Quantum Electronics **48** (6) 532–536 (2018)

EXTREMELY SHORT PULSES

https://doi.org/10.1070/QEL16619

Propagation of a light pulse with a duration of less than one period in a resonant amplifying medium

R.M. Arkhipov, M.V. Arkhipov, I. Babushkin, A.V. Pakhomov, N.N. Rosanov

Abstract. **The propagation of a unipolar light pulse with a duration of less than one period in a two-level resonant amplifying medium is studied theoretically. In the process of amplification, the unipolar pulse becomes bipolar. The effect of the relaxation time on the shape and duration of the amplified pulse is demonstrated. It is found that the electric pulse area (integral of electric field strength with respect to time) is conserved, in contrast to the area under the long pulse envelope (integral of slowly varying field amplitude with respect to time), which, in the case of long pulses, satisfies the McCall–Hahn area theorem.**

Keywords: *subcycle pulses, attosecond pulses, attosecond science, coherent effects, amplification of unipolar pulses, electric pulse area.*

1. Introduction

Presently, extremely short pulses (ESPs) with a duration on the order or even less than the period of light wave oscillations have been obtained [1–5]. The generation of such pulses has led to an active study of their interaction with matter [6–11]. Short pulses with a duration of less than one period (subcycle pulses) made it possible to use them to control the dynamics of wave packets in matter and contributed to the birth of attosecond science [12–14]. Simultaneously, the issue of obtaining pulses with a high degree of unipolarity ζ is being currently discussed (see reviews [15, 16], and also works [17–21] and references therein):

$$\zeta = \frac{\left|\int_{-\infty}^{\infty} E\mathrm{d}t\right|}{\int_{-\infty}^{\infty}|E|\mathrm{d}t} \approx 1. \tag{1}$$

R.M. Arkhipov St. Petersburg State University, Universitetskaya nab. 7/9, 199034 St. Petersburg, Russia; ITMO University, Kronverskii prosp. 49, 197101 St. Petersburg, Russia;
e-mail: arkhipovrostislav@gmail.com;
M.V. Arkhipov St. Petersburg State University, Universitetskaya nab. 7/9, 199034 St. Petersburg, Russia;
I. Babushkin Institute of Quantum Optics, Leibniz University, 30167, Hannover, Germany;
A.V. Pakhomov ITMO University, Kronverskii prosp. 49, 197101 St. Petersburg, Russia;
N.N. Rosanov Vavilov State Optical Institute, Kadetskaya liniya 5, stroenie 2, 199053 St. Petersburg; ITMO University, Kronverskii prosp. 49, 197101 St. Petersburg, Russia; Ioffe Institute, Polytekhnicheskaya ul. 26, 194021 St. Petersburg Russia

Received 31 January 2018; revision 20 March 2018
Kvantovaya Elektronika **48** (6) 532–536 (2018)
Translated by M.A. Monastyrskiy

Strictly unipolar pulses (UPs) contain a splash of a single-polarity field (half-wave), in contrast to the ordinary multicycle bipolar pulses, in which the degree of unipolarity ζ is close to zero. Because of the presence of a constant component, UPs can be used to effectively control the dynamics of wave packets in matter, exerting a unidirectional action on electric charges [15].

The duration of such subcycle UPs may be less than not only the polarisation relaxation time T_2 and the population difference T_1 in a resonant medium, but also than the period of natural oscillations of resonant transitions in a medium. In the first case, coherent interaction of pulses with a medium occurs and self-induced transparency (SIT) may appear when a 2π SIT pulse propagates in a resonant medium without losses [22–24]. Coherent propagation in resonant media is well studied in the case of long [22–24] and extremely short bipolar pulses [25–30] with a zero electric area (integral of electric field strength). Despite these advantages of UPs, their interaction with resonant amplifying media has not been fully investigated so far. The UP propagation in absorbing media was mainly studied [31–34]. Analytical results were obtained in a number of works, but they mainly concern the stationary soliton solutions [35–38]. Moreover, the solutions are found using different approximations and do not describe the UP propagation dynamics. The propagation of video pulses in various nonlinear absorbing media with no regard to the media relaxation and in the approximation of unidirectional propagation was considered in [6].

