pm A_{\max}\,\mathrm{sn}\left[\frac{-px_0}{\sqrt{1+\xi^2}};\xi\right]\exp(qx), \quad x < 0,\\
&\tilde{E}_z^{(2)}(x) = \pm A_{\max}\,\mathrm{sn}\left[\frac{p(x-x_0)}{\sqrt{1+\xi^2}};\xi\right], \quad 0 \leqslant x \leqslant h,\\
&\tilde{E}_z^{(3)}(x) = \pm A_{\max}\,\mathrm{sn}\left[\frac{p(h-x_0)}{\sqrt{1+\xi^2}};\xi\right]\exp[q(h-x)], \quad x > h
\end{aligned} \tag{12}$$

so that $\tilde{E}_z^{(1)}(0) = \tilde{E}_z^{(2)}(0)$ and $\tilde{E}_z^{(2)}(h) = \tilde{E}_z^{(3)}(h)$, i.e., as required, the tangential components of the electric field strengths are continuous at the interfaces between the media.

In the case of the focusing medium of the waveguide core, the distribution of the z-component of the electric field follows from (12) when using the substitution $\xi = \mathrm{i}\kappa$, where $0 \leqslant \kappa \leqslant 1$.

The components $\tilde{H}_y$ and $\tilde{E}_x$ can be expressed through $\tilde{E}_z$ using the formulas:

$$\begin{aligned}
&\tilde{H}_y = \frac{\mathrm{i}k_0\varepsilon_{\mathrm{e}}(x)}{k_0^2\varepsilon_{\mathrm{e}}(x) - \beta^2}\frac{\partial}{\partial x}\tilde{E}_z,\\
&\tilde{E}_x = \frac{\mathrm{i}\beta}{k_0^2\varepsilon_{\mathrm{e}}(x) - \beta^2}\frac{\partial}{\partial x}\tilde{E}_z.
\end{aligned} \tag{13}$$

These expressions follow from system (3) and assumption (4).

It is interesting to analyse how system (12) is transformed in the linear case, when the value of ε_{K} is negligible. Then, for the parameter ξ we have

$$\xi^2 = \frac{p^2 - \sqrt{p^4 - G(2\alpha)^2}}{p^2 + \sqrt{p^4 - G(2\alpha)^2}} \to 0 \quad (\alpha \to 0),$$

$$\mathrm{sn}[p(x-x_0);0] \to \sin[p(x-x_0)].$$

Consequently, the maximum value of $\tilde{E}_z^{(1,2,3)} = A_{\max}$ takes the form

$$A_{\max}^2 = \frac{p^2}{\alpha^2}\frac{\xi^2}{1+\xi^2} \to \frac{G}{p^2},$$

where we used the definition of ξ and L'Hospital's rule. Thus, the constant G is related to the maximum intensity of the field in the linear case.

4. Dispersion relations

Taking into account the continuity conditions for $\tilde{H}_y$ at the boundaries $x = 0$, $x = h$, the following relations can be obtained from system (12) and equations (13):

$$-\frac{\varepsilon_o}{\varepsilon_i}\frac{p}{q}\sqrt{1+\xi^2} = \frac{\mathrm{cn}(-px_0/\sqrt{1+\xi^2};\xi)\mathrm{dn}(-px_0/\sqrt{1+\xi^2};\xi)}{\mathrm{sn}(-px_0/\sqrt{1+\xi^2};\xi)},$$

$$\frac{\varepsilon_o}{\varepsilon_i}\frac{p}{q}\sqrt{1+\xi^2} \tag{15}$$

$$= \frac{\mathrm{cn}[p(h-x_0)/\sqrt{1+\xi^2};\xi]\mathrm{dn}[p(h-x_0)/\sqrt{1+\xi^2};\xi]}{\mathrm{sn}[p(h-x_0)/\sqrt{1+\xi^2};\xi]}.$$

Comparing the equations of this system, one can note that $x_0 = h/2 + NT\sqrt{1+\xi^2}/p$, where T is the period of the elliptic function $\mathrm{sn}(x,\xi)$, and N is an integer constant. The periods of the elliptic functions $\mathrm{sn}(x,\xi)$ and $\mathrm{cn}(x,\xi)$ are the same, and the period of the function $\mathrm{dn}(x,\xi)$ is equal to $T/2$. Therefore, the pair of equations (15) reduces to a single equation:

$$\frac{\varepsilon_o}{\varepsilon_i}\frac{p}{q}\sqrt{1+\xi^2} = \frac{\mathrm{cn}(ph/2\sqrt{1+\xi^2};\xi)\mathrm{dn}(ph/2\sqrt{1+\xi^2};\xi)}{\mathrm{sn}(ph/2\sqrt{1+\xi^2};\xi)}. \tag{16}$$

Expression (16) relates the propagation constant β contained in the parameters p and q to the radiation frequency ω, which is included in the definition of k_0. Thus, the equation obtained is the dispersion relation. The substitution $\xi = i\kappa$ from (16) yields the dispersion relation in the case of the focusing medium of the waveguide layer ($\varepsilon_K > 0$).

The parameter ξ takes into account the nonlinear properties of the medium of the waveguide layer. It follows from (11) that its value increases with increasing $A^2_{\max}$.

In the linear case ($\xi = 0$), the elliptic functions are replaced by their trigonometric counterparts and the system of equations (15) leads to the dispersion equation obtained previously [18]:

$$hk_0\sqrt{\varepsilon_i - n^2_{\mathrm{eff}}} = -2\arctan\left(\sqrt{\frac{\varepsilon_i^2}{|\varepsilon_o|\varepsilon_e}\frac{\varepsilon_e - n^2_{\mathrm{eff}}}{\varepsilon_i - n^2_{\mathrm{eff}}}}\right) + \pi m. \tag{17}$$

where m is an integer that can be used as a mode marker (it is usually called the mode index or mode order).

In the further analysis, instead of the propagation constant β, we will use the value $n_{\mathrm{eff}} = \beta k_0$, called the effective refractive index of the mode. In the previous section, it was shown that the parameter q must satisfy the inequality $q^2 > 0$. In the case of the hyperbolic environment with principal permittivities $\varepsilon_o < 0$ and $\varepsilon_e > 0$, this condition restricts the possible values of the effective refractive index n_{eff}:

$$0 \leqslant n^2_{\mathrm{eff}} \leqslant \min(\varepsilon_e, \varepsilon_i), \tag{18}$$

where $n_{\mathrm{eff}} = 0$ corresponds to the zero value of the propagation constant β, i.e., the formation of a standing wave between the core bounding media. In the beam representation, $n_{\mathrm{eff}} = \sqrt{\varepsilon_i}\sin\alpha_{\mathrm{in}}$, where α_{in} is the angle of incidence of radiation on the dielectric–hyperbolic medium interface. Thus, for $\varepsilon_e < \varepsilon_i$, it follows from (18) that there is a critical angle of incidence α_{cr} that determines the condition of total internal reflection $\sin\alpha_{\mathrm{in}} \leqslant \sin\alpha_{\mathrm{cr}} = \sqrt{\varepsilon_e/\varepsilon_i}$; therefore, the total reflection of radiation from the dielectric–hyperbolic medium interface is observed for all angles of incidence that are less than critical. In this case, hyperbolic media differ from ordinary anisotropic media.

