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Generalised quantised spiral beams

V.G. Volostnikov

Abstract. **A generalisation is found of previously known quantised spiral beams in the form of closed non-self-intersecting plane curves. An analytic expression is obtained for the distributions of the complex amplitudes of the fields of these beams. It is shown that the intensity of such light fields can reach zeros on the generating curve and the generalisation allows the number of zeros to be controlled inside the closed curves, which is important for applied problems. The issue of the extremal properties of the orbital angular momentum of generalised quantised spiral beams is investigated.**

Keywords: *coherent optics, spiral light beams, optical vortices.*

1. Introduction

Spiral beams are light fields that retain their intensity distribution to within scale and rotation during their propagation and focusing. These light fields are modes of specific laser cavities with field rotation. Work [1, 2] considered spiral beams, whose intensity distributions in planes orthogonal to the propagation direction have the form of plane closed curves. These beams will be called bundles in the form of closed curves. Figure 1 shows an example of the initial generating curve and the distribution of the intensity and phase of the corresponding beam.

It was also shown [1, 2] that such light fields admit the existence of a certain quantisation condition relating the area under the curve and the Gaussian beam parameter. Quantised beams have found wide application in various problems of analysis and synthesis of light fields [2]. The characteristic properties of these beams are, firstly, the absence of zeros of the intensity on the generating curve, and secondly, the independence of the intensity distribution from the choice of the initial point on this curve.

Figure 1. Generating curve and its corresponding distributions of the intensity and phase of the spiral beam.

V.G. Volostnikov Samara Branch, P.N. Lebedev Physical Institute, Russian Academy of Sciences, ul. Novo-Sadovaya 221, 443011 Samara, Russia; e-mail: coherent@fian.smr.ru

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The distribution of the complex amplitude of the spiral light fields contains, as a rule, optical vortices, or wave front dislocations [3], and they have a nonzero orbital angular momentum (OAM). Light fields with a nonzero OAM attract attention of opticians of the most diverse directions. For example, an international conference was held in 2017, the main topic being devoted to such light fields [4]. For example, Vallone [5] studied the issues of the formation of light fields with a nonzero OAM and discussed various applications of these fields, Wang [6] considered the problem of quantum information processing, and Banzer [7] described the application of the fields with a nonzero OAM for ultrahigh resolution microscopy.

The authors of Refs [1, 2] presented analytical expressions for the complex amplitudes of the beam fields in the form of closed curves and obtained conditions for their quantisation. One of the obvious methods of experimental realisation of such fields is the calculation and formation of amplitude and phase masks, as well as illumination of their 'sandwich' by a uniform-intensity beam.

The aim of this paper is to demonstrate that for closed curves one can obtain some generalisation of spiral light fields. These generalised fields possess properties that are both known for and different from ordinary quantised spiral fields.

2. Quantised spiral beams

Ordinary quantised beams have a characteristic property consisting in the fact that there are no zeros of the complex amplitude of the field $\zeta(t) = \xi(t) + \mathrm{i}\eta(t)$ on the generating curve $S(z, z^*|\zeta(t), t \in [0, T])$ (see Appendix) [1, 2]:

$$|S(z, z^*|\zeta(t), t \in [0, T])| \neq 0, \quad z = x + \mathrm{i}y,$$

where T is the period.

There are various formulations of this statement. Suppose, for example, that for some closed curve the following conditions hold [1]:

1) the curve starts and ends at the origin:

$$x(0) = x(T) = 0, \quad y(0) = y(T) = 0;$$

2) the function $\dot{\sigma}(t)$ is sign-constant (which is equivalent to the absence of self-intersecting curves), where σ is the area bounded by the contour $\zeta(t)$;

3) the area of the region bounded by the curve satisfies the quantisation condition

$$2\int_0^t [x(\tau)\dot{y}(\tau) - \dot{x}(\tau)y(\tau)]\mathrm{d}\tau = 4\sigma = 2\pi n,$$

where $n = 0, 1, 2, \ldots$ is also equal to the number of zeros inside the curve $\zeta(t)$.