A separate issue is the propagation of subcycle UPs in a resonant amplifying medium. Coherent amplification of bipolar ESPs was considered earlier [39–43]. To date, the dynamics of long pulses in a coherent amplifying medium, when the slowly varying envelope approximation (SVEA) and the rotating wave approximation (RWA) are valid, is well studied (see reviews [23, 44, 45] and references therein). A characteristic feature of amplification is that, in the case of coherent propagation of a long pulse in an amplifying medium, when the SVEA and RWA are valid, it is possible to form a 2π-pulse which transforms the medium into the ground state with a simultaneous decrease in the pulse duration. However, in the case of subcycle UPs, the amplification dynamics can be considerably more complicated due to the short duration of UPs and the inapplicability of SVEA and RWA. For such pulses, because of the inapplicability of the pulse envelope notion, the McCall–Hahn area theorem loses its meaning [26–30]. In this case, as shown in [43, 46–48], the electric area of the pulse is preserved (see Section 2). Moreover, analytical and numerical solutions of the Maxwell–Bloch equations [36–38, 45] were derived under a number of assumptions and are unsuitable for the correct description of

UP amplification. For example, the medium relaxation and rapid oscillations at the optical frequency of resonance transitions were neglected (see Section 2).

To obtain a more detailed picture of the subcycle pulse propagation in a resonant medium with a minimal number of simplifying assumptions, numerical calculations are necessary. In this connection, we consider the UP propagation dynamics of subcycle (attosecond) duration in a two-level resonant amplifying medium for the case when the input pulse duration τ_p is less than the period $T_0 = 2\pi/\omega_0$ of natural oscillations of the resonance transition.

The calculation is based on the numerical solution of the system of Maxwell–Bloch equations that do not contain the above approximations. It is found that the coherent amplification dynamics of a subcycle pulse is essentially different from that for long pulses, when the SVEA and RWA are valid. In particular, for the first time, the rule of the electric pulse area conservation obtained analytically from Maxwell's equations [43, 46–48] for the amplifying medium is illustrated on the basis of numerical simulation. It is shown that the dynamics of the amplified pulse essentially depends on the medium relaxation time, which was not previously taken into account.

2. Theoretical model and basic notions

The basic notion widely used in describing coherent resonant interactions of long pulses (containing a large number of periods of field oscillations) with resonant media is the notion of the area under the pulse envelope (integral of the slowly varying field amplitude with respect to time) [22–24]:

$$\Theta(z) \equiv \frac{d_{12}}{\hbar}\int_{-\infty}^{\infty} \varepsilon(z,t)\,\mathrm{d}t, \tag{2}$$

where d_{12} is the dipole moment of the transition, and $\varepsilon(t)$ is the slow envelope of the pulse. The pulse area evolution in the coherent propagation of long pulses is described by the McCall–Hahn area theorem [22–24]. If the pulse duration is of the order of the field oscillation period and less, SVEA and RWA, and, as a consequence, the area theorem, are not applicable. For ESPs, we can speak about the pulse electric area, which, within the framework of one-dimensional consideration, retains its value for any longitudinal coordinates [43, 46–48]:

$$S_E \equiv A\int_{-\infty}^{\infty} E(t)\,\mathrm{d}t. \tag{3}$$

Here, A is a constant. Further, as in [36–38], we choose $A = 2d_{12}/\hbar$. To study the dynamics of subcycle pulse amplification in a resonant amplifying medium, we use the Maxwell–Bloch system of equations. Due to the small (subcyclic) duration of the incident pulse, SVEA and RWA are not used in this system of equations, which has the form [17–34]:

$$\frac{\partial\rho_{12}(z,t)}{\partial t} = -\frac{\rho_{12}(z,t)}{T_2} + \mathrm{i}\omega_0\rho_{12}(z,t) - \frac{\mathrm{i}}{\hbar}d_{12}E(z,t)n(z,t), \tag{4}$$

$$\frac{\partial n(z,t)}{\partial t} = -\frac{n(z,t)-n_0}{T_1} + \frac{4}{\hbar}d_{12}E(z,t)\,\mathrm{Im}\rho_{12}(z,t), \tag{5}$$

$$P(z,t) = 2N_0 d_{12}\,\mathrm{Re}\rho_{12}(z,t), \tag{6}$$

$$\frac{\partial^2 E(z,t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E(z,t)}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P(z,t)}{\partial t^2}, \tag{7}$$

where P is the polarisation of medium, N_0 is the concentration of active centres, E is the electric field intensity with a fixed linear polarisation; c is the speed of light in vacuum, ω_0 is the resonant transition frequency of a medium ($\lambda_0 = 2\pi c/\omega_0$ is the resonance transition wavelength), and n_0 is the population difference of two operating levels in the electric field absence.