It should be noted that in the case of a conventional dielectric waveguide, the possible values of n_{eff} lie in the interval

$$\varepsilon_e \leqslant n^2_{\mathrm{eff}} \leqslant \varepsilon_i. \tag{19}$$

The total reflection from the interface of two dielectrics occurs when $\sin\alpha_{\mathrm{in}} \geqslant \sin\alpha_{\mathrm{cr}} = \sqrt{\varepsilon_e/\varepsilon_i}$. Figure 2 shows the dispersion curves for TM modes of a dielectric waveguide with parameters $\varepsilon_i = 4.0$ and $\varepsilon_e = \varepsilon_o = 2.25$. Each curve has an initial point (cut-off frequency), where $n^2_{\mathrm{eff}} = \varepsilon_e$, and then tends to the common value for all modes, $n^2_{\mathrm{eff}} = \varepsilon_i$, which is reached at $hk_0 \to \infty$.

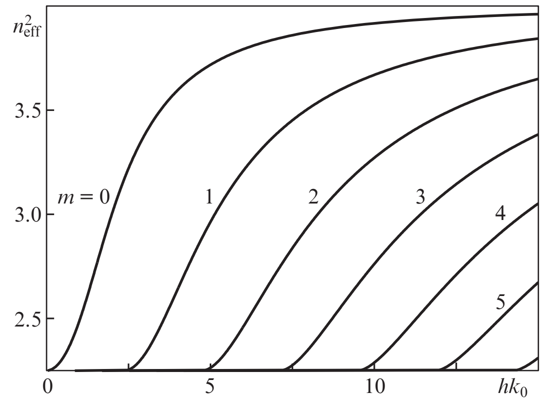

Figure 2. Dispersion curves for TM modes of a conventional dielectric waveguide with $m = 0–5$.

The elliptic functions in (16) are periodic for all ξ, except for $\xi = 1$. Therefore, we can assume that equation (16), like (17), has a set of branches of solutions $n^2_{\mathrm{eff}} = \varepsilon_i$, which are the dispersion characteristics of the waveguide modes. Each mode is also denoted with an integer index m.

In the case of the hyperbolic medium, at $\varepsilon_e < \varepsilon_i$, the value $n^2_{\mathrm{eff}} = \varepsilon_i$ is not attainable (18) and the behaviour of the dispersion curves differs markedly from that presented in the case of a conventional waveguide (Fig. 2).

Figure 3 shows the functions $n^2_{\mathrm{eff}}(hk_0)$, which are the numerical solutions of equation (16) for several values of ξ. The solutions for $\xi = 0$ correspond to the case of (17). It was assumed that the linear permittivities of the waveguide layers are as follows: $\varepsilon_i = 4.0$, $\varepsilon_e = 3.0$ and $\varepsilon_o = -3.5$. It was shown in [17, 18] that in a waveguide surrounded by a hyperbolic material there is no fundamental mode with $m = 0$. Equations (16) and (17) also do not have solutions for $m = 0$. Thus, in all the cases presented in Fig. 3, the mode index starts with $m = 1$.

It follows from the dependences shown in Fig. 3 that the TM modes in each case have two cut-off frequencies. With a change in frequency at $n^2_{\mathrm{eff}}(hk_0) = 0$, the mode appears in the waveguide, and at $n^2_{\mathrm{eff}}(hk_0) = \varepsilon_e$ it disappears, i.e. – the radiation is coupled out of the waveguide core (6). This phenomenon is absent in a conventional dielectric waveguide. Thus, the presence of an additional cut-off frequency for directional TM modes [18], detected in the linear case, also occurs for an optically nonlinear waveguide layer.

Comparing the results presented in Fig. 3, one can note that with increasing nonlinear parameter, the mode density per unit reduced thickness hk_0 decreases, and each dispersion curve shifts towards larger values of the reduced thickness. This suggests that with an increase in the field intensity in the case of the defocusing medium, the effective value of n_{eff} decreases and eventually reaches zero (at a constant radiation frequency). Because the propagation constant β is propor-

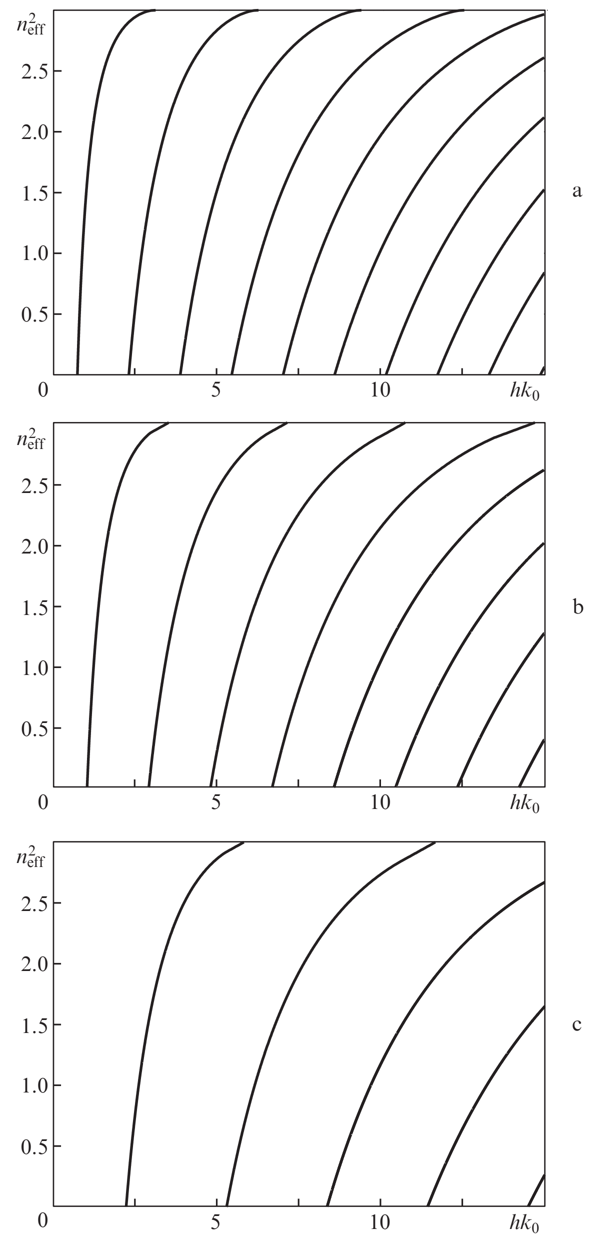

Figure 3. Dispersion curves for TM modes in the case of a defocusing medium, $\varepsilon_K < 0$, $\varepsilon_e < \varepsilon_i$, with mode indices m = (a) 1–9, (b) 1–8 and (c) 1–5; ξ = (a) 0, (b) 0.5 and (c) 0.9.