Then, the following inequality holds:

$$I = \int_0^T \exp[-x^2(t) - y^2(t)]\cos\left\{2\int_0^t [x(\tau)\dot{y}(\tau) - \dot{x}(\tau)y(\tau)]\mathrm{d}\tau\right\}\times$$

$$\times\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\,\mathrm{d}t \neq 0.$$

Let us show this. Integrating this expression, we obtain

$$\begin{aligned} I = L &+ \int_0^T \exp[-x^2(t) - y^2(t)] \\ &\times\cos\left\{2\int_0^t [x(\tau)\dot{y}(\tau) - \dot{x}(\tau)y(\tau)]\mathrm{d}\tau\right\} \\ &\times l(t)[2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)]\mathrm{d}t \\ &+ \int_0^T \exp[-x^2(t) - y^2(t)]\sin\left\{2\int_0^t [x(\tau)\dot{y}(\tau) - \dot{x}(\tau)y(\tau)]\mathrm{d}\tau\right\} \\ &\times l(t)[2x(t)\dot{y}(t) - 2\dot{x}(t)y(t)]\mathrm{d}t, \end{aligned} \tag{1}$$

where L is the total length of the curve, and $l(t)$ is its current length. Transforming (1) in the standard way, we obtain the expression

$$\begin{aligned} I = L &+ \int_0^T \exp[-x^2(t) - y^2(t)] \\ &\times R(t)\cos\left\{\left\{2\int_0^t [x(\tau)\dot{y}(\tau) - \dot{x}(\tau)y(\tau)]\mathrm{d}\tau\right\}\right. \\ &\left. - \arccos\left[\frac{\dot{R}(t)}{\dot{l}(t)}\right]\right\}2l\dot{l}(t)\mathrm{d}t. \end{aligned} \tag{2}$$

Here $\dot{R}(t) = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$.

The function $2l\dot{l}(t)$ has a constant sign; therefore, according to the integral mean value theorem, expression (2) takes the form:

$$\begin{aligned} I = L &+ R(t_1)\exp[-x^2(t_1) - y^2(t_1)] \\ &\times\cos\left\{\left\{2\int_0^{t_1} [x(t)\dot{y}(t) - \dot{x}(t)y(t)]\mathrm{d}t\right\}\right. \\ &\left. - \arccos\left[\frac{\dot{R}(t_1)}{\dot{l}(t_1)}\right]\right\}L^2, \end{aligned} \tag{3}$$

where $t_1 \in [0, T]$. Taking L through the parentheses, we obtain the relation

$$I = L\Big(1 + R(t_1)L\exp[-x^2(t_1) - y^2(t_1)]\times$$

$$\times\cos\left\{\left\{2\int_0^{t_1} [x(t)\dot{y}(t) - \dot{x}(t)y(t)]\mathrm{d}t\right\} - \arccos\left[\frac{\dot{R}(t_1)}{\dot{l}(t_1)}\right]\right\}\Big). \tag{4}$$

Now let $\zeta_1(t)$ be the quantised curve for $n = 1$; then, $\sqrt{n}\,\zeta_1(t)$ is the n-quantised curve, and expression (4) takes the form

$$I = L\Big(1 + R(t_1)Ln\exp[-nx^2(t_1) - ny^2(t_1)]$$

$$\times\cos\left\{\left\{2\int_0^{t_1} [x(t)\dot{y}(t) - \dot{x}(t)y(t)]\mathrm{d}t\right\} - \arccos\left[\frac{\dot{R}(t_1)}{\dot{l}(t_1)}\right]\right\}\Big).$$

Let us investigate the expression $R(t)Ln\exp[-nx^2(t_1) - ny^2(t_1)]$. It can be seen that, starting from some n, it becomes less than unity, and then relation (4) cannot be equal to zero. Many numerical experiments have shown that this property is typical of quantised beams with any value of the quantisation parameter n. Unfortunately, no strict proof of this fact has been obtained so far.

Now let the curve be self-intersecting. Without loss of generality, we choose the position of the curve and its parametrisation so that the origin of the curve coincides with the origin and the self-intersection point (Fig. 2). It follows from [1] that other cases can be reduced to the above-mentioned ones. Thus, $\zeta(0) = \zeta(t_0) = \zeta(T) = 0$, where t_0 is the coordinate of the self-intersection point.

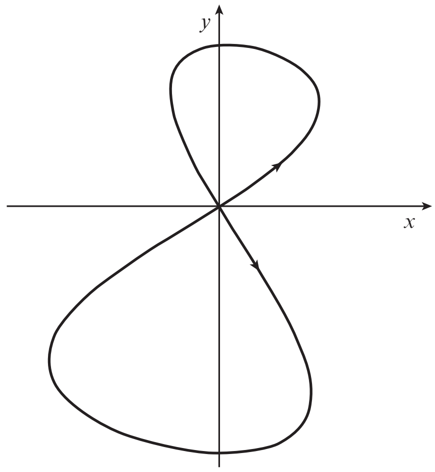

Figure 2. Appearance of a self-intersecting curve.

