Quantum Electronics **48** (2) 95–104 (2018)

REVIEW

https://doi.org/10.1070/QEL16555

Modelling of rotation-induced frequency shifts in whispering gallery modes

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Abstract. **We study the angular velocity sensors based on whispering gallery mode resonators. Rotation of such resonators gives rise to various effects that can cause a spectral shift of their modes. Optical methods allow this shift to be determined with high precision, which can be used practically to measure the angular velocity in inertial orientation and navigation systems. The basic principles of constructing the angular velocity sensors utilising these effects are considered, their advantages and drawbacks are indicated. We also study the interrelation between the effects and the possibility of their mutual influence on each other. Based on the analytical studies of the effects, we consider the possibility of their combined application for angular velocity measurements.**

Keywords: *sagnac effect, whispering gallery modes, morphologically dependent resonator, angular velocity sensor.*

Figure 1. Echo Wall surrounding the Imperial Temple of Heaven (left) and the whispering gallery of St. Paul's Cathedral in London (right).

1. Introduction

The whispering gallery phenomenon has already been known for several centuries and is originally related to historic landmarks located in different countries (China, Italy, etc.), e.g., the so-called 'Echo Wall' surrounding the yard of the Imperial Temple of Heaven in Beijing (Fig. 1) [1]. The wall is built of special bricks from the city of Linqing, Shandong Province. These bricks are carefully fabricated, possess identical structure and differ from the silicate bricks commonly used in house building. They have almost no pores. Such bricks perfectly reflect acoustic waves and do not absorb acoustic energy. The phenomenon is that the sound (whisper) propagates along the concave surface of the wall rather than along the shortest path.

One can face similar phenomena under the domes of some temples, such as St. Paul's Cathedral in London (Fig. 1), where Lord Rayleigh studied and described the effect from a scientific point of view in the nineteenth century. He discovered that the acoustic radiation concentrates in a thin layer near the concave surface and glides along it. Thus, the words pronounced near the closed surface return back after a roundtrip. It looks as if the gallery whispered. To describe this phenomenon the term 'whispering gallery mode' (WGM) was introduced. Later, in the beginning of the 20th century, the existence of electromagnetic WGM was proved. For a long time (until the last decade of the 20th century), these types of waves did not attract much attention [2]. However, at present the interest of researchers in optical WGM resonators is growing [3, 4].

The WGM resonators are dielectric axially symmetric resonators with smooth edges, supporting the existence of WGMs due to the total internal reflection from the resonator surface. To date, different types of such resonators have been developed, e.g., spherical, disk-shaped, toroidal, bottle-shaped, etc. [3]. An increased interest in them is due to their unique characteristics. They possess an ultrahigh Q-factor (in some cases exceeding 10^9), small dimensions (from a few centimetres to a few micrometres), and a limited number of eigenfrequencies. These properties allow such resonators to be used as sensitive elements of compact high-accuracy instruments. The rotation of WGM resonators gives rise to various effects that can be used to measure the angular velocity [5–8]. The effects that lead to the WGM spectral shift (the Sagnac effect and the impact of centrifugal forces on the resonator morphology) seem to be most promising for the design of angular velocity sensors. The completeness of study in this case requires not only an analysis of the applicability of these effects to the angular velocity measurement, but also the consideration of their mutual influence.

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Received 28 September 2017; revision received 8 November 2017
Kvantovaya Elektronika **48** (2) 95–104 (2018)
Translated by V.L. Derbov

2. Angular velocity sensors based on the Sagnac effect

We assume that the optical radiation with the intensity I_0 constant in a certain spectral range is launched into a WGM resonator by means of an auxiliary waveguide (Fig. 2) in two opposite directions, clockwise (CW) and counterclockwise (CCW). If the resonator does not move with respect to an inertial frame, then the frequencies of the counterpropagating waves in the WGM resonator are equal. In this case the intensities of the CW wave (I_1) and CCW wave (I_2) correspond to curve (*1*) in Fig. 3. Imagine that the resonator began to rotate clockwise with the angular velocity Ω. It is well known that in rotating ring cavities the Sagnac effect causes a shift of the eigenfrequencies [9]. This is true for the WGM resonators too. If the resonator rotates, then according to the Sagnac effect theory, the frequencies of WGMs with opposite directions of resonator roundtrip split and their difference (Sagnac frequency shift) is determined by the formula:

$$\Delta f_S = f_{m\mathrm{CCW}} - f_{m\mathrm{CW}} = \frac{4Af_m}{cL}\Omega = \frac{2Rf_m}{c}\Omega, \quad (1)$$

where $f_{m\mathrm{CW}}$ and $f_{m\mathrm{CCW}}$ are the WGM frequencies for the clockwise and counterclockwise waves, respectively; $f_m = c/(\lambda_{m0}n)$ is the WGM frequency in the nonrotating resonator; n is the refractive index of the resonator material; λ_{m0} is the wavelength of the mth mode in vacuum; A is the area of the operating cross section of the resonator; c is the speed of light in vacuum; L is the path length passed by the mode in a single roundtrip of the resonator (approximately equal to the perimeter of its operating cross section); Ω is the angular velocity of rotation about the axis perpendicular to the operating cross section of the resonator; and R is the resonator radius. In this case, the intensity I_1 corresponds to curve (*2*) in Fig. 3, and the intensity I_2 corresponds to curve (*3*). Equation (1) shows that the effect is nonreciprocal and linear with respect to the resonator radius and angular velocity. The sign of the frequency shift is determined by the direction of the resonator rotation. To find the angular velocity the resonator is frequency scanned in two mutually opposite directions (CW and CCW), the nonreciprocal frequency shift Δf_S is determined, and the angular velocity that is proportional to it is calculated [10].

Recently, an experimental study of angular velocity sensors based on the nonreciprocal WGM frequency shift caused by the Sagnac effect has been reported [11, 12]. First, a prototype of the angular velocity sensor based on the WGM resonator was demonstrated and tested [11]. In the prototype the authors used a disk-shaped crystal resonator made of calcium fluoride with a Q-factor of $\sim 10^9$, a disk diameter of ~ 1 cm, and a thickness of 0.2 mm. The radiation with a wavelength of 1.5 μm and a power of 0.075 mW was launched into the resonator in counterpropagating directions. To reduce the external factor effect, the WGM resonator was isolated from the environment by a hermetic case. The limiting sensitivity of the sensor to the angular velocity was measured and appeared to be equal to 2.3 deg h^{-1}. Then an experimental prototype of the sensor was fabricated [12], in which a similar WGM resonator having a diameter of 7 mm was used. The total volume of the experimental prototype, including the resonator, the semiconductor laser, the photodiodes, etc., amounted to 15 cm^3. Its limiting sensitivity was equal to about 3 deg h^{-1}. Besides the above passive optical angular velocity sensors, the Sagnac effect can be used in the construction of active angular velocity sensors (laser gyroscopes). However, the limiting sensitivity of laser gyroscopes based on WGM resonators is worse, and their size is larger. To date, the limiting sensitivity to the angular velocity achieved in a sensor based of the WGM resonator having a diameter of 18 mm amounts to 22 deg h^{-1} [13].