The medium is described by Eqns (4)–(7) in the two-level approximation using the density matrix formalism. Equation (4) describes the evolution of the off-diagonal element of the density matrix ρ_{12}, while Eqn (5) – the behaviour of the population difference $n \equiv \rho_{11} - \rho_{22}$ between the ground and excited states of the two-level system. The medium polarisation P is related to the off-diagonal element of the density matrix ρ_{12} by expression (6).

Currently, the Maxwell–Bloch system of equations (4)–(7) is actively used to study the coherent propagation of ESPs in a resonance medium [17–34]. If the pulse duration reaches a subcycle value, the question arises as to the applicability of the two-level approximation in describing the resonant medium. It should be noted that any state of a quantum system can be described on the basis of states with certain energy, which constitute a complete set. Up to the ionisation threshold, the states of the continuous spectrum are excluded from consideration, so that only the discrete spectrum remains. In this case, it is necessary to find out how many levels are sufficient to describe the phenomenon in question with certain accuracy. On the basis of numerical calculations conducted in [30, 41–43], it has been shown that the main features of the coherent propagation of ESPs with single-cycle and subcycle durations (in particular in the SIT regime) in two-level amplifying and absorbing media are qualitatively similar to the case of multilevel systems and with taking into account the inhomogeneous broadening [49]. Moreover, the two-level system is the simplest object that has been used for many years in studying the coherent interaction of short pulses with matter [21–34]. This model makes it possible to understand at an elementary level the main features of the coherent UP propagation in a resonance medium, and may serve as a first approximation for more complex models. Therefore, in the present work, for simplicity and clarity, a two-level approximation is also used, with inhomogeneous broadening of the spectral transition being neglected. In addition, we neglect the diffraction of radiation, which is justified in the propagation of beams at the distances smaller than the diffraction length, and solve the one-dimensional propagation problem.

As in work [30–34], equations (4), (5) for the density matrix were solved by the fourth-order Runge–Kutta method. The wave equation (7) was solved by the method of finite differences described in [50]. The spatial integration domain had a length $L = 30\lambda_0$. The resonance medium was located along the z axis at the centre of the region between the points with coordinates $z_1 = 9\lambda_0$ and $z_2 = 23\lambda_0$.

In the case of subcycle UPs, the authors of [6, 35–38] investigated analytically and numerically the UP dynamics in amplifying and absorbing media using Eqns (3)–(7). In this case, we made a number of assumptions: we used the unidirectional propagation approximation [6] and neglected the relaxation terms in Eqns (4) and (5) as well as the term that

$i\omega_0\rho_{12}(z,t)$ oscillates rapidly compared to the derivative $\partial\rho_{12}(z,t)/\partial t$ in Eqn (4) [35–38]. The latter assumption was justified by the condition of a small pulse duration: $\omega_0\tau_p \ll 1$. As a result, the system of equations (3)–(7) was reduced to the sine-Gordon equation describing the pulse electric area evolution in space and time. The total area of such a complex wave packet becomes equal to π, and the packet takes in the entire energy stored in the medium. Moreover, the carrier frequency of the pulse increases and shifts to the blue region in the course of its propagation. It is clear that within these approximations the pulse electric area (3) will also be a variable quantity. However, a more general consideration [43, 46–48] being free of these approximations shows that this area is preserved within the framework of the one-dimensional wave equation (7). The pulse area conservation in the amplifying medium leads to a loss of unipolarity during the pulse propagation, which is confirmed by the results of the numerical calculations given below.3.

3. Results of numerical calculations

A series of numerical calculations was performed by using the system of equations (3)–(7). A subcycle UP with an envelope in the form of hyperbolic secant was directed from left to right into the medium:

$$E(t) = E_0 \mathrm{sech}(t/\tau_p). \tag{8}$$

The electric area of such a pulse is given by the expression

$$S_E = 2d_{12}E_0\tau_p\pi/\hbar. \tag{9}$$

We studied the dynamics of the field, polarisation, and population difference during the pulse propagation in the amplifying medium, with variations in the input pulse area, medium relaxation time, and input pulse duration.