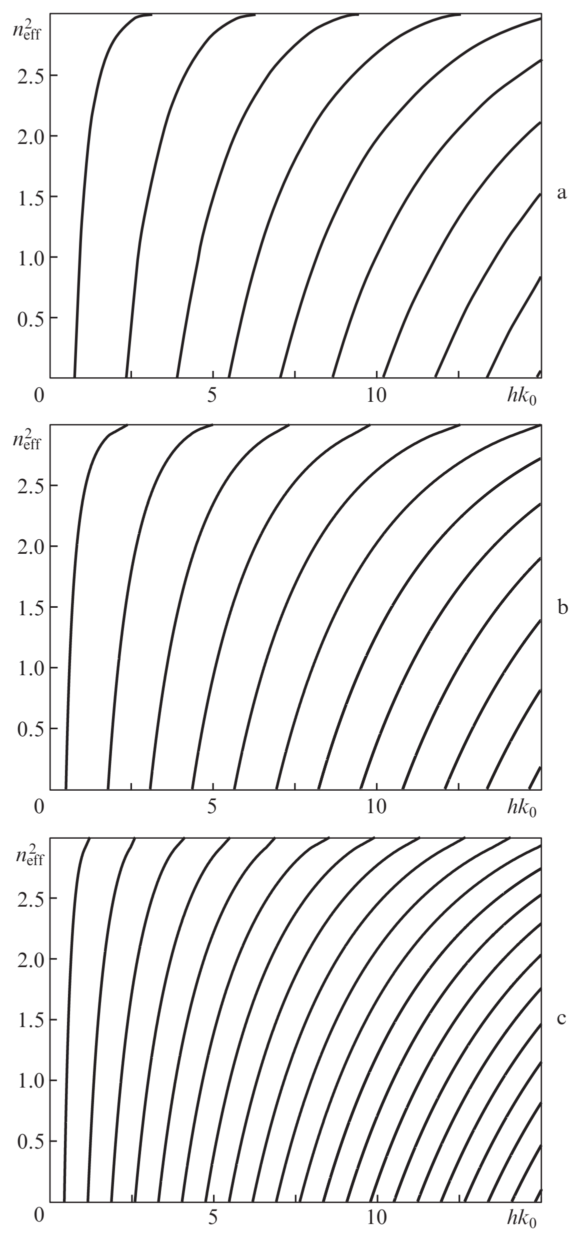

Figure 4. Dispersion curves for TM modes in the case of a focusing medium, $\varepsilon_K > 0$, $\varepsilon_e < \varepsilon_i$, with mode indices m = (a) 1–9, (b) 1–12 and (c) 1–21; κ = (a) 0, (b) 0.5 and (c) 0.85.

tional to $n_{\rm eff}$, the case $n_{\rm eff} = 0$ corresponds to the formation of a standing wave in such a waveguide, i.e., to the absence of a travelling wave along the waveguide axis.

Figure 4 shows the dispersion curves for TM waves in the case of the focusing medium ($\varepsilon_K > 0$) for several values of the nonlinear parameter κ. It follows from these dependences that in this case each TM mode also has two cut-off frequencies. However, in contrast to the situation with the defocusing medium, with increasing parameter κ, the density of the excited modes (the repetition rate of the dispersion curves) increases. And each of the dispersion curves is shifted towards smaller values of hk_0. Therefore, in the case $\varepsilon_K > 0$, the value of $n_{\rm eff}^2$ increases with increasing radiation intensity at a constant frequency. As a result, the value of $n_{\rm eff}^2 = \varepsilon_e$ is attained, corresponding to the cut-off frequency, and the considered mode is no longer retained by the waveguide.

Comparing the results for $\varepsilon_K < 0$ and $\varepsilon_K > 0$, we can conclude that the case $\varepsilon_K < 0$ is more interesting: the number of possible modes decreases with increasing intensity. This can mean a decrease in the number of excited modes for a given width of the waveguide core and a lesser probability of the appearance of intermode dispersion. In addition, an increase in intensity contributes to the slowing down of the propagating mode.

In what follows, the case of the defocusing medium of the dielectric layer will be considered in more detail.

We note that for the maximum possible intensity of the guided wave, the nonlinear parameter ξ becomes equal to unity and the elliptic functions entering into the dispersion relation (16) reduce to the corresponding hyperbolic functions [25]. Equation (16) then takes the form:

$$\tanh\left(\frac{hk_0}{2\sqrt{2}}\sqrt{\varepsilon_i - n_{\rm eff}^2}\right)\cosh^2\left(\frac{hk_0}{2\sqrt{2}}\sqrt{\varepsilon_i - n_{\rm eff}^2}\right)$$
$$= -\frac{\varepsilon_i}{\sqrt{|\varepsilon_o|\,\varepsilon_e}}\sqrt{\frac{\varepsilon_e - n_{\rm eff}^2}{\varepsilon_i - n_{\rm eff}^2}}\,. \qquad (20)$$

Here, the signs of the principal permittivities of the hyperbolic material, $\varepsilon_o < 0$ and $\varepsilon_e > 0$, are taken into account, and explicit expressions for the parameters p and q are also used. Because $hk_0 \geqslant 0$, equation (20) has a solution only in the case

$$n_{\rm eff}^2 = \varepsilon_e, \quad hk_0 = 0.$$

However, these conditions are devoid of physical meaning, since they are realized at $h = 0$.

5. Effect of the field energy density on the characteristics of the waveguide modes

This section is devoted to the analysis of the influence of the energy density of the waveguide mode on the parameters of its propagation, such as the effective refractive index, the cut-off frequency and the width of the transverse distribution of the electric field of the mode in the case $\varepsilon_K < 0$.