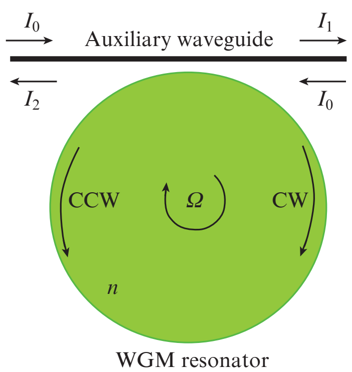

Figure 2. WGM resonator optically coupled with an auxiliary waveguide.

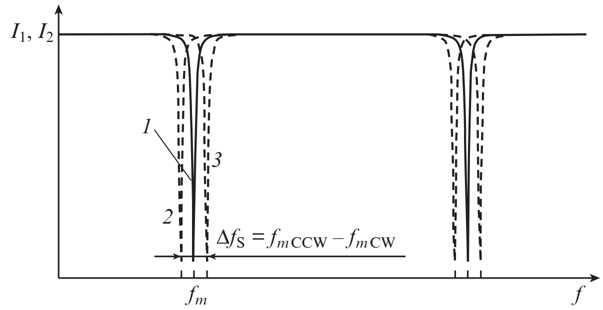

Figure 3. Intensities of the CW wave (I_1) and CCW wave (I_2) at the outputs of (*1*) nonrotating and (*2*, *3*) rotating resonator; curves (*2*) (I_1) and (*3*) (I_2) are a result of splitting of curve (*1*), due to the Sagnac effect.

It is clear that in the angular velocity sensors based on the Sagnac effect one can use ring waveguide resonators of any type instead of a WGM resonator [9]. However, the use of WGM resonators offers certain advantages. It is known that the higher the Q-factor of a ring resonator, the higher the limiting sensitivity of the corresponding angular velocity sensor [14]. To date, the Q-factor achieved in WGM resonators considerably exceeds that of the waveguide resonators comparable in size. The maximal Q-factor achieved in the calcium fluoride WGM resonator amounted to 10^{11} [15]. The Q-factor typical for WGM resonators is $\sim 10^9$. At the same time only for the best waveguide resonators, the Q-factor exceeds 10^6 [9]. Moreover, the use of WGM resonators instead of the waveguide ones allows the reduction of errors caused by polarisation noises and Faraday effect [12].

Despite the above advantages, the application of WGM resonators suffers from serious drawbacks. In contrast to single-mode waveguide resonators, the spectrum of a common WGM resonator is weakly degenerate and usually has a few modes with the frequencies close to the frequency of the used mode. The coupling of these modes is a source of error [11]. As a rule, the specific shape of the WGM resonators complicates their integration with other elements of the angular velocity sensor. For some types of WGM resonators (e.g., the spherical ones), it is difficult to provide a stable input and output of radiation under the conditions of their motion in the process of the angular velocity measurement. To prevent the displacement of the input/output element with respect to the resonator, the system is coated with an optical polymer having a low refractive index. Usually, this measure leads to the reduction of the Q-factor and, consequently, to lower sensitivity of the sensor [16]. It is known that the higher the power of input radiation, the higher the limiting sensitivity of the angular velocity sensor based on the Sagnac effect [14]. However, due to the ultrahigh Q-factor of WGM resonators and the fact that the radiation in them is concentrated in a strongly confined volume, an increase in power leads to the manifestation of various nonlinear effects, leading to additional errors in measurements. This circumstance imposes additional limitations on the input power of radiation and makes it unproductive to reduce the diameter of WGM resonators used in the above sensors to a few millimetres and smaller [12].

3. Angular velocity sensors based on morphologically dependent resonators

The centrifugal forces caused by the rotation of material objects can lead to mechanical deformation of the latter and to the growth of mechanical stress in their material. This is also valid for WGM resonators. The rotation can cause the change in the radius R of the operating cross section (due to the mechanical deformation) and the refractive index n (due to the mechanical stress) in WGM resonators. The WGM frequencies are known to be inversely proportional to the product of the operating cross section radius and the refractive index of the resonator material [2]. Thus, the effect of centrifugal forces gives rise to a reciprocal (similar for the opposite directions of the resonator roundtrip) shift of the resonator eigenfrequencies $\Delta f_{\rm c}$ (Fig. 4). It is worth noting that commonly the relative change in the refractive index is negligibly small as compared to the relative change in the radius, and its contribution to the WGM frequency shift can be neglected [8].

Recently, it has been proposed to use this effect for designing an angular velocity sensor based on a morphologically dependent resonator. The expression relating the reciprocal frequency shift caused by the rotation of a spherical WGM resonator about the axis, perpendicular to the operating cross section and passing through its centre (about the principal axis of sensitivity) was derived and experimentally tested [8]:

$$\Delta f_{\rm c} = f_m - f_{m\rm CW} = f_m - f_{m\rm CCW} = \frac{f_m \Delta R}{R_0}$$

$$= \frac{R_0^2 f_m \rho (17 + (6 - 5\nu)\nu)}{30G(1+\nu)(7+5\nu)} \Omega^2, \qquad (2)$$

where R_0 is the radius of the operating cross section in a nonrotating resonator; and ρ, ν and G are the density, Poisson coefficient, and shear modulus of the resonator material, respectively. By fixing the value of the mutual shift of the WGM resonator, one can determine the angular velocity of the resonator. Practically, by the example of soft polymer spherical WGM resonators with the radius 0.5 mm it was demonstrated that the reciprocal shift caused by the centrifugal forces has the value sufficient for detection [8]. However, it is clear that there are still a lot of problems related to the implementation of the angular frequency measurement using a reciprocal WGM frequency shift. First, they are due to the influence of multiple external factors (variations of the environment temperature, pressure, humidity, etc.). In this connection in the angular velocity sensors based on morphologically dependent resonators, it is necessary either to eliminate the influence of these factors, or to estimate and compensate for them in the calculation.