3.1. The case of small relaxation times of the medium

Consider the pulse amplification with a large input area. An example of electric field evolution of a pulse with an input electric area $S_E = 2d_{12}E_0\tau_p\pi/\hbar = 0.9\pi$ is shown in Fig. 1. The calculation parameters are as follows: $\lambda_0 = 700$ nm, $d_{12} = 5$ D, $T_1 = 100$ fs, $T_2 = 0.005$ ps, $N_0 = 10^{21}$ cm^{-3}, $E_0 = 2.4\times10^5$ CGSE units, $S_E = 0.9\pi$, $\tau_p = T_0/6 = 388$ as, and $n_0 = 1$.

We first consider the case when the relaxation times of the medium are small. It can be seen from Fig. 1 that the pulse becomes bipolar as it propagates, which is manifested in the appearance of subpulses in the form of tails. For clarity, Fig. 2 shows the time dependences of the field at the entrance to the medium (Fig. 2a), at the medium centre (Fig. 2b), and at the exit from the medium (Fig. 2c). During the pulse propagation in the amplifying medium, its amplitude increases at the leading edge, while long tails appear at the trailing edge, which gradually attenuate. This dynamics is significantly different from the case of coherent amplification of long pulses [23, 44, 45], when SVEA and RWA are valid, and the amplification is only accompanied by a decrease in the pulse duration, an increase in the amplitude, and the formation of a π-pulse. The appearance of tails is due to the fact that a short pulse propagating in the medium causes oscillations of macroscopic polarisation at the resonant frequency of the medium transition. This polarisation re-emits the field in antiphase with the field of the incident pulse – 'coherent ringing of the medium', which forms the tails of opposite polarity at the trailing edge [51]. This field of the medium re-radiation propagates behind the incident pulse and can amplify, since the medium is amplifying. Such a result is qualitatively consistent with the analytical result derived in [36–38]. The difference is that, due to the medium relaxation, these tails attenuate as the pulse propagates.

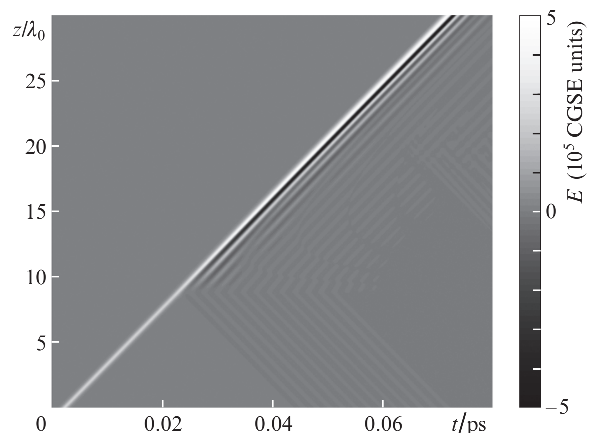

Figure 1. Pulse evolution with an initial electric area of $S_E = 0.9\pi$. The calculation parameters are indicated in the text. The resonance medium was located along the z axis between the points with coordinates $z_1 = 9\lambda_0$ and $z_2 = 23\lambda_0$.

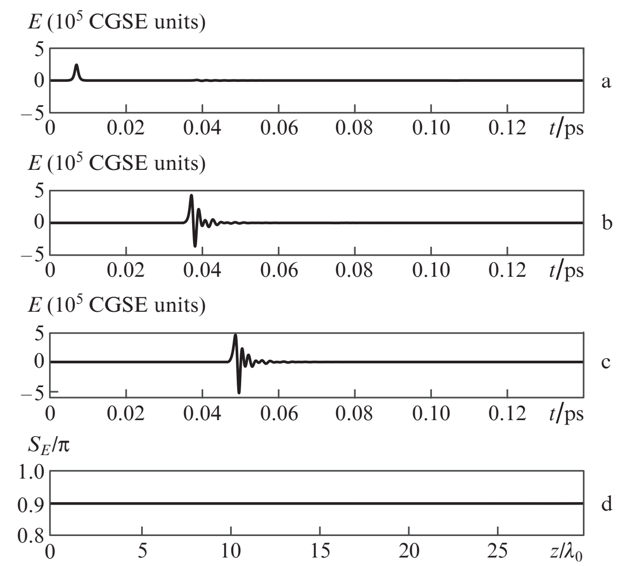

Figure 2. Time dependences of the electric field at (a) the medium entrance, (b) the medium centre for $z = 15\lambda_0$, and (c) the medium exit for $z = 20\lambda_0$ (c), and also (d) the pulse electric area $S_E(z)$ at each point inside the integration region.