In the previous sections, in order to take into account nonlinear effects, in all equations describing the guided wave, we used the dimensionless parameter ξ. However, the quantity ξ does not have a simple physical meaning and, therefore, in the subsequent analysis, the density of the energy transferred by the wave, W, will be considered as a parameter determining the degree of the waveguide nonlinearity.

The energy density of a plane monochromatic wave in a medium at a given frequency is determined using the Brillouin formula:

$$W = \frac{1}{16\pi}\int_{-\infty}^{\infty}(\varepsilon_e(x)|\tilde{E}_x|^2 + \varepsilon_o(x)|\tilde{E}_z|^2 + |\tilde{H}_y|^2)\,{\rm d}x, \qquad (21)$$

where the components of the electric field of the TM wave are determined by equations (12) and (13), and the transverse distributions of the principal permittivities $\varepsilon_e(x)$ and $\varepsilon_o(x)$ are determined by expression (2).

5.1. Effective refractive index

We consider a planar waveguide with a fixed thickness of the waveguide layer. The radiation frequency is assumed to be constant and such that $hk_0 = 5$. The values of the linear permittivities are the same as in the previous section, and the Kerr constant is $\varepsilon_K = -10^{-9}$ cgs units (in SI units this value corresponds to -10^{-18} m^2 V^{-2}). Figure 5a shows the dispersion curves for the linear case. The dashed vertical line corresponds to $hk_0 = 5$. One can see that in this case for a given width of the waveguide layer and the radiation frequency in the waveguide, TM modes with indices $m = 2$ and 3 can be excited. Comparing the dependences shown in Figs 3a–3c, it can be seen that with an increase in the nonlinearity parameter ξ the dispersion curves with indices $m = 2$ and 3 cease to intersect with the line $hk_0 = 5$. This means that, starting from certain values of ξ, the corresponding modes in the linear waveguide are not excited. Also, for sufficiently large values of ξ in the waveguide with $hk_0 = 5$, a single mode with the index $m = 1$ can be excited, whereas modes with $m = 2$ and 3 cannot be retained.

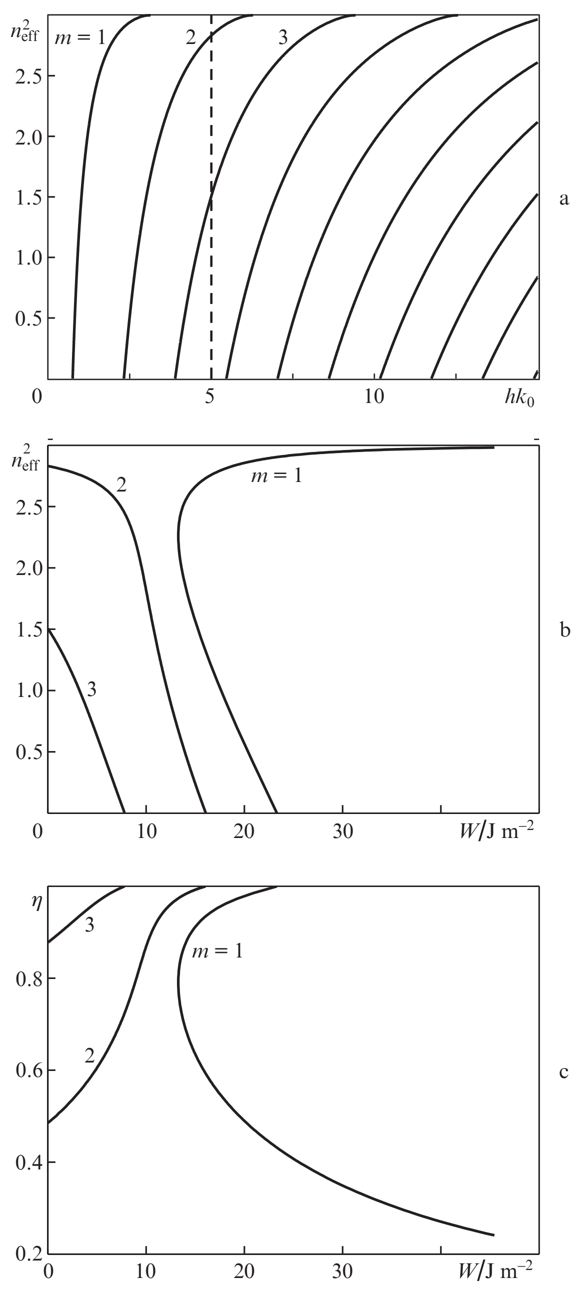

Figure 5. Behaviour of the modes in a hyperbolic waveguide at $hk_0 = 5$: (a) dispersion curve at $\xi = 0$, (b) dependence of the effective refractive index on the density of the energy carried by the wave and (c) dependence of the fraction of the energy contained in the waveguide core on the density of the transferred energy.

A similar behaviour of the dispersion curves is presented in Fig. 5b, which shows how the square of the effective refractive index of all possible modes at $hk_0 = 5$ varies with increasing energy density W. For $W = 0$, the values of $n_{\rm eff}^2$ for the indices $m = 2$ and 3 are the same as in the linear case for $hk_0 = 5$ (see Fig. 5a).

With increasing energy density W, the values of $n_{\rm eff}^2$ for $m = 2$ and 3 decrease. At a certain value of W, propagation of the mode with the index $m = 3$ ceases, because the corresponding effective exponent $n_{\rm eff}$ or, equivalently, the propagation constant becomes zero. With a further increase in W, the mode with $m = 3$ becomes damped. With an even larger increase in W, the effective refractive index of the mode with the index $m = 2$ becomes zero and the mode leaves the waveguide.

The mode with the index $m = 1$ is a special case. It is absent in the linear case, but if W exceeds a certain threshold, this mode can be excited. Another feature of the function $n_{\text{eff}}^2(W)$ for $m = 1$ is that it is two-valued. One branch of the curve behaves in the same way as for $m = 2$ and so for $m = 3$: n_{eff}^2 decreases to zero, the other branch tending to the maximum possible value $n_{\text{eff}}^2 = \varepsilon_{\text{e}}$.

Figure 5c shows the dependence of the energy fraction η of the guided wave contained in the waveguide core on W, defined as

$$h = W_{\text{core}}/W,$$

where W_{core} is the density of the transferred energy concentrated in the waveguide core. Comparing Figs 5b and 5c, one should pay attention to the fact that with decreasing n_{eff} the guided wave is better concentrated in the core of the waveguide. This fact will be discussed in more detail in the next subsection of the paper. If n_{eff}^2 tends to ε_{e} (mode with $m = 1$), then the guided wave gradually leaves the waveguide. Two branches of the curve with $m = 1$ correspond to two propagating modes. The field of one of them is mainly contained in the core of the waveguide, and the field of the second one is concentrated mainly in the surrounding hyperbolic medium.