In this case, since $\dot{\sigma}(t)$ is a sign-variable function (the round-trip direction changes), expression (2) will be the sum of two integrals:

$$\begin{aligned} I = L &+ \int_0^{t_0} \exp[-x^2(t) - y^2(t)] \\ &\times R(t)\cos\left\{\left\{2\int_0^t [x(\tau)\dot{y}(\tau) - \dot{x}(\tau)y(\tau)]\mathrm{d}\tau\right\}\right. \\ &\left. - \arccos\left[\frac{\dot{R}(t)}{\dot{l}(t)}\right]\right\}2l\dot{l}(t)\mathrm{d}t + \int_{t_0}^T \exp[-x^2(t) - y^2(t)] \\ &\times R(t)\cos\left\{\left\{2\int_0^t [x(\tau)\dot{y}(\tau) - \dot{x}(\tau)y(\tau)]\mathrm{d}\tau\right\}\right. + \end{aligned}$$

$$+\arccos\left[\frac{\dot{R}(t)}{\dot{l}(t)}\right]\right\}2l\dot{l}(t)\,\mathrm{d}t.$$

Each integral, like the case described by expression (3), becomes modulo less than unity for some n, which is no longer possible to say about the sum. This is the reason for the absence of a constant sign of the function $\dot{\sigma}(t)$. Therefore, it is impossible to judge the absence or presence of zeros on the generating curve.

3. Generalised quantised spiral beams

Let

$$S(z,z^*|\zeta(t),t\in[0,T]) = \exp\left(-\frac{zz^*}{\rho^2}\right)f\left(\frac{z}{\rho}\right) \tag{5}$$

be a quantised spiral beam in the form of the curve $\zeta(t)$. We introduce the parameter A; then,

$$S^{(A)}(z,z^*|\zeta(t),t\in[0,T]) = \exp\left(-\frac{zz^*}{\rho^2}\right)f\left(\frac{Az}{2\rho}\right) \tag{6}$$

is a generalised quantised beam, and the zeros of its field will be at points

$$\{z_j^{(A)}\} = z_j(2/A), \tag{7}$$

where z_j are the zeros of the function $f(z/\rho)$.

Without loss of generality, we assume that A is positive and real. It is seen from (6) and (7) that for a closed curve there is always an A (if $z_j \neq 0$) such that $z_j(2/A) = \zeta(t_j)$. This will happen (Fig. 3) if the two conditions:

$$\frac{2\pi t_j}{T} = \arg z_j = \arg z_j^{(A)},$$

$$|z_j^{(A)}| = |\zeta(t_j)| \tag{8}$$

are met. It is obvious that as the parameter A decreases, zero vanishes from the region of the closed curve. Figure 4 shows the results of numerical simulation of the intensity and phase distributions of spiral beams for different values of A. From the comparison of Figs 1 and 4 it is clear that the number of zeros inside the curve is really different for different values of the parameter A. This means that the rigid relationship between the number of zeros inside the generating curve and the quantisation parameter vanishes.

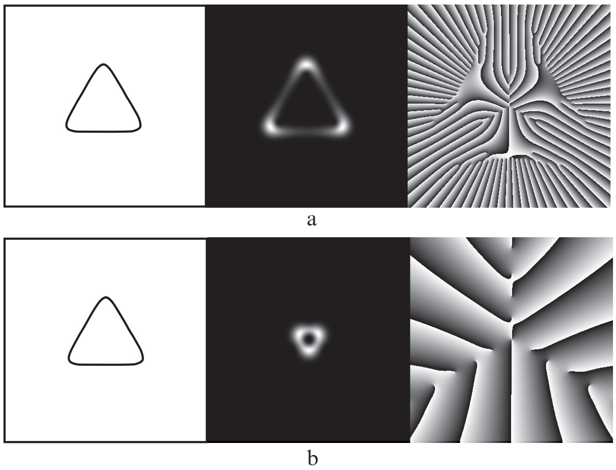

Figure 4. Generating curve and its corresponding distributions of the intensities and phases of the spiral beams for A = (a) 2.2 and (b) 1.