Finally, the electric area of the pulse is conserved, which is illustrated in Fig. 2d, which shows the electric area dependence on the coordinate, obtained in numerical calculations. This dependence agrees with rule of pulse area conservation derived in [43, 46–48]. Calculations show that the field unipolarity degree (1) at the entrance to the medium is $\zeta = 0.9$. It is less than unity as a result of the appearance of reflected radiation of opposite polarity. At the exit from the medium, the unipolarity degree decreases to $\zeta = 0.26$. Thus, the bipolar component becomes dominant after amplification in the medium.

The rule of the pulse electric area conservation turns out essential for the dynamics of the subcycle pulse amplification, whereas, in the case of amplification of long pulses, it is quite acceptable to use the McCall–Hahn area theorem, according to which the area under the pulse envelope (2) changes during the pulse propagation.

3.2. The case of large relaxation times of the medium

With increasing medium relaxation time, the oscillating polarisation does not have time to attenuate during the pulse propagation and emits a field with a phase shift π with respect to the incident pulse field. This is manifested in the fact that a long undamped tail of opposite polarity appears in the pulse during its propagation (Figs 3 and 4). The presence of such undamped sign-alternating tails is qualitatively consistent with the results of numerical calculations from paper [38] in disregard of the medium relaxation, and with approximate analytical results [36–38], also obtained without allowance for relaxation.

The population difference dynamics in the medium in the case of large relaxation times is shown in Figs 5 and 6. At the exit from the medium, a long pulse is observed, containing a long tail of opposite polarity (see Fig. 4c). Accordingly, the inversion has a complex dynamics (Fig. 6c). After the action of such a long pulse terminates, the medium evolves virtually to the ground state with an inversion very close to +1. In this case, the field unipolarity degree at the entrance to the medium is $\zeta = 0.27$. It is again less than unity because of the appearance of reflected radiation of opposite polarity. At the exit from the medium, the unipolarity degree is even smaller: $\zeta = 0.05$.

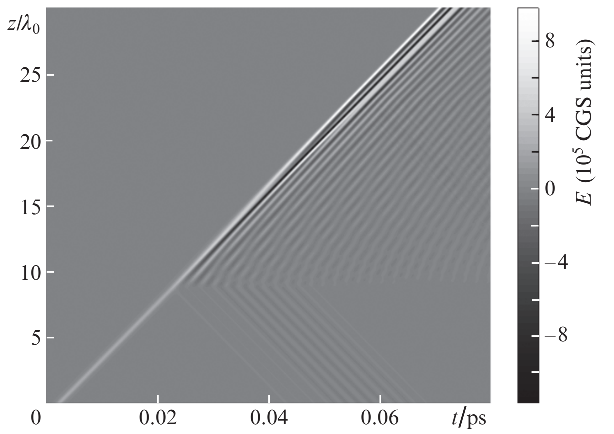

Figure 3. Pulse evolution with an initial electric area of $S_E = 0.9\pi$ for long relaxation times: $T_1 = 1$ ps, $T_2 = 0.5$ ps. Other calculation parameters are indicated in the text. The resonant medium was located along the z axis between the points $z_1 = 9\lambda_0$ and $z_2 = 23\lambda_0$.