As mentioned above, Eqn (2) is valid only for a spherical WGM resonator, rotating about the principal axis of sensitivity. Consider how the fixation of the resonator at the rotating base affects its response and how the resonator reacts to the centrifugal forces in the case of rotation about the axis, perpendicular to the principal one. For this goal, we use the data modelled using the OOFELIE::Multiphysics suite.

An external view of the spherical WGM resonator model, generated using the OOFELIE::Multiphysics suite, is presented in Fig. 5a. In the modelling, we assumed that the light is launched into the resonator via an optical element, rigidly attached to the surface of the sphere along the thick line encircling it (Fig. 5a). The resonator model consists of a sphere with a radius of 5 mm mounted on a cylindrical stem with the radius varied in the process of modelling in the interval $r =$

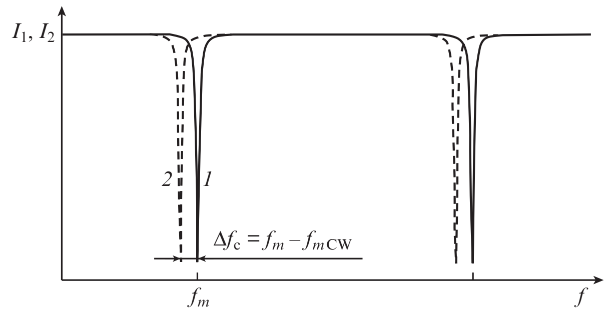

Figure 4. Intensities of CW (I_1) and CCW (I_2) waves at the outputs from (*1*) nonrotating and (*2*) rotating resonators; curve (*2*) is a result of the shift of curve (*1*) caused by the centrifugal forces.

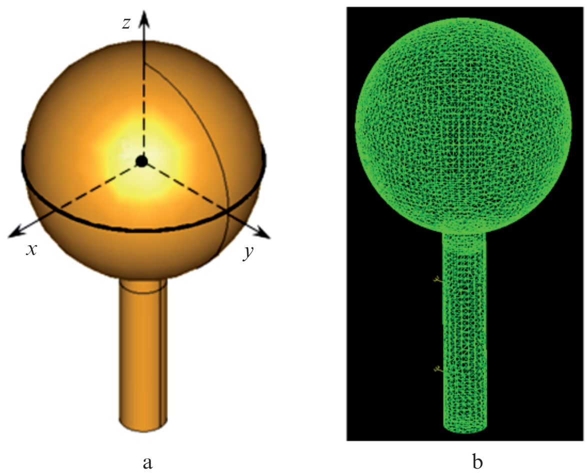

Figure 5. (a) External view of the spherical WGM resonator model and (b) the mesh of its finite elements.

[0.1; 4] mm. The resonator material was either the soft optical polymer polydimethylsiloxane (PDMS) with hardener (in the proportion 60 : 1), or fused silica. The material of the stem was fused silica. In the modelling, we used the material parameters presented in Table 1.

Table 1. Material parameters of WGM resonators.

Material	Density/ kg m^{-3}	Poisson coefficient	Shear modulus/Pa
Fused silica	2200	0.17	31×10^9
Calcium fluoride	3180	0.28	34×10^9
PDMS with hardener (60 : 1)	960	0.46	1×10^3

The outer surface of the stem was clamped.. It was sectioned into a mesh of finite elements as shown in Fig. 5b. The Cartesian system of coordinates was introduced such that its origin coincided with the geometric centre of the resonator and the z axis coincided with the symmetry axis of the resonator stem.

First, the verification of the constructed model was performed. The rotation of spherical WGM resonators about their principal axis of sensitivity (the z axis) was modelled for different angular velocities within the interval Ω = [–50; 50] rad s^{-1}. In this case, as shown in Fig. 6, the resonator was stretched in the horizontal plane (xy plane) and compressed along the rotation axis. For clarity, the contour along which the light propagates around the resonator is shown as a convex belt. The modelling has shown that in the case of rotation about the principal axis of sensitivity the path length of the light roundtrip increases, and the WGM frequency decreases (shifted towards lower values). Thus, qualitatively the results of the modelling correspond to the expectations.

At the second stage of verifying the model, we compared the WGM frequency shifts obtained from the results of modelling with those obtained from the analytical expression (2). By modelling, we determined the change in the light roundtrip path length. Using the fact that the relative variation of the path length causes an equal relative shift of the WGM frequency, we calculated the reciprocal WGM frequency shift. As seen from Fig. 7, the frequency shift, determined by the modelling, depends on the angular velocity in a way analogous to that given by Eqn (2). It is worth noting that in the derivation of Eqn (2) the attachment of the resonator to the base was not taken into account. Therefore, the smaller the radius of the resonator stem, the closer the spectral shift given by the modelling to that given by Eqn (2).

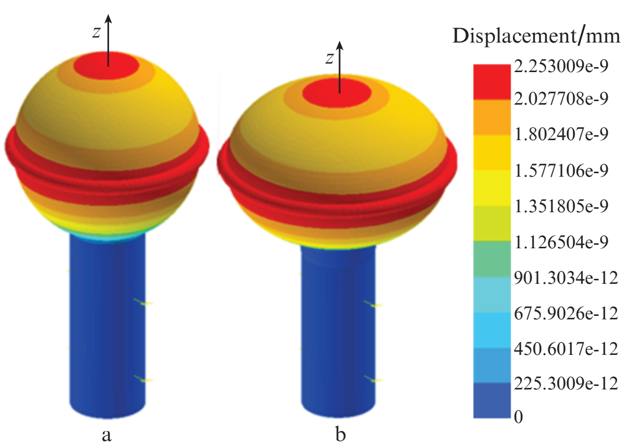

Figure 6. (Colour online) (a) Deformation of the resonator made of fused silica with a stem radius of 2 mm under the rotation about the z axis with the velocity 50 rad s^{-1} and (b) the same resonator much stronger deformed for clarity (compresses along the z axis and stretched in the horizontal plane).

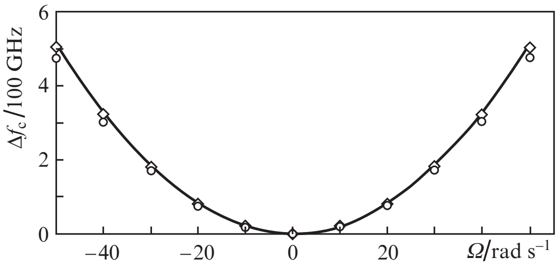

Figure 7. Eigenfrequency shifts in the spherical PDMS resonator due to its rotation about the z axis, calculated using Eqn (2) (solid curve) and the modelling results at a stem radius of (◇) 0.1 and (○) 4 mm.