Figures 6 and 7 show several examples of the field distributions of TM modes at $hk_0 = 5$ and $\varepsilon_{\text{K}} = -10^{-9}$ cgs units. Figure 6 demonstrates the distribution of the electric field of the mode with $m = 2$ at the minimum and maximum values of W. The cases in Figs 6a and 6c correspond to $W \leqslant 1$ J m^{-2}, $n_{\text{eff}}^2 = 2.83$, the cases in Figs 6b and 6d correspond to $W \geqslant 15$ J m^{-2}, $n_{\text{eff}}^2 = 0.05$ (see Fig. 5b). It follows from the presented dependences that the length of the damping field in the substrate and in the coating layer of the waveguide is smaller for small values of n_{eff}.

Figure 7 shows the distribution of electric and magnetic fields for the nonlinear mode with $m = 1$. The cases in Figs 7a and 7c correspond to $n_{\text{eff}}^2 = 2.88$, and the cases in Figs 7b and 7d correspond to $n_{\text{eff}}^2 = 1.66$.

For comparison, Fig. 8 shows similar (see Fig. 5) dependences for a conventional dielectric waveguide with $\varepsilon_{\text{i}} = 4.0$, $\varepsilon_{\text{e}} = \varepsilon_{\text{o}} = 2.25$ and $\varepsilon_{\text{K}} = -10^{-9}$ cgs units. The reduced thickness of the waveguide layer (the waveguide core) hk_0 was assumed equal to 10. In the case of arbitrarily weak fields, the modes with indices $m = 0-4$ can be excited in the waveguide. With increasing field intensity, the linear values of n_{eff}^2 for each mode decrease to its minimal limit: $n_{\text{eff}}^2 = \varepsilon_{\text{e}} = \varepsilon_{\text{o}}$. No additional modes arise in this case.

5.2. Width of the transverse distribution of the electric field strength

It follows from expressions (6) that the parameter q determines the damping of the field channelled along the X axis outside the waveguide layer. Let us determine the radiation damping length outside the waveguide layer in units of the wavelength λ as

$$L_{\text{d}} = \frac{1}{4\pi\tilde{q}},$$

where $\tilde{q} = q/k_0$ is the normalised value of q. Consequently, at a distance L_{d}, the field intensity $|E_z|^2$ decreases by a factor of e. Because n_{eff} depends on the density of the transferred energy W, the length L_{d} will also vary with W.

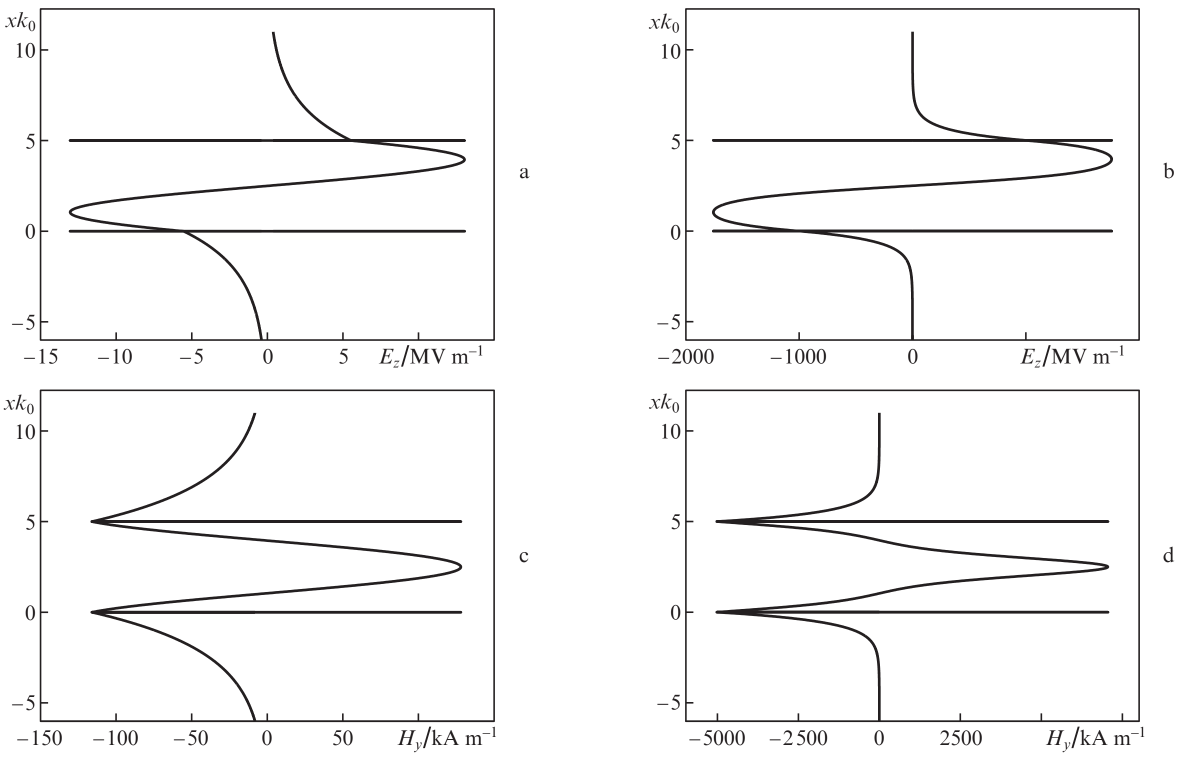

Figure 6. Distribution of the electric (E_z) and magnetic (H_y) fields in a hyperbolic waveguide with $hk_0 = 5$ for the TM mode with $m = 2$ for (a, c) small and (b, d) high intensities.