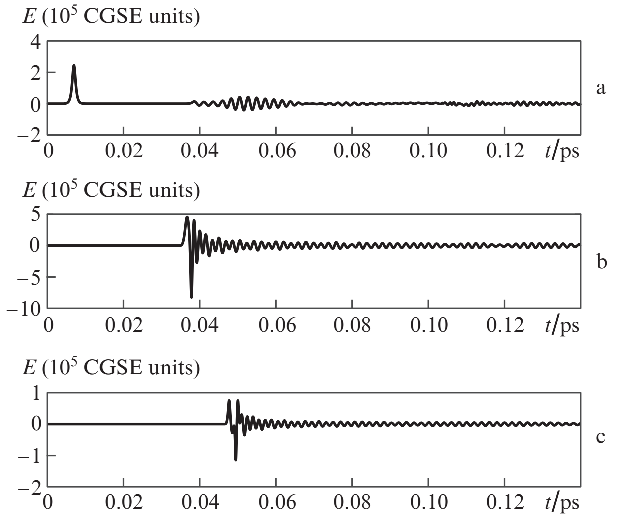

Figure 4. Time dependences of the electric field at (a) the medium entrance, (b) the medium centre for $z = 15\lambda_0$, and (c) the medium exit for $z = 20\lambda_0$, corresponding to Fig. 3.

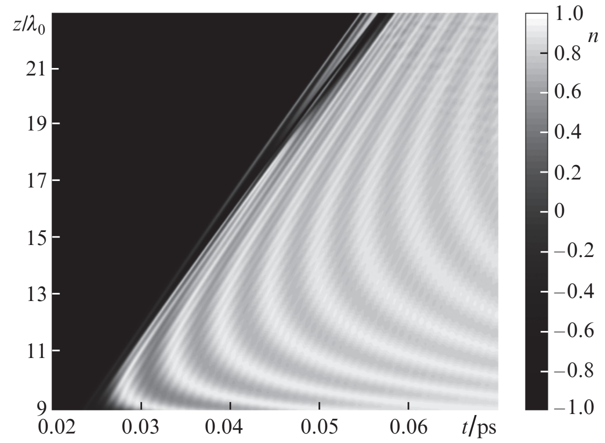

Figure 5. Spatiotemporal dependence of the population difference in the medium corresponding to Figs 3 and 4.

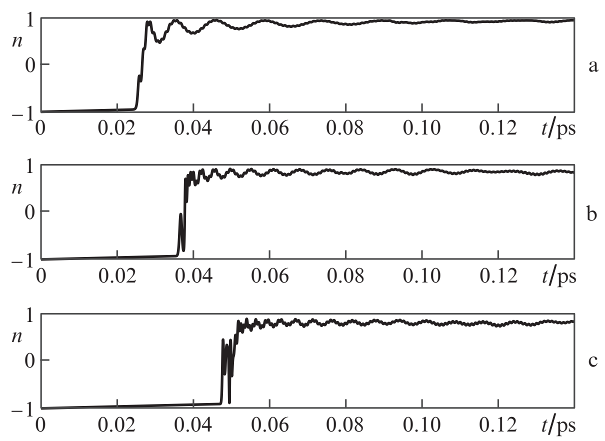

Figure 6. Time dependences of the population difference for two different sections of the medium corresponding to Fog. 5: at (a) the medium entrance, (b) the medium centre for $z = 15\lambda_0$, and (c) the medium exit for $z = 20\lambda_0$.

Consequently, for large relaxation times, a complex wave packet is formed in the medium, which acts on the medium like a π-pulse, removing virtually the entire energy stored in the active medium. However, the pulse electric area is also preserved in this case.

4. Conclusions

Thus, the propagation dynamics of a subcycle UP in a coherent resonant amplifying medium was studied by solving numerically the Maxwell – Bloch system of equations without the use of SVEA and RWA. The results of numerical calculations allow for a more complete analysis of the field dynamics beyond the framework of a number of approximations adopted in previous analytical studies. It is shown that, as the pulse propagates, the unipolarity degree decreases: it becomes bipolar, and part of the energy obtained from the active medium at the leading edge passes over into the rapidly oscillating tails.

It is found that the dynamics of the subcycle pulse propagation is more complicated than in the case of long pulses in SVEA and RWA. For long pulses, the area under the pulse envelope changes as the pulse propagates, while in general case, including the ESP, the pulse electric area is conserved. The latter imposes significant limitations on the possibility of amplification of subcycle UPs.

Significant differences in the dynamics of amplification of long pulses and UPs have been revealed. If, for long pulses, the amplification process is characterised by pulse compression, an increase in the amplitude, and convergence of the area under the envelope to π, then in the case of a subcycle UP propagating in the amplifying medium, the components of opposite polarity appear. The duration and form of these tails depend on the media relaxation time. At long relaxation times, the presence of a long, slowly damping tail, consisting of a large number of oscillation cycles, is characteristic.