Figure 8 presents the shifts of the spherical resonator eigenvalues obtained from the results of the modelling with different values of the stem radius. The increase in the spherical resonator stem radius reduces its reaction to rotation and, therefore, the sensitivity to the angular velocity in the devices on its base. The closer the stem radius to that of the resonator, the stronger the dependence of the sensitivity on the stem radius. From Fig. 8 it is also seen that under equal conditions the frequency shift caused by centrifugal forces in the resonators made of fused silica is almost by seven orders of magnitude smaller than that in the PDMS resonators. However, until now the size of fabricated high-Q PDMS resonators does not exceed 0.5 mm [8], and the accuracy of measuring the frequency shift in the resonators is proportional to their Q-factor and radius [14]. To date, in the spherical resonators made of fused silica the attainable Q-factor is by more than three orders of magnitude higher than that in the resonators made of PDMS (60 : 1) [17].

Then we modelled the action of centrifugal forces on a spherical resonator rotating about the x axis. Under such rotation, the circular contour along which the light travels around the modelled resonator turns into an ellipse (Fig. 9). Its minor axis corresponds to the axis of rotation (x axis) and the major axis is perpendicular to it (y axis). The higher the angular velocity, the greater the ellipse excentricity. The modelling has shown that the path length passed by the waves during one roundtrip is decreased. The WGM frequency of the spherical resonator shifts in the direction opposite to that corresponding to the rotation about the principal axis of sensitivity (z axis). Since in the measurement of the angular velocity the orientation of the rotation axis is generally unknown, the sensitivity to the angular velocity component perpendicular to the z axis leads to an uncertainty of the results of the angular velocity measurement using the reciprocal shift of WGM frequencies. It is expected that the use of triads of WGM resonators with mutually orthogonal operating cross sections would eliminate this drawback.

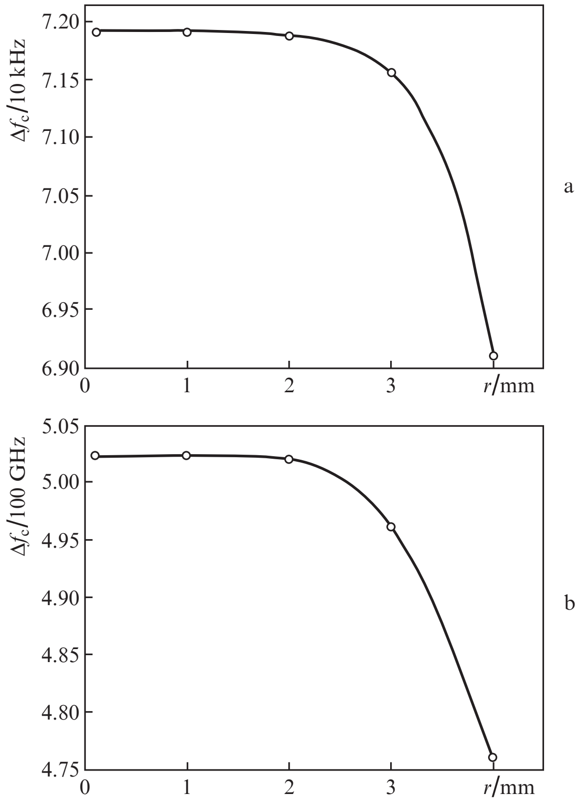

Figure 8. Eigenfrequency shift in spherical resonators made of (a) fused silica and (b) PDMS (60 : 1) rotating about the z axis with the angular velocity 50 rad s^{-1} and having different radii of their stems. The sphere radius is 5 mm.

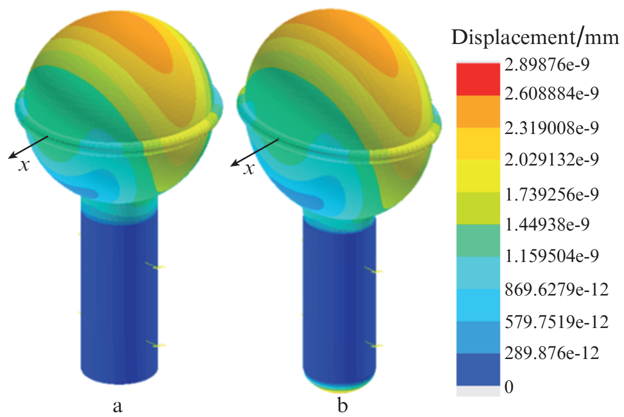

Figure 9. (Colour online) (a) Deformation of the resonator made of fused silica with a stem radius of 2 mm under the rotation about the x axis with the angular velocity 50 rad s^{-1} and (b) the same resonator essentially more deformed for clarity.

Now let us consider the effect of centrifugal forces on 'flat' (toroidal and disk-shaped) WGM resonators at different heights of the resonators and radii of stems. The models of 'flat' WGM resonators generated in the OOFELIE::Multiphysics suite are presented in Fig. 10a. The edge of the resonator with a radius of 5 mm was ended either by a torus with the radius of generating circle 10 μm, or by a wedge with the angle ~23°. The height of the resonator (its membrane) in the modelling varied within the interval h = [2; 9] μm, and the stem radius varied within the interval r = [0.5; 4] mm. The material of the resonator was fused silica. The soft optical polymer PDMS (60 : 1) was not considered as a material for the resonator, since to date the possibility of fabricating WGM resonators of this shape is not yet proved. As shown in Figs 10b and 10c, the models were sectioned into a mesh of finite elements. The Cartesian coordinate frame was introduced such that its origin coincides with the geometric centre of the resonator, and the z axis coincides with the symmetry axis of the resonator stem.

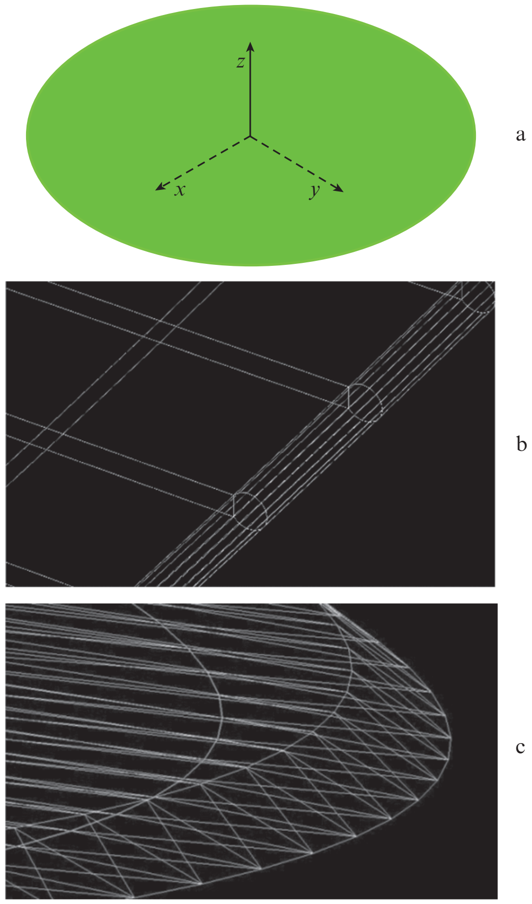

Figure 10. (a) Model of 'flat' (toroidal and disk-shaped) WGM resonators and meshes of finite elements for (b) the toroidal resonator and (c) the disk-shaped one.