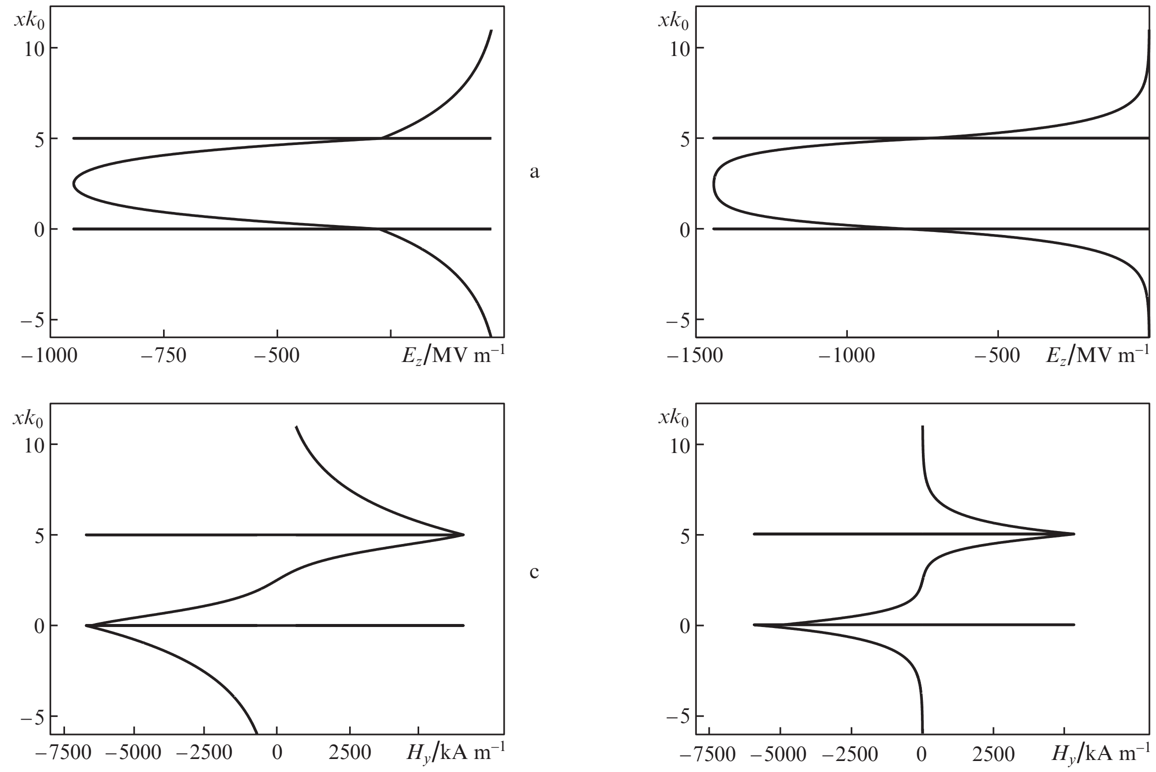

Figure 7. Distribution of the electric (E_z) and magnetic (H_y) fields in a hyperbolic waveguide with $hk_0 = 5$ for the TM mode with $m = 1$ for $n_{\rm eff}^2 =$ (a, c) 2.88 and (b, d) 1.66.

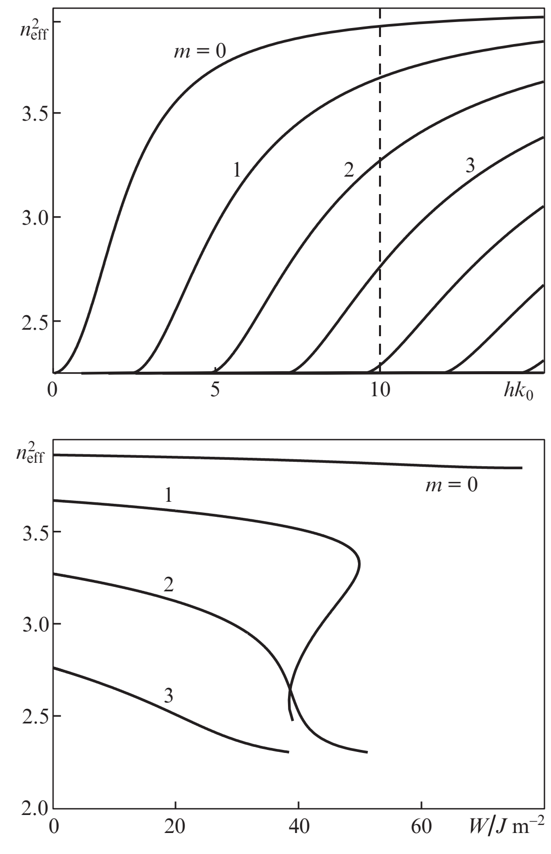

Figure 8. (a) Dispersion curves for $\xi = 0$ and (b) dependence of the effective refractive index on the transferred energy density for a conventional dielectric waveguide at $hk_0 = 10$.

Figure 9 shows the dependence of $L_{\rm d}$ on W for symmetric hyperbolic and ordinary dielectric waveguides. For illustration, in both cases a mode with the index $m = 2$ was chosen. As was shown above, in the case of the defocusing Kerr medium, the effective refractive index decreases with increasing W, and for the hyperbolic medium with $\varepsilon_{\rm o} < 0$ and $\varepsilon_{\rm e} > 0$ the parameter q increases with increasing W. In turn, $q \sim 1/L_{\rm d}$. Thus, with increasing W, the concentration of radiation in the waveguide core increases and the width of the transverse distribution of the field strengths, which can be called the mode width, decreases. In the case of the conventional dielectric waveguide, the parameters $n_{\rm eff}$ and q vary identically and, consequently, as the field intensity W increases, the mode width increases and the radiation penetrates deeper into the cladding.

5.3. Cut-off frequencies

It was shown above that each TM mode in the hyperbolic waveguide with $\varepsilon_{\rm e} < \varepsilon_{\rm i}$ and a fixed thickness of the waveguide layer exists only in a certain frequency interval. Thus, each mode is characterised by two cut-off frequencies. Next, we consider the dependence of the cut-off frequencies on the density of the energy carried by the wave. For illustration, the mode with the index $m = 2$ was chosen.

The cut-off frequencies are defined as follows. The reduced thicknesses hk_0, for which $n_{\rm eff}^2 = 0$, and $\varepsilon_{\rm e}$ are denoted by $V_{\rm cm}^{(1)}$ and $V_{\rm cm}^{(2)}$, respectively; therefore, the hyperbolic waveguide with the thickness of the waveguide layer h retains the TM mode with the index m only for frequencies lying in the interval

$$\frac{c}{h} V_{\rm cm}^{(1)} \leqslant \omega \leqslant \frac{c}{h} V_{\rm cm}^{(2)}. \qquad (22)$$