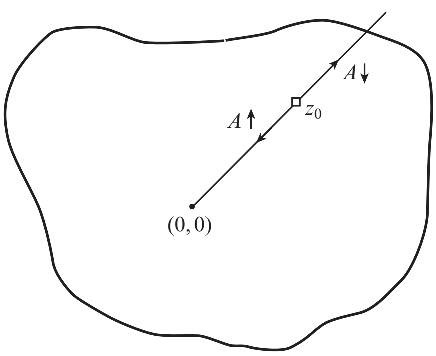

Figure 3. Scheme of motion of zeros with changing the parameter A.

The number of zeros inside the curve is an important characteristic for applied problems. It sets the OAM value for optical manipulation of microscopic objects. When processing contour images [8], it is essential to determine the angle of rotation of the recognisable contour with respect to contours from the database. To do this, in the case of ordinary quantised beams, curves with a small value of the quantisation parameter are taken, when the number of zeros inside the corresponding curve is small: according to Vieta's theorem, the expansion coefficients of the entire function $f(z)$ are found more accurately, and therefore the angle of rotation is also obtained more accurately. Thus, changing the parameter A is an alternative way of changing the number of zeros inside the curve.

The generalisation of (A1) in Appendix will be the formula

$$S^{(A)}(z,z^*|\zeta(t),t\in[0,T]) = \exp\left(-\frac{zz^*}{\rho^2}\right)$$
$$\times\int_0^T \exp\left\{-\frac{\zeta(t)\zeta^*(t)}{\rho^2} + \frac{2z(A/2)\zeta^*(t)}{\rho^2}\right.$$
$$\left.+\frac{1}{\rho^2}\int_0^t[\zeta^*(\tau)\dot{\zeta}(\tau) - \zeta(\tau)\dot{\zeta}^*(\tau)]\mathrm{d}\tau\right\}|\dot{\zeta}(t)|\,\mathrm{d}t. \tag{9}$$

It is characteristic that the quantisation condition for (9) will be the same as for (A1). Indeed, by writing conditions (A2), (A3) and (A4) for (9), we obtain an equality equivalent to (A7) with the substitutions $z \to z(A/2)$, $\Phi(a) \to \Phi(a,A)$ and $F_1(a)$, $F_2(a) \to F_1(a,A)$, $F_2(a,A)$. Thus, we find the quantisation condition equivalent to (A7).

4. OAM of generalised quantised spiral beams

The generalised quantised spiral beams depend on the parameter A, and so it makes sense to investigate the extremum conditions for their specific OAM.

It is known [1, 2] that for ordinary spiral beams the value of the specific OAM is determined by the expression ($\rho = 1$, $n = 0$)

$$L_1 = \frac{\sum_{m=0}^{\infty} m \frac{m!}{2^m} |c^1_m|^2}{\sum_{m=0}^{\infty} \frac{m!}{2^m} |c^1_m|^2}, \quad (10)$$

where c^1_m are the coefficients of the expansion of the spiral beam in the Laguerre–Gauss modes LG_{0m}.

For generalised spiral beams, the specific OAM

$$L^{(1)}_A = \frac{\sum_{m=0}^{\infty} m \frac{m!}{2^m} \frac{A^{2m}}{2^{2m}} |c^1_m|^2}{\sum_{m=0}^{\infty} \frac{m!}{2^m} \frac{A^{2m}}{2^{2m}} |c^1_m|^2}. \quad (11)$$

For the sake of simplicity, we make a substitution

$$\frac{m!}{2^{3m}} |c^1_m|^2 = |c_m|^2, \; L_A = \sum_{m=0}^{\infty} m A^{2m} |c_m|^2, \; E_A = \sum_{m=0}^{\infty} A^{2m} |c_m|^2. \quad (12)$$

Now the extremum condition for the specific OAM will have the form

$$\frac{\mathrm{d}}{\mathrm{d}A}\left(\frac{L_A}{E_A}\right) = 0 \Rightarrow \frac{\mathrm{d}L_A}{\mathrm{d}A} E_A - \frac{\mathrm{d}E_A}{\mathrm{d}A} L_A = 0. \quad (13)$$

It is clear from (13) that

$$\frac{\mathrm{d}E_A}{\mathrm{d}A} = \sum_{m=0}^{\infty} 2mA^{2m-1} |c_m|^2 = \frac{2L_A}{A}. \quad (14)$$