Using the above model we studied the effect of centrifugal forces on the toroidal and disk-shaped resonator rotating about the principal axis of sensitivity (z axis) with the angular velocity Ω = 50 rad s^{-1}. Similar to spherical resonators, the rotation of 'flat' resonators (Fig. 11) about the principal axis of sensitivity increases the light roundtrip path length and decreases the WGM frequency (shifting the latter towards lower values). The response to the rotation is practically similar in disk-shaped and toroidal resonators of equivalent size, except a small quantitative difference. The qualitative discrepancy between these two resonators is observed only when their size (radius) is reduced to a few tens of microns. The practical detection of the rotation-induced spectral shift of eigenfrequencies in such resonators is still impossible; therefore, in the present paper the resonators of this size are not considered. In correspondence with the abovementioned

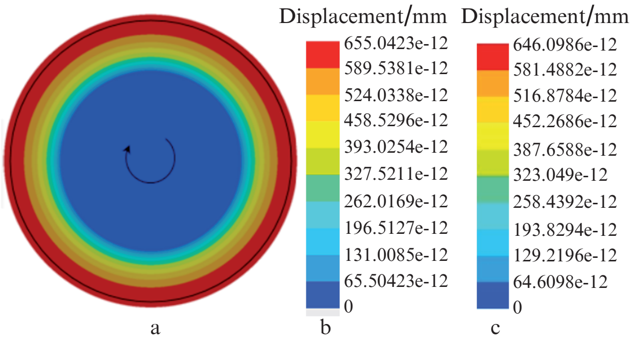

Figure 11. (Colour online) (a) Magnified displacements (the black circle is the edge of immobile resonator) in 'flat' (toroidal and disk-shaped) resonators having a height of 3 μm and a stem radius of 3 mm rotating about the z axis with the angular velocity 50 rad s^{-1}, and the displacement scales for (b) the toroidal resonator and (c) the disk-shaped one.

facts, the effect of centrifugal forces on 'flat' WGM resonators is considered below only by the example of toroidal resonators.

Figure 12 presents the values of eigenfrequency shifts in toroidal resonators with different heights and stem radii obtained from the results of the modelling. The increase in the stem radius and the resonator height reduces its response to the rotation. In contrast to the spherical resonator, the response of which weakly depends on the stem radius and when the latter is small enough ($r < 0.4R$) becomes practically constant (see Fig. 8), in toroidal resonator this dependence is more essential for any value of the stem radius (Fig. 12). For toroidal and disk-shaped resonators, the dependence of the spectral shift on the stem radius is closer to linear. An interesting feature of 'flat' resonators is the possibility to increase the sensitivity of devices on their basis to the centrifugal forces at the expense of reducing the resonator height (membrane thickness).

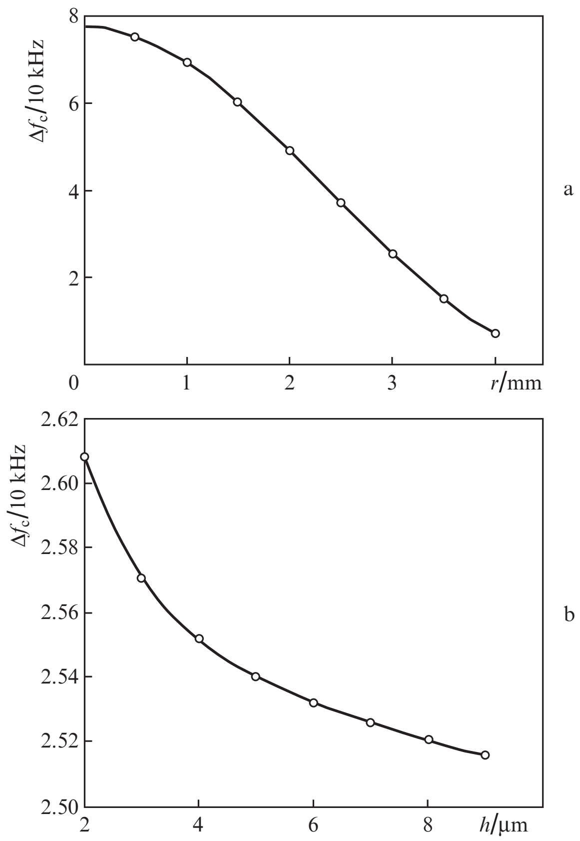

Figure 12. Eigenfrequency shift in toroidal resonators due to their rotation about the z axis with the angular velocity 50 rad s^{-1}, the resonator height 3 μm, and different radii of stems (a) and for the stem radius 3 mm and different resonator heights (b).

Next, we modelled the action of centrifugal forces on the toroidal resonator rotating about the x axis. In this case, similar to the spherical resonator, the circular contour of light roundtrip turns into an ellipse (Fig. 13). Its minor axis is parallel to the rotation axis (x axis) and the major axis is perpendicular to the rotation axis (y axis); the excentricity grows with increasing angular velocity. However, the modelling has shown that, in contrast to the spherical resonator, in the toroidal resonator this rotation increases the WGM roundtrip path length instead of decreasing it. Hence, the frequency of WGMs in the toroidal resonator shifts in the direction coincident with that for the rotation about the principal axis of sensitivity (z axis).

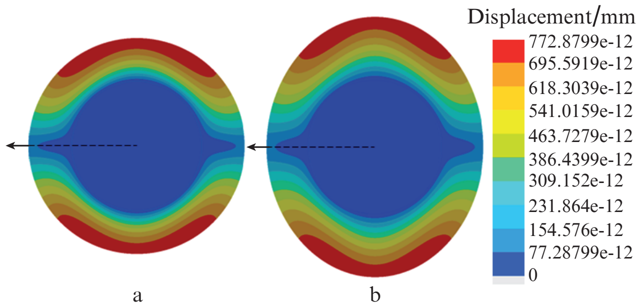

Figure 13. (Colour online) (a) Deformation of toroidal resonator having a height of 3 μm with a stem radius of 3 mm rotating about the x axis with the angular velocity 50 rad s^{-1} and (b) the same resonator much more deformed for clarity.