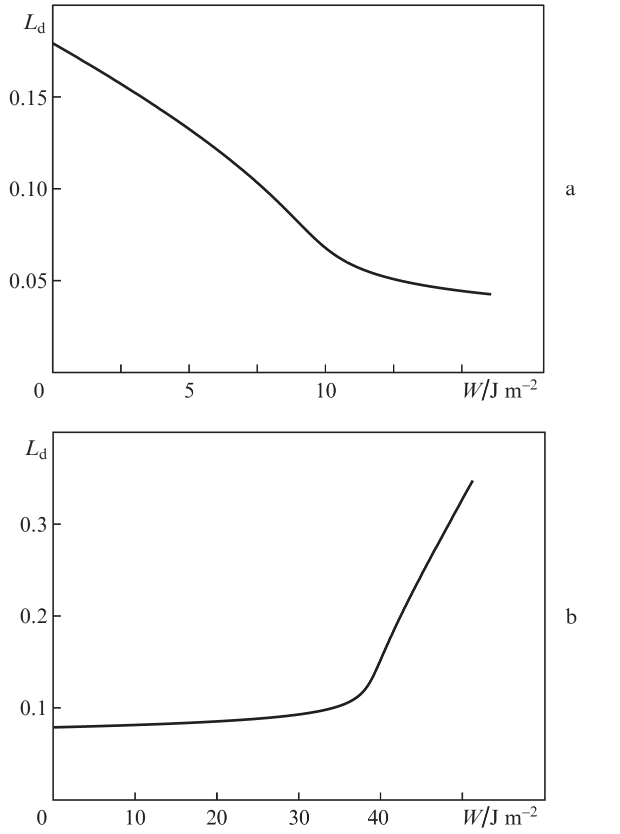

Figure 9. Dependences of the length of the energy damping of the mode with $m = 2$ outside the waveguide core in units of λ on the energy density (a) for the hyperbolic waveguide and (b) for the conventional dielectric waveguide.

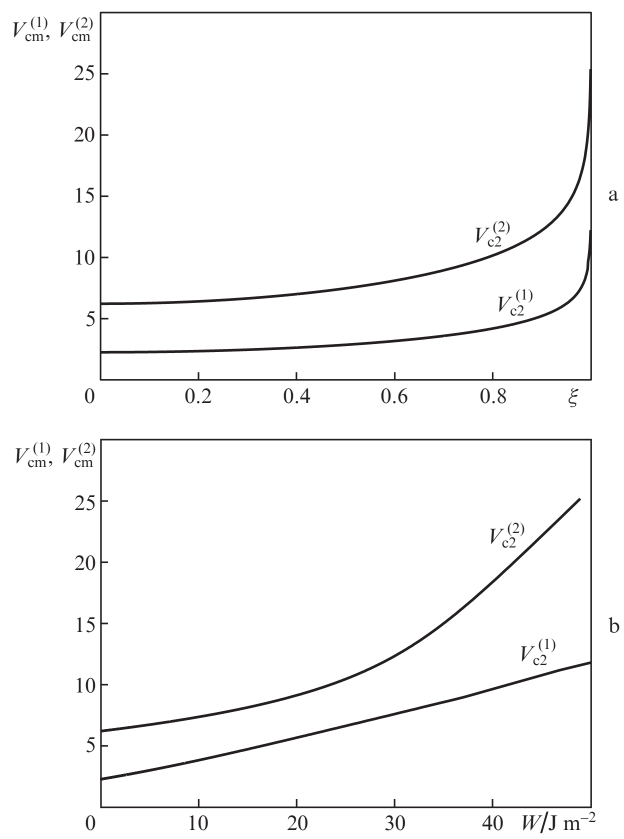

Figure 10. Dependences of the cut-off frequencies on (a) ξ and (b) W.

The cut-off frequencies can be obtained from Eqn (16) as a function of the nonlinear parameter ξ at suitable values of $p(n_{\rm eff}^2)$ and $q(n_{\rm eff}^2)$. The results for the mode with $m = 2$ are shown in Fig. 10, and the initial values of $V_{c2}^{(1)}$ and $V_{c2}^{(2)}$ for $\xi = 0$ were found in the linear case (17). The change in the cut-off frequencies with increasing field energy W is shown in Fig. 10b. When constructing dependences, the permittivities were chosen the same as before, and $\varepsilon_{\rm K} = -10^{-9}$ cgs units.

It follows from Fig. 10 that both cut-off frequencies increase with increasing nonlinearity parameter ξ (or field intensity). The width of the interval ($V_{\rm cm}^{(1)}$, $V_{\rm cm}^{(2)}$) increases insignificantly. For $\xi \to 1$, both frequencies tend to infinity, and this means that it is no longer possible to excite guided waves in the waveguide.

6. Effect of losses in the surrounding hyperbolic medium

Hyperbolic media, as a rule, are composite metamaterials, the structure of which includes conductive components, such as metal plates or nanowires. This leads to the appearance of energy losses of electromagnetic radiation passing through such a medium. For a real hyperbolic medium, the principal permittivities $\varepsilon_{\rm o}$ and $\varepsilon_{\rm e}$ are complex quantities. If the radiation frequency is far from all resonances characteristic of the medium, the imaginary parts of $\varepsilon_{\rm o}$ and $\varepsilon_{\rm e}$ are insignificant compared with the real ones [19]. However, because of the complexity of $\varepsilon_{\rm o}$ and $\varepsilon_{\rm e}$, the propagation constant β also becomes a complex quantity: $\beta = k_0(n_{\rm eff} + {\rm i}\delta)$. The coefficient δ determines the absorption of the energy carried by the wave in the linear case when $W(z) \sim \exp(-2k_0\delta z)$.

Figure 11 shows the dispersion curves for TM modes at $\varepsilon_{\rm e} = 3 + 0.05{\rm i}$, $\varepsilon_{\rm o} = -3.5 + 0.1{\rm i}$, $\varepsilon_{\rm i} = 4$ and $\varepsilon_{\rm K} = 0$. Imaginary parts of $\varepsilon_{\rm o}$, $\varepsilon_{\rm e}$ of this order can be obtained in real hyperbolic media without applying any means to compensate for energy losses [26]. The dependences of the real and imaginary parts of the parameter q on hk_0 are shown in the same figure. The values of $n_{\rm eff}^2$ are the squares of the real part of the normalised propagation constant ${\rm Re}\beta/k_0$.

In the presence of energy losses, the cut-off frequencies are defined as follows: $V_{\rm cm}^{(1)}$ is defined as before, the reduced thickness hk_0 for which ${\rm Re}\,q = {\rm Im}\,q$ will be denoted by $V_{\rm cm}^{(2)}$. In the presence of losses, the maximum possible value of $n_{\rm eff}^2$ is

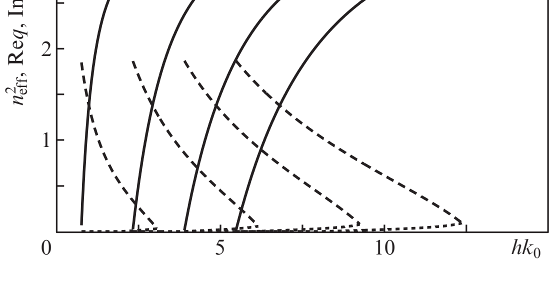

Figure 11. Dependences of $n_{\rm eff}^2$ (solid curves), Re$\,q$ (dashed curves) and Im$\,q$ (dotted curves) on the reduced thickness hk_0 of the waveguide for the first four modes.

somewhat greater than $\mathrm{Re}\,\varepsilon_e$. As was shown in [19], the second cut-off frequency disappears with strong dissipation, when $\mathrm{Re}\,\varepsilon_{e,o} \sim \mathrm{Im}\,\varepsilon_{e,o}$.