Then from (7) and (8) we obtain the relations:

$$E_A = \frac{L_A \mathrm{d}E_A/\mathrm{d}A}{\mathrm{d}L_A/\mathrm{d}A} = \frac{2L_A^2}{A \mathrm{d}L_A/\mathrm{d}A},$$
$$\frac{A}{2} \frac{E_A}{\mathrm{d}A} = \sum_{m=0}^{\infty} m A^{2m} |c_m|^2 = L_A. \quad (15)$$

From (15), after simple transformations, we find the extremum condition in terms of E_A or L_A:

$$\frac{\mathrm{d}^2 E_A}{\mathrm{d}A^2} A E_A + \frac{\mathrm{d}E_A}{\mathrm{d}A} E_A - A\left(\frac{\mathrm{d}E_A}{\mathrm{d}A}\right)^2 = 0,$$
$$\frac{\mathrm{d}^2 L_A}{\mathrm{d}A^2} A L_A + \frac{\mathrm{d}L_A}{\mathrm{d}A} L_A - A\left(\frac{\mathrm{d}L_A}{\mathrm{d}A}\right)^2 = 0. \quad (16)$$

These ordinary second-order nonlinear differential equations can easily be reduced to first-order equations by using substitutions

$$u = \frac{\mathrm{d}E_A}{\mathrm{d}A}/E_A, \quad v = \frac{\mathrm{d}L_A}{\mathrm{d}A}/L_A. \quad (17)$$

In this case we obtain the relations

$$\frac{\mathrm{d}}{\mathrm{d}A}\left(A \frac{\mathrm{d}E_A}{\mathrm{d}A}/E_A\right) = 0, \; \frac{\mathrm{d}}{\mathrm{d}A}\left(A \frac{\mathrm{d}L_A}{\mathrm{d}A}/L_A\right) = 0. \quad (18)$$

Returning now to formulas (8) and (9), we find the expressions

$$\frac{A}{2} \frac{\mathrm{d}E_A}{\mathrm{d}A} = \sum_{m=0}^{\infty} m A^{2m} |c_m|^2, \; \frac{A}{2} \frac{\mathrm{d}L_A}{\mathrm{d}A} = \sum_{m=0}^{\infty} m^2 A^{2m} |c_m|^2. \quad (19)$$

Then condition (9) takes the form

$$\sum_{m=0}^{\infty} m^2 A^{2m} |c_m|^2 \sum_{m=0}^{\infty} A^{2m} |c_m|^2 - \left(\sum_{m=0}^{\infty} m A^{2m} |c_m|^2\right)^2 = 0. \quad (20)$$

Thus, formula (20) is the difference of products of series.

We use the obvious identity:

$$\sum_{k=0}^{\infty} a_k \sum_{k=0}^{\infty} b_k \equiv \sum_{k=0}^{\infty} b_k \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} d_k. \quad (21)$$

According to [9], the terms of the last sum in (21) can be represented in the form

$$d_k = \sum_{l=0}^{k} a_l b_{k-1} = \sum_{l=0}^{k} b_l a_{k-1}. \quad (22)$$

Then, using formulas (18) and (19) for the terms on the left-hand side of (21), we obtain the relation

$$d_m = \frac{1}{2} \sum_{k=0}^{m} A^{2m} |c_{m-k}|^2 |c_k|^2$$
$$\times [(m-k)^2 + k^2 - 2(m-k)k] > 0. \quad (23)$$

A strict inequality is due to the fact that the square brackets contain a doubled difference between the arithmetic mean and the geometric mean of two integers.

Equality (20) will be valid only if one summand ($m = m_0$) is present in the sums. In this case, from (16) and (22) we find the expressions

$$\frac{A}{2} \frac{\mathrm{d}E_A}{\mathrm{d}A}/E_A = \frac{A}{2} \frac{\mathrm{d}L_A}{\mathrm{d}A}/L_A = m_0, \;\; L_A/E_A = m_0. \quad (24)$$

In other cases, the ratio L_A/E_A increases monotonically with increasing A, and conditions (16) and (24) are satisfied only for spiral beams constructed for circles centred at the origin. Of course, this is in complete agreement with (7) for $z_j = 0$ ($j = 0, 1, 2, \ldots$).

5. Conclusions

Thus, a new method for constructing quantised spiral beams with a different number of zeros inside the generating curve and a different value of the OAM is found. Extremal properties of the specific OAM of the light fields of such beams are also studied. Changing the parameter A is an alternative way to changing the number of zeros inside a closed curve, which is important in solving the problem of recognising contour images using spiral beams [8]. From the point of view of speed, the change in the quantisation parameter n and the introduction of the parameter A are equivalent approaches. A comparative analysis of both approaches requires a further study.