4. Comparison and mutual influence of the effects causing a frequency shift in the rotating resonator

Let us compare the frequency shifts caused by the effects described above. We restrict ourselves to considering only the shifts in spherical WGM resonators. The shift due to the Sagnac effect in the first approximation does not depend on the mechanical properties of the resonator material [see Eqn (1)]. However, as seen from Eqn (2), the value of the shift caused by centrifugal forces, on the contrary, is largely determined by the properties of the material. Therefore, to compare the effects, one should consider the WGM resonators made of different materials. It is known that spherical resonators are most often made of fused silica or calcium fluoride, and more seldom of optical polymers [13]. The main characteristics of the materials, necessary to compare the effects, are listed in Table 1.

Figure 14 presents the dependence of the WGM frequency shifts on the angular velocities for the resonators with the radius R_0 equal to 5 mm in a nonrotating resonator, obtained from Eqn (1) and Eqn (2). It is seen that at small angular velocities the influence of centrifugal forces is minimal. In this case the Sagnac frequency shift dominates. Thus, the devices that measure the angular velocities by the frequency shift caused by the centrifugal forces are practically not sensitive to slow rotations. They are also unable to determine the rotation direction, since the effect is reciprocal for the opposite directions of the resonator roundtrip [curves (*1*, *2*, *3*) in Fig. 14 are symmetric with respect to the zero angular velocity]. From Eqn (2) and Fig. 14, one can see that the softer and heavier the material of the resonator, the more essential the effect of the centrifugal forces on its radius. Using Eqn (2) we have found that at any size of the resonator the WGM frequency shift caused by centrifugal forces in the calcium fluoride resonators is by 15% greater than in the fused silica resonators. The shift of WGM frequencies in the resonators made of PDMS (60 : 1) is higher almost by seven orders of magnitude. Thus, the sensitivity of the devices measuring the angular velocity using the frequency shift caused by centrifugal forces can be essentially enhanced by using soft optical polymers. However, as mentioned above, the size and Q-factor of real polymer WGM resonators existing nowadays restrict their use, in spite of the sensitivity advantage. Moreover, at high angular velocities the resonators made of soft materials [e.g., PDMS (60 : 1)] can be torn to pieces. This feature essentially restricts their operation range. One can enhance the resistance of such resonators against the centrifugal forces by reducing their dimensions or increasing their rigidity (increasing the fraction of hardener in the polymer composition). However, in this case, naturally, the sensitivity to the angular velocity is reduced too.

The angular velocity sensors based on the Sagnac effect are free of these drawbacks. They are equally sensitive in the entire operating range of angular velocities and allow the detection of the rotation direction (the sign of the frequency shift is determined by the direction of rotation). However, with an increase in the angular velocity of WGM resonator rotation, the value of the frequency shift caused by the centrifugal forces grows faster than the Sagnac shift. At a certain value of the angular velocity Ω' the curves of the dependences in Fig. 14 intersect, and the shifts become comparable. Using Eqn (1) and Eqn (2), one can find this value of the angular velocity:

$$\Omega' = \frac{60G(1+\nu)(7+5\nu)}{R_0\rho c[17+(6-5\nu)\nu]}. \quad (3)$$

Figure 15 presents the dependences of Ω' on the size of the resonators made of different materials. In the area below the curves in Fig. 15, the Sagnac effect dominates, while above the curves the centrifugal forces begin to dominate. One can

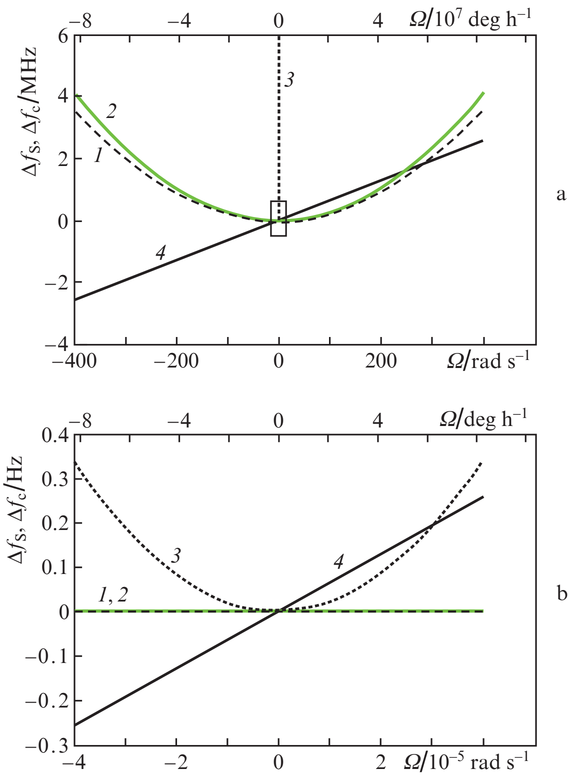

Figure 14. (a) Dependence of frequency shifts Δf_c (*1*, *2*, *3*) and Δf_S (*4*) on the angular velocity of rotation in the resonators with R_0 = 5 mm and (b) the magnified segment of these dependences (here and in Figs 16, 17 selected by a rectangle). Dependences (*1*, *2*, *3*) are plotted for the resonators made of fused silica, calcium fluoride and PDMS (60 : 1), respectively.