The propagation length of the guided mode in units of the emission wavelength λ in the linear case is defined as $L_p = 1/(4\pi\delta)$. Dependences of L_p on the reduced thickness hk_0 of the waveguide layer for several first TM modes are presented in Fig. 12. For their construction, the principal permittivities of the hyperbolic medium were chosen to be the same as in [14], where a waveguide with a hyperbolic core was considered: $\varepsilon_e = 5.97 + 0.065i$, $\varepsilon_o = -3.44 + 0.15i$. For the waveguide considered in this section, we used $\varepsilon_i = 7$. It was also assumed that the radiation frequency remains unchanged.

Figure 12 shows that for a waveguide with a hyperbolic medium, the propagation length is equal to tens of wavelengths. For a waveguide with a hyperbolic core, considered in [14], the propagation length was less than 4λ. In the case considered here, the radiation is localised mainly in a transparent dielectric, and only the exponentially decreasing tails of the field distribution are in the hyperbolic material.

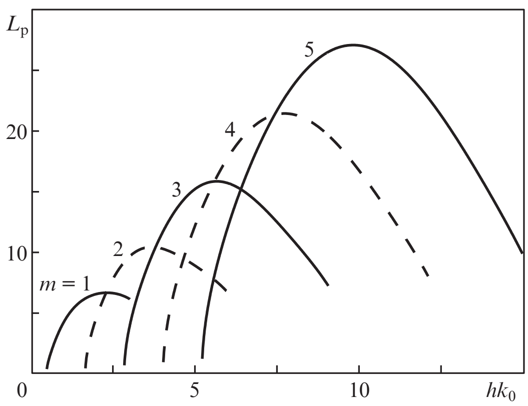

Figure 12. Dependences of the propagation length L_p in units of λ on the reduced thickness hk_0 for the first five modes.

In conclusion, it should be noted that for hyperbolic materials with weak dissipation, when $\mathrm{Im}\,\varepsilon_{e,o} < 0.1\mathrm{Re}\,\varepsilon_{e,o}$, all the results are preserved. The absorption in the surrounding dielectric medium leads mainly to losses of the transferred energy and has little effect on the phase relationships that dictate the properties of the mode. As usual, the losses lead to the finite propagation length of the radiation in the waveguide.

7. Conclusions

In this paper, the properties of the guided TM modes of a planar hyperbolic waveguide are theoretically investigated. The planar waveguide is formed by a nondissipative dielectric layer and is surrounded by a hyperbolic medium with an anisotropy axis directed along the normal to the interfaces between media. The main attention is paid to the role of the cubic-nonlinear response of the dielectric layer.

A dispersion relation for guided modes of a symmetric nonlinear hyperbolic waveguide is obtained. It is shown that if the permittivity of the waveguide layer exceeds the extraordinary component of the permittivity tensor of the surrounding media, then each TM mode is characterised by two cut-off frequencies, as in the cases considered previously [18, 19]. This leads to the fact that each mode can be excited only in a certain frequency interval. In contrast to the waveguide considered in [19], in the present work the main part of the energy of the guided radiation is concentrated in a nonlinear dielectric medium. For this reason, the mode density per unit of reduced thickness hk_0 depends on the radiation intensity in the waveguide. In the case of a focusing medium, the mode density increases with the field intensity, and in the case of a defocusing medium, it decreases.

In the case of a defocusing medium, the mode propagation constant decreases with increasing electromagnetic field intensity. In addition, for some of its values, the propagation constant vanishes. With a further increase in the field intensity, the mode under consideration becomes exponentially damped and ceases to propagate in the waveguide. Thus, we can talk about a new mechanism of the light deceleration, which is realised in a nonlinear hyperbolic waveguide. The degree of deceleration is controlled by the radiation intensity in the waveguide. An optically controlled mode density can form the basis for optical modulation devices and switches.

Dependences of the mode width and cut-off frequencies on the density of the energy transferred by the wave are also analysed. It is shown that the width of the transverse mode distribution decreases with increasing electromagnetic field energy. In the case of a conventional dielectric waveguide, this dependence is inverse.

The effect of dissipation in the hyperbolic medium of the covering layer of a planar waveguide was discussed in [19]. Both in [19] and this paper it is shown that the presence of losses leads to a decrease in the amplitude of the guided wave. The effect of energy losses in the considered hyperbolic waveguides is in many respects similar to the effect that takes place in the case of waveguides with metal walls [27–30].

For practical implementation of the waveguide structure in question, it is necessary to fulfil a number of basic requirements. Energy losses in the hyperbolic cladding should be small ($\mathrm{Im}\,\varepsilon_{o,e} < 0.1\mathrm{Re}\,\varepsilon_{o,e}$), the linear permittivity of the core should exceed the main unusual component of the permeability of the hyperbolic medium ($\varepsilon_i > \varepsilon_e$), and the core should be a cubically nonlinear medium with a sufficiently large modulus of the Kerr constant ε_K. The considered waveguide can be realised, for example, using the following media: As a hyperbolic cladding of the waveguide, use can be made of the metamaterials discussed in [11], formed by alternating sub-wavelength Ag/SiO_2 layers with a silver volume fraction of 0.4. In this case, the permittivities are as follows: $\varepsilon_e = 3.7 + 0.01i$, $\varepsilon_o = -8.8 + 0.24i$ (at $\lambda = 0.75$ μm). Additional cut-off frequencies exist if the permittivity of the core exceeds ε_e. Semiconductors possess sufficiently high values of permittivities, for example, $\varepsilon_i = 11.68$ (Si), 13.1 (GaAs), 9.2 (ZnSe), 10 (SiC) (values are given for low frequencies). In addition, semiconductors have a significant nonlinear response: $\varepsilon_K \sim 10^{-10} - 10^{-9}$ cgs units ($10^{-19} - 10^{-18}$ $m^2 V^{-2}$), which allows us to consider them as a possible core of the hyperbolic waveguide. At the same time, the required radiation power densities will be 50–500 MW cm^{-2}.

Acknowledgements. The authors are grateful to I.R. Gabitov and K. Bayun for numerous fruitful discussions. The study was supported by the Russian Foundation for Basic Research (Grant No. 15-02-02764).

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