Acknowledgements. The study on the area conservation of a subcycle pulse during its propagation was supported by the Russian Foundation for Basic Research (Grant No. 16-02-00762). The study on the amplification dynamics of a subcycle pulse was supported by the Russian Science Foundation (Grant No. 17-19-01097).

References

1. Krausz F., Ivanov M. *Rev. Mod. Phys.*, **81**, 163 (2009).
2. Manzoni C., Mücke O.D., Cirmi G., Fang S., Moses J., Huang S.-W., et al. *Laser Photonics Rev.*, **9**, 129 (2015).
3. Calegari F., Sansone G., Stagira S., Vozzi C., Nisoli M. *J. Phys. B: At. Mol. Opt. Phys.*, **49**, 062001 (2016).
4. Hassan M.T. et al. *Nature*, **530** (7588), 66 (2016).
5. Babushkin I. et al. *Phys. Rev. Lett.*, **105** (5), 053903 (2010).
6. Maimistov A.I. *Quantum Electron.*, **30** (4), 287 (2000) [*Kvantovaya Elektron.*, **30** (4), 287 (2000)].
7. Maimistov A.I. *Quantum Electron.*, **40** (9), 756 (2010) [*Kvantovaya Elektron.*, **40** (9), 756 (2010)].
8. Skryabin D.V., Gorbach A.V. *Rev. Mod. Phys.*, **82**, 1287 (2010).
9. Dudley J.M., Genty G., Coen S. *Rev. Mod. Phys.*, **78**, 1135 (2006).
10. Leblond H., Mihalache D. *Phys. Rep.*, **523**, 61 (2013).
11. Mihalache D.U. *Roman. Rep. Phys.*, **69**, 403 (2017).
12. Gallmann L., Cirelli C., Keller U. *Annu. Rev. Phys. Chem.*, **63**, 447 (2012).
13. Ramasesha K., Leone S.R., Neumark D.M. *Annu. Rev. Phys. Chem.*, **67**, 41 (2016).
14. Ciappina M.F., Pérez-Hernández J.A., Landsman A.S. *Rep. Prog. Phys.*, **80**, 054401 (2017).
15. Arkhipov R.M., Pakhomov A.V., Arkhipov M.V., et al. *JETP Lett.*, **105** (6), 408 (2017) [*Pis'ma Zh. Eksp. Teor. Fiz.*, **105** (6), 388 (2017)].
16. Arkhipov R.M., Pakhomov A.V., Arkhipov M.V., Babushkin I., et al. *Laser Phys.*, **27** (5), 053001 (2017).
17. Kalosha V.P., Herrmann J. *Phys. Rev. Lett.*, **83**, 544 (1999).
18. Song X., Yang W., Zeng Z., et al. *Phys. Rev. A*, **82** (5), 053821 (2010).
19. Kozlov V.V., Rosanov N.N., Angelis C.D., Wabnitz S. *Phys. Rev. A*, **84**, 023818 (2011).
20. Arkhipov R.M., Arkhipov M.V., Belov P.A., Tolmachev Yu.A., Babushkin I. *Laser Phys. Lett.*, **13** (4), 046001 (2016).
21. Pakhomov A.V., Arkhipov R.M., Babushkin I.V., Arkhipov M.V., Tolmachev Yu.A., Rosanov N.N. *Phys. Rev. A*, **95**, 013804 (2017).
22. McCall S.L., Hahn E.L. *Phys. Rev.*, **183**, 457 (1969).
23. Kryukov P.G., Letokhov V.S. *Sov. Phys. Usp.*, **12**, 641 (1970) [*Usp. Fiz. Nauk*, **99**, 169 (1969)].
24. Allen L., Eberly J.H. *Optical Resonance and Two-Level Atoms* (New York: Wiley, 1975; Moscow: Mir, 1978).
25. Hughes S. *Phys. Rev. Lett.*, **81** (16), 3363 (1998).
26. Xiao J., Wang Z., Xu Z. *Phys. Rev. A*, **65**, 031402 (2002).
27. Tarasishin A.V., Magnitskii S.A., Shuvaev V.A., Zheltikov A.M. *Opt. Express*, **8**, 452 (2001).