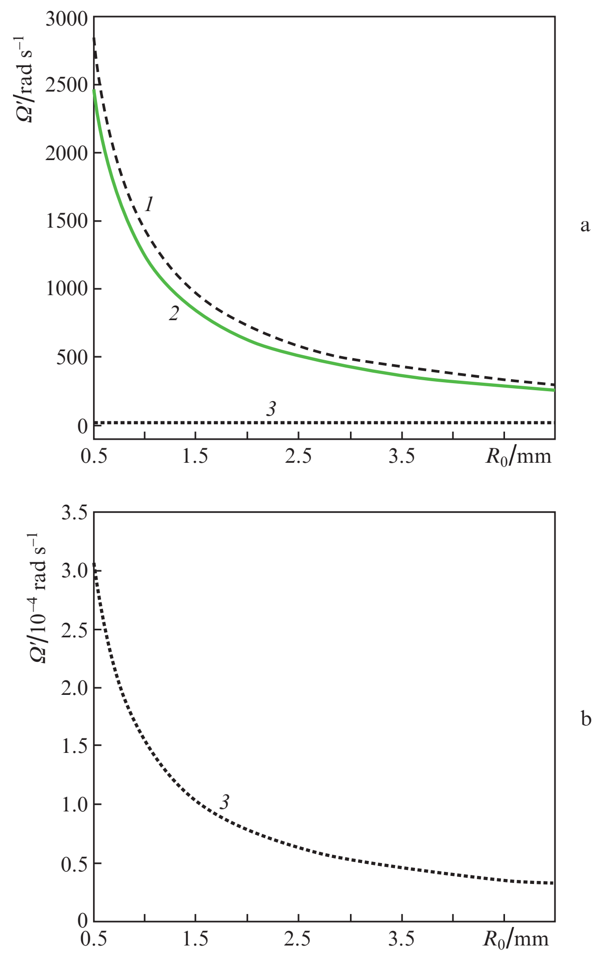

Figure 15. (a) Dependences of the angular velocity at which the frequency shifts caused by the Sagnac effect become equal to those caused by centrifugal forces on the resonator size, and (b) a segment of this dependence magnified in the direction of the ordinate axis. Dependences (*1*, *2*, *3*) are plotted for resonators made of fused silica, calcium fluoride and PDMS (60 : 1), respectively.

see that the softer and heavier the material and the greater the size of the resonator, the smaller the angular velocity, at which the frequency shifts of the WGM become equal.

If the angular velocity of rotation exceeds Ω', the devices based on the Sagnac effect are less sensitive than the devices based on the frequency shift caused by centrifugal forces. With a further increase in angular velocity, the frequency shift caused by centrifugal forces more and more exceeds the Sagnac one. In this connection, the idea of using both effects simultaneously for measuring angular velocities seems rather attractive. However, before considering possible ways of its implementation, one has to analyse how these phenomena can affect each other.

When the angular velocity is determined from the frequency shift caused by centrifugal forces, the radiation is launched into the WGM resonator via one end of the auxiliary waveguide, and from the other end of the waveguide it is coupled into a photodetector. The resonator is frequency scanned, and from the extremal signal at the photodetector the frequency is calculated for the WGM travelling clockwise or counterclockwise in the resonator (depending on the output at which the detector is installed). From the value of the frequency shift, the angular velocity is determined using Eqn (2) [8]. However, due to the influence of the Sagnac effect, the resulting total value of the frequency shift differs from that described by Eqn (2):

$$\Delta f_\Sigma = \Delta f_c \pm \frac{\Delta f_S}{2} = \frac{R_0^2 f_m \rho[17 + (6 - 5\nu)\nu]}{30G(1+\nu)(7+5\nu)}\Omega^2 \pm \frac{R_0 f_m}{c}\Omega, \quad (4)$$

where the sign '±' before the Sagnac shift is chosen according to the choice of the end of the coupling waveguide, where the detector is installed. As shown above, to measure the angular velocity using the frequency shift caused by centrifugal forces, it is preferable to employ soft polymer resonators. In this case, the Sagnac shift is, as a rule, relatively small and can be neglected. However, at small angular velocities, when the sensitivity of the considered device is low anyway, the Sagnac shift gives rise to a significant additional error (Fig. 16). It is clear that when the resonator size is reduced, the relative value of the Sagnac effect contribution to the total frequency shift grows and the error component caused by it increases. By modifying the method of the angular velocity measurement, one can exclude the additional error caused by the Sagnac effect. For this purpose it is required to measure the intensity of radiation at both outputs of the WGM resonator and record the frequencies of the WGMs, travelling around the resonator in the opposite directions, $f_{m\mathrm{CW}}$ and $f_{m\mathrm{CCW}}$. Then the WGM frequency shift due to the effect of centrifugal forces is determined from the expression $2\Delta f_c = 2f_m - (f_{m\mathrm{CW}} + f_{m\mathrm{CCW}})$.

In the design of devices using the Sagnac effect, it is necessary to eliminate the effect of centrifugal forces. It is interesting that as such the WGM frequency shift caused by them does not affect the angular velocity measurement. The explanation is that the centrifugal effect is reciprocal and is cancelled out in the calculation of the frequency difference (the Sagnac shift). Nevertheless, the centrifugal forces lead to the change in the WGM resonator radius [see Eqn (2)], which enters the Sagnac scaling factor [see Eqn (1)]. As a

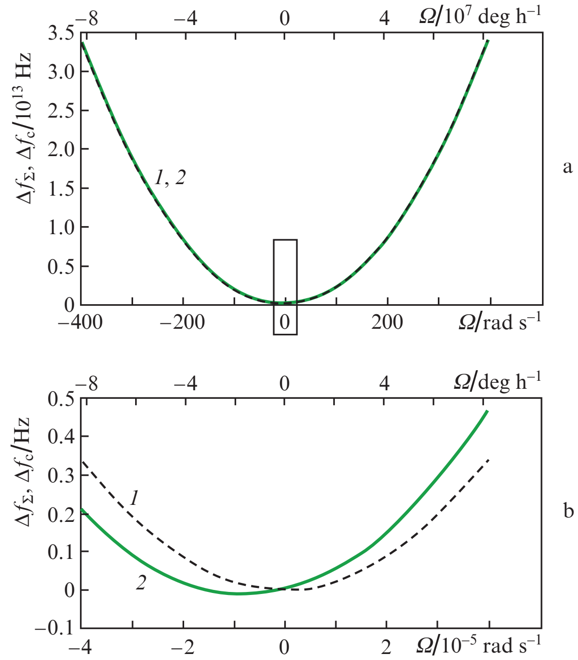

Figure 16. (a) WGM frequency shifts (*1*) Δf_c and (*2*) Δf_Σ vs. rotation angular velocity in the PDMS resonator (60:1) having a radius of 5 mm and (b) a fragment of these dependences magnified in the direction of the ordinate axis.

result, with the effect of centrifugal forces taken into account, the Sagnac shift appears to be determined by the expression:

$$\Delta f_S = \frac{2(R_0 + \Delta R) f_m}{c}\Omega = \frac{2R_0 f_m}{c}\Omega + \frac{2R_0^3 f_m \rho[17 + (6-5\nu)\nu]}{30G(1+\nu)(7+5\nu)c}\Omega^3.$$

Thus, the scaling factor becomes nonlinear. If we do not allow for the dependence of the radius on the angular velocity (due to centrifugal forces), then an additional systematic error appears, the amount of which is significant when using resonators made of soft optical polymers (Fig. 17). As already mentioned above, in the angular velocity measurement based on the Sagnac effect it is preferable to use rigid resonators (made of fused silica or calcium fluoride). In this case, the systematic error caused by the effect of centrifugal forces is small and becomes essential only at very high angular velocities (above 10^4 rad s^{-1}). If necessary, it can be eliminated. For this purpose, using the sum of frequencies of the counterpropagating WGMs one should calculate the frequency shift Δf_c; then, the correction ΔR to the radius of the WGM resonator can be calculated using Eqn (2).