Appendix

According to [1, 2], the complex amplitude of the field of a spiral beam in the form of a plane (generating) curve $\zeta(t) = \xi(t) + \mathrm{i}\eta(t)$ in the waist can be represented in the form

$$S(z, z^* | \zeta(t), t \in [0, T]) = \exp\left(-\frac{zz^*}{\rho^2}\right) \times$$

$$\times\int_0^T \exp\left\{-\frac{\zeta(t)\zeta^*(t)}{\rho^2}+\frac{2z\zeta^*(t)}{\rho^2}\right.$$
$$\left.+\frac{1}{\rho^2}\int_0^t[\zeta^*(\tau)\dot\zeta(\tau)-\zeta(\tau)\dot\zeta^*(\tau)]\mathrm{d}\tau\right\}|\dot\zeta(t)|\mathrm{d}t, \qquad (\mathrm{A1})$$

$$z=x+\mathrm{i}y.$$

Let us find the condition under which the intensity distributions of the spiral beams constructed for the closed curves $\zeta(t)$ and $\zeta(t+a)$ coincide:

$$|S(z,z^*|\zeta(t),t\in[a,a+T])|\equiv|S(z,z^*|\zeta(t),t\in[0,T])|. \quad (\mathrm{A2})$$

This identity can be written in the form

$$\exp[\mathrm{i}\Phi(a)]S(z,z^*|\zeta(t),t\in[a,a+T])$$
$$\equiv S(z,z^*|\zeta(t),t\in[0,T]), \qquad (\mathrm{A3})$$

where $\Phi(a)$ is some real function that does not depend on z [otherwise, dividing both sides of (A3) by the Gaussian function, we obtain that Φ is an analytic function of z and, consequently, cannot be a real function for all z]. Differentiating (A3) with respect to a and using the periodicity of $\zeta(t)$, we find the expression

$$\exp[-\mathrm{i}\Phi(a)]S(z,z^*|\,\zeta(t),t\in[a,a+T])$$
$$\times\left[\mathrm{i}\dot\Phi(a)-\frac{\zeta^*(a)\dot\zeta(a)-\zeta(a)\dot\zeta^*(a)}{\rho^2}\right]$$
$$+\exp\left[\mathrm{i}\Phi(a)-\frac{zz^*-2z\zeta^*(a)+\zeta(a)\zeta^*(a)}{\rho^2}\right]$$
$$\times\left\{\exp\left[\frac{1}{\rho^2}\int_0^T[\zeta^*(\tau)\dot\zeta(\tau)-\zeta(\tau)\dot\zeta^*(\tau)]\mathrm{d}\tau\right]-1\right\}|\dot\zeta(a)|=0. \qquad (\mathrm{A4})$$

Replacing the spiral beam in the first term in (A4) in accordance with (A3) and dividing the result by the Gaussian function, we rewrite equation (A4) in a symbolic form:

$$f(z)F_1(a)+\exp\left[\frac{2z\zeta^*(a)}{\rho^2}\right]F_2(a)=0, \qquad (\mathrm{A5})$$

where $f(z)$ is an entire analytic function; and $F_1(a)$ and $F_2(a)$ are some functions of a. This equality holds for all z and a only for $F_1(a)=F_2(a)\equiv 0$ [if $f(z)$ has a zero, this is obvious; the case where $f(z)$ does not have zeros is also simple]. Therefore,

$$\Phi(a)=\frac{1}{\mathrm{i}\rho^2}\int_0^a[\zeta^*(\tau)\dot\zeta(\tau)-\zeta(\tau)\dot\zeta^*(\tau)]\mathrm{d}\tau,$$
$$\exp\left\{\frac{1}{\rho^2}\int_0^T[\zeta^*(\tau)\dot\zeta(\tau)-\zeta(\tau)\dot\zeta^*(\tau)\mathrm{d}\tau]\right\}=1. \qquad (\mathrm{A6})$$

Hence the quantisation condition has the form

$$\frac{1}{\mathrm{i}\rho^2}\int_0^T[\zeta^*(\tau)\dot\zeta(\tau)-\zeta(\tau)\dot\zeta^*(\tau)]\mathrm{d}\tau=\frac{4\sigma}{\rho^2}=2\pi n. \qquad (\mathrm{A7})$$

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