28. Xie X.T., Macovei M.A. *Phys. Rev. Lett.*, **104**, 073902 (2010).
29. Novitsky D.V. *Phys. Rev. A*, **86**, 063835 (2012).
30. Arkhipov R.M., Pakhomov A.V., Arkhipov M.V., Babushkin I., Demircan A., et al. *Sci. Rep.*, **7** (1), 12467 (2017).
31. Arkhipov R.M., Arkhipov M.V., Pakhomov A.V., Babushkin I., Rosanov N.N. *Quantum Electron.*, **47** (7), 589 (2017) [*Kvantovaya Elektron.*, **47** (7), 589 (2017)].
32. Arkhipov R.M., Arkhipov M.V., Pakhomov A.V., Babushkin I., Rosanov N.N. *Laser Phys. Lett.*, **14** (9), 1 (2017).
33. Arkhipov R.M., Arkhipov M.V., Pakhomov A.V., Babushkin I., Rosanov N.N. *Opt. Spectrosc.*, **123** (4), 610 (2017) [*Opt. Spektroskop.*, **123** (4), 600 (2017)].
34. Arkhipov R.M., Arkhipov M.V., Pakhomov A.V., Zhiguleva D.O., Rosanov N.N. *Opt. Spectrosc.*, **124** (4), 541 (2018) [*Opt. Spektroskop.*, **124** (4), 510 (2018)].
35. Bullough R.K., Ahmad F. *Phys. Rev. Lett.*, **27**, 330 (1971).
36. Belenov E.M., Kryukov P.G., Nazarkin A.V., Oraevsky A.N., Uskov A.V. *Pis'ma Zh. Eksp. Teor. Fiz.*, **47**, 442 (1988).
37. Belenov E.M., Nazarkin A.V. *JETP Lett.*, **51**, 288 (1990) [*Pis'ma Zh. Eksp. Teor. Fiz.*, **51**, 252 (1990)].
38. Belenov E.M., Nazarkin A.V., Ushchapovskii V.A. *Sov. Phys. JETP*, **73**, 422 (1991) [*Zh. Eksp. Teor. Fiz.*, **100**, 762 (1991)].
39. Tarasishin A.V., Magnitskii S.A., Zheltikov A.M. *Laser Phys.*, **11**, 620 (2001).
40. Tarasishin A.V., Magnitskii S.A., Zheltikov A.M. *Opt. Commun.*, **193**, 187 (2001).
41. Rosanov N.N., Semenov V.E., Vysotina N.V. *Quantum Electron.*, **38** (2), 137 (2008) [*Kvantovaya Elektron.*, **38** (2), 137 (2008)].
42. Vysotina N.V., Rosanov N.N., Semenov V.E. *Opt. Spectrosc.*, **106** (5), 713 (2009) [*Opt. Spektroskop.*, **106** (5), 793 (2009)].
43. Rosanov N.N. *Dissipativnye opticheskie solitony. Ot mikro k nano- i atto-* (Dissipative Optical Solitons. From Micro to Nano- and Atto-) (M: Fizmatlit, 2011) Ch. 17.
44. Lamb G.L. *Rev. Mod. Phys.*, **43**, 99 (1971).
45. Belenov E.M., Kryukov P.G., Nazarkin A.V., Oraevsky A.N., Uskov A.V. *J. Opt. Soc. Am. B*, **5** (5), 946 (1988).
46. Rosanov N.N. *Opt. Spectrosc.*, **107** (5), 721 (2009) [*Opt. Spektroskop.*, **107** (5), 761 (2009)].
47. Rosanov N.N., Kozlov V.V., Wabnitz S. *Phys. Rev. A*, **81**, 043815 (2010).
48. Rosanov N.N. *Opt. Spectrosc.*, **107** (5), 721 (2009) [*Opt. Spektroskop.*, **118** (6), 975 (2015)].
49. Kozlov V.V., Rosanov N.N. *Phys. Rev. A*, **87** (4), 043836 (2013).
50. Taflove A., Hagness S.C. *Computational Electrodynamics: the Finite-Difference Time-Domain Method* (Boston–London: Artech House, 2005).
51. Siddons P. *J. Phys. B: At. Mol. Opt. Phys.*, **47** (9), 093001 (2014).