Other ways of joint application of the Sagnac effect and the effect of centrifugal forces on the resonator morphology are possible. For example, it is possible to switch between the used effects at achieving a certain value of angular velocity depending on the material and size of the resonator. In other words, the idea is to perform the measurement using the Sagnac effect at small angular velocities and variable direction of rotation and then to switch to the measurement using

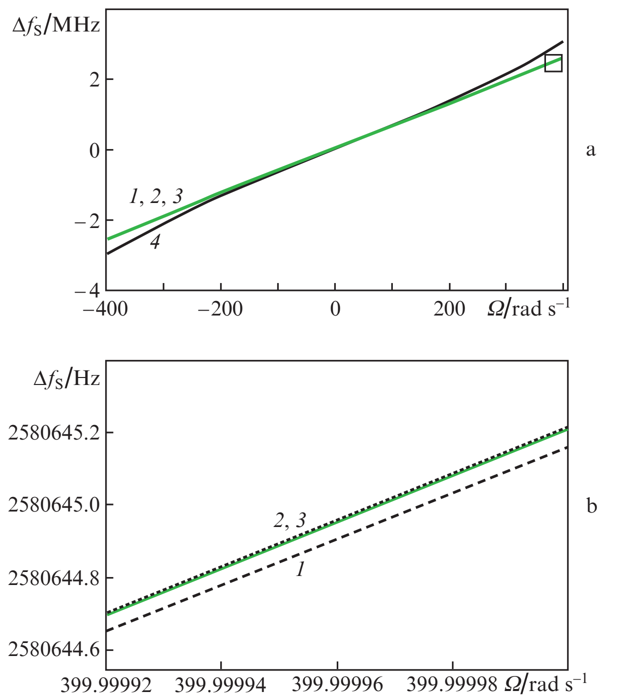

Figure 17. (a) Dependences of Sagnac shifts for the resonators having a radius of 5 mm on the rotation angular velocity and (b) a magnified fragment of the plot; (*1*) the frequency shift caused by the Sagnac effect without the centrifugal forces taken into account; (*2*, *3*, *4*) frequency shifts caused by the Sagnac effect with the effect of centrifugal forces taken into account for the resonators made of fused silica, calcium fluoride and PDMS (60 : 1), respectively.

different effects, when the angular velocity attains the value at which the frequency shifts caused by these effects become equal to those caused by the Sagnac effect [$|\Delta f_S| = \Delta f_c$ (see Fig. 15)]. At large angular velocities, one should perform the angular velocity measurements using the frequency shift due to the effect of centrifugal forces.

It is also possible to implement a simultaneous measurement of the angular velocity using both effects and the fact that one of the effects is reciprocal and the other effect is not. In this case, one should determine the Sagnac frequency shift from the WGM frequency difference, i.e., $\Delta f_S = f_{m\mathrm{CW}} - f_{m\mathrm{CCW}}$, and simultaneously calculate the frequency shift caused by centrifugal forces from the sum of WGM frequencies, i.e., $2\Delta f_c = 2f_m - (f_{m\mathrm{CW}} + f_{m\mathrm{CCW}})$. As a result, there will be two devices operating simultaneously, based on different physical effects, but using one resonator. This will allow one to exploit the advantages of both effects and enhance the accuracy of the angular frequency determination at the expense of the measurement data complexing.

5. Conclusions

In the present paper, the effects causing the mode frequency shift in WGM resonators due to their rotation are fully considered. These effects can be used to determine the angular velocity by the value of the WGM frequency shift. To date, the prototypes of such sensors are already developed. It is shown that the devices measuring the angular velocity by the reciprocal frequency shift (based on morphologically dependent WGM resonators), which is caused by centrifugal forces, are virtually insensitive to low velocities of rotation. They are also unable to determine the rotation direction. Instead, these devices are characterised by high sensitivity in the measurement of high angular velocities, particularly, when using soft polymer resonators. Besides that, their characteristics depend on the type of the used WGM resonator and its geometric parameters. The sensitivity of the angular velocity sensors based on morphologically dependent WGM resonators decreases with increasing resonator stem radius. For spherical resonators the influence of the stem radius sharply grows as the radius of the stem approaches that of the resonator. For toroidal and disk-shaped resonators the dependence of sensitivity to the rotation on the stem radius is closer to linear. In this case, the sensitivity to the angular velocity can be increased by reducing the resonator thickness. On the contrary, the devices for the angular velocity measurement based on the Sagnac effect are equally sensitive within the entire range of angular velocities and allow the determination of the rotation direction. In the first approximation, their accuracy characteristics do not depend on the WGM resonator shape and material. It is worth noting that at high rotation velocities their sensitivity is lower than that of the angular velocity sensors based on morphologically dependent resonators.

It is shown that the phenomena described above can affect each other. As a result, an additional systematic error can arise in the measurement of the angular velocity caused by the influence of a parasitic effect (not taken into account). In the angular velocity sensors based on morphologically dependent resonators, it is better to use soft resonators. In this case, the Sagnac WGM shift, as a rule, is relatively small and can be neglected. However, in the measurement of small angular velocities, when the sensitivity of these devices is anyway reduced, the Sagnac shift causes a significant additional error. Moreover, when the resonator size is decreased, the relative value of the contribution from the Sagnac effect into the total frequency shift increases, and the error component caused by it grows. In the paper, we show that the influence of the Sagnac effect can be algorithmically compensated for, if necessary. In the devices based on the Sagnac effect, the centrifugal forces can affect the scaling coefficient. When using sufficiently rigid resonators, the arising systematic error is small and becomes significant only at very high angular velocities (above 10^4 rad s^{-1}). In conclusion, we would like to note that there are different ways of combined usage of both effects for measuring the angular velocity, which are planned to be developed and practically used in future.

Acknowledgements. The authors are grateful to the Russian Foundation for Basic Research for supporting this work (Project No. 16-02-01002). V.Yu. Venediktov is an executor of research within the State Task 'Organisation of Scientific Research'. E.V. Shalymov thanks Saint Petersburg State Electrotechnical University "LETI" for support within the project 'Organisation and Conduction of the Competition of Research Projects by Students, Postgraduates, and Young Researchers and Educators'